MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 3, 2009
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QUIZ 3

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A SUMMARY OF USEFUL INFORMATION IS AT THE END OF THE EXAM

Problem	**Maximum**	**Score**
1	25	
2	35	
3	40	
TOTAL	100	

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Your Name

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%).

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB.

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\text{bary}}/n_\gamma$.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct.

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5 (ii) 1/50 (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or F ?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. T or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

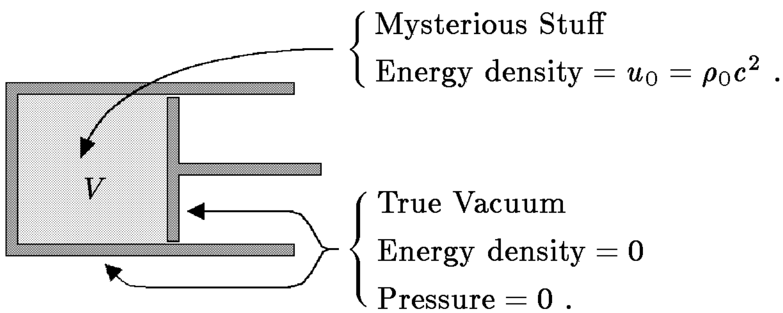
(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. T or F ?

PROBLEM 2: PRESSURE, ENERGY DENSITY, AND COSMIC EVOLUTION WITH MYSTERIOUS STUFF *(35 points)*

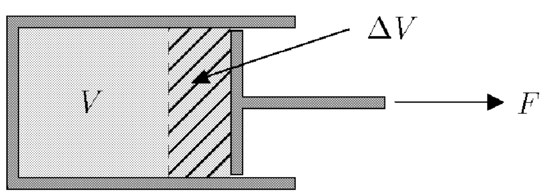
The following problem is a conglomeration of Problem 6 and 7 of the Review Problems for Quiz 3.

On Problem Set 7, a thought experiment involving a piston was used to show how pressure can be calculated by using the principle of energy conservation. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(10 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

Now consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced above. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/a^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(d) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{a(t)}{a(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(e) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

PROBLEM 3: A NEW THEORY OF THE WEAK INTERACTIONS *(40 points)*

Suppose a New Theory of the Weak Interactions (NTWI) was proposed, which differs from the standard theory in two ways. First, the NTWI predicts that the weak interactions are somewhat weaker than in the standard model. In addition, the theory implies the existence of new spin-$\frac{1}{2}$ particles (fermions) called the R^+ and R^-, with a rest energy of 50 MeV (where 1 MeV $= 10^6$ eV). This problem will deal with the cosmological consequences of such a theory.

The NTWI will predict that the neutrinos in the early universe will decouple at a higher temperature than in the standard model. Suppose that this decoupling takes place at $kT \approx 200$ MeV. This means that when the neutrinos cease to be thermally coupled to the rest of matter, the hot soup of particles would contain not only photons, neutrinos, and e^+-e^- pairs, but also μ^+, μ^-, π^+, π^-, and π^0 particles, along with the R^+-R^- pairs. (The muon is a particle which behaves almost identically to an electron, except that its rest energy is 106 MeV. The pions are the lightest of the mesons, with zero angular momentum and rest energies of 135 MeV and 140 MeV for the neutral and charged pions, respectively. The π^+ and π^- are antiparticles of each other, and the π^0 is its own antiparticle. Zero angular momentum implies a single spin state.) You may assume that the universe is flat.

(a) *(10 points)* According to the standard particle physics model, what is the mass density ρ of the universe when $kT \approx 200$ MeV? What is the value of ρ at this temperature, according to NTWI? Use either g/cm^3 or kg/m^3. (If you wish, you can save time by not carrying out the arithmetic. If you do this, however, you should give the answer in "calculator-ready" form, by which I mean an expression involving pure numbers (no units), with any necessary conversion factors included, and with the units of the answer specified at the end. For example, if asked how far light travels in 5 minutes, you could answer $2.998 \times 10^8 \times 5 \times 60$ m.)

(b) *(10 points)* According to the standard model, the temperature today of the thermal neutrino background should be $(4/11)^{1/3} T_\gamma$, where T_γ is the temperature of the thermal photon background. What does the NTWI predict for the temperature of the thermal neutrino background?

(c) *(10 points)* According to the standard model, what is the ratio today of the number density of thermal neutrinos to the number density of thermal photons? What is this ratio according to NTWI?

(d) *(10 points)* Since the reactions which interchange protons and neutrons involve neutrinos, these reactions "freeze out" at roughly the same time as the neutrinos decouple. At later times the only reaction which effectively converts neutrons to protons is the free decay of the neutron. Despite the fact that neutron decay is a weak interaction, we will assume that it occurs with the usual 15 minute mean lifetime. Would the helium abundance predicted by the NTWI be higher or lower than the prediction of the standard model? To within 5 or 10%, what would the NTWI predict for the percent abundance (by weight) of helium in the universe? (As in part (a), you can either carry out the arithmetic, or leave the answer in calculator-ready form.)

USEFUL INFORMATION:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\text{coord}} = \frac{c}{a(t)} \ .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\, \rho_r(t_i) \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$):

$$a(t) \propto t^{2/3} \qquad \text{(matter-dominated)} \ ,$$

$$a(t) \propto t^{1/2} \qquad \text{(radiation-dominated)} \ ,$$

$$\Omega = 1 \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

$$\text{or:}\qquad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2} \ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ ,$$

$$g_{e^+e^-}^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \ .$$

CHEMICAL EQUILIBRIUM:

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

HORIZON DISTANCE:

$$\begin{aligned}\ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}}c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2} ,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z} ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} .$$

Age of universe:

$$\begin{aligned}t_0 &= \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}} \\ &= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .\end{aligned}$$

Look-back time:

$$t_{\text{look-back}}(z) =$$

$$\frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}} .$$

PHYSICAL CONSTANTS:

$$G = 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2} = 6.674 \times 10^{-8}\ \mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-23}\,\text{joule/K} \\ &= 1.381 \times 10^{-16}\,\text{erg/K} \\ &= 8.617 \times 10^{-5}\,\text{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-34}\ \text{joule} \cdot \text{s} \\ &= 1.055 \times 10^{-27}\ \text{erg} \cdot \text{s} \\ &= 6.582 \times 10^{-16}\ \text{eV} \cdot \text{s} \end{aligned}$$

$$\begin{aligned} c &= 2.998 \times 10^{8}\ \text{m/s} \\ &= 2.998 \times 10^{10}\ \text{cm/s} \end{aligned}$$

$$\hbar c = 197.3\ \text{MeV-fm}, \quad 1\ \text{fm} = 10^{-15}\ \text{m}$$

$$1\ \text{yr} = 3.156 \times 10^{7}\ \text{s}$$

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{joule} = 1.602 \times 10^{-12}\ \text{erg}$$

$$\begin{aligned} 1\ \text{GeV} = 10^{9}\ \text{eV} &= 1.783 \times 10^{-27}\ \ \text{kg (where } c \equiv 1) \\ &= 1.783 \times 10^{-24}\ \text{g}\ . \end{aligned}$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\begin{aligned} \ell_P &= \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-35}\ \text{m}\ , \\ &= 1.616 \times 10^{-33}\ \text{cm}\ , \\ t_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44}\ \text{s}\ , \\ m_P &= \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-8}\ \text{kg}\ , \\ &= 2.177 \times 10^{-5}\ \text{g}\ , \\ E_P &= \sqrt{\frac{\hbar c^5}{G}} = 1.221 \times 10^{19}\ \text{GeV}\ . \end{aligned}$$