MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 11, 2011
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QUIZ 1 SOLUTIONS

Quiz Date: October 6, 2011

PROBLEM 1: DID YOU DO THE READING? *(25 points)*†

(a) *(10 points)* To determine the distance of the galaxies he was observing Hubble used so called *standard candles*. Standard candles are astronomical objects whose intrinsic luminosity is known and whose distance is inferred by measuring their apparent luminosity. First, he used as standard candles variable stars, whose intrinsic luminosity can be related to the period of variation. Quoting Weinberg's *The First Three Minutes*, chapter 2, pages 19-20:

In 1923 Edwin Hubble was for the first time able to resolve the Andromeda Nebula into separate stars. He found that its spiral arms included a few bright variable stars, with the same sort of periodic variation of luminosity as was already familiar for a class of stars in our galaxy known as Cepheid variables. The reason this was so important was that in the preceding decade the work of Henrietta Swan Leavitt and Harlow Shapley of the Harvard College Observatory had provided a tight relation between the observed periods of variation of the Cepheids and their absolute luminosities. (Absolute luminosity is the total radiant power emitted by an astronomical object in all directions. Apparent luminosity is the radiant power received by us in each square centimeter of our telescope mirror. It is the apparent rather than the absolute luminosity that determines the subjective degree of brightness of astronomical objects. Of course, the apparent luminosity depends not only on the absolute luminosity, but also on the distance; thus, knowing both the absolute and the apparent luminosities of an astronomical body, we can infer its distance.) Hubble, observing the apparent luminosity of the Cepheids in the Andromeda Nebula, and estimating their absolute luminosity from their periods, could immediately calculate their distance, and hence the distance of the Andromeda Nebula, using the simple rule that apparent luminosity is proportional to the absolute luminosity and inversely proportional to the square of the distance.

He also used particularly bright stars as standard candles, as we deduce from page 25:

Returning now to 1929: Hubble estimated the distance to 18 galaxies from the apparent luminosity of their brighest stars, and compared these distances with the galaxies' respective velocities, determined spectroscopically from their Doppler shifts.

Note: since from reading just the first part of Weinberg's discussion one could be induced to think that Hubble used just Cepheids as standard candles, students who mentioned only Cepheids got 9 points out of 10. In fact, however,

Hubble was able to identify Cepheid variables in only a few galaxies. The Cepheids were crucial, because they served as a calibration for the larger distances, but they were not in themselves sufficient.

(b) *(5 points)* Quoting Weinberg's *The First Three Minutes*, chapter 2, page 21:

We would expect intuitively that at any given time the universe ought to look the same to observers in all typical galaxies, and in whatever directions they look. (Here, and below, I will use the label "typical" to indicate galaxies that do not have any large peculiar motion of their own, but are simply carried along with the general cosmic flow of galaxies.) This hypothesis is so natural (at least since Copernicus) that it has been called **the** *Cosmological Principle by the English astrophysicist Edward Arthur Milne.*

So the Cosmological principle basically states that the universe appears as homogeneous and isotropic (on scales of distance large enough) to any typical observer, where typical is referred to observers with small local motion compared to the expansion flow. Ryden gives a more general definition of Cosmological Principle, which is valid as well. Quoting Ryden's *Introduction to Cosmology*, chapter 2, page 11 or 14 (depending on which version):

However, modern cosmologists have adopted the **cosmological principle,** *which states: There is nothing special about our location in the universe. The cosmological principle holds true only on large scales (of 100 Mpc or more).*

(c) *(10 points)* Quoting again Ryden's *Introduction to Cosmology*, chapter 2, page 9 or 11:

Saying that the universe is **isotropic** *means that there are no preferred directions in the universe; it looks the same no matter which way you point your telescope. Saying that the universe is* **homogeneous** *means that there are no preferred locations in the universe; it looks the same no matter where you set up your telescope.*

(i) $\boxed{\text{False.}}$ If the universe is isotropic around one point it does not need to be homogeneous. A counter-example is a distribution of matter with spherical symmetry, that is, with a density which is only a function of the radius but does not depend on the direction: $\rho(r, \theta, \phi) \equiv \rho(r)$. In this case for an observer at the center of the distribution the universe looks isotropic but it is not homogeneous.

(ii) $\boxed{\text{True.}}$ For the case of Euclidean geometry isotropy around two or more distinct points does imply homogeneity. Weinberg shows this in chapter 2, page 24. Consider two observers, and two arbitrary points A and B which we would like to prove equivalent. Consider a circle through point A, centered on observer 1, and another circle through point B, centered on observer 2. If C is a point on the intersection of the two circles, then isotropy about the two observers implies that $A = C$ and $B = C$, and hence $A = B$. (This argument was good enough for Weinberg and hence good enough to deserve full credit, but it is actually incomplete: one can find points A and B for which the two circles will not intersect. On your next problem set you will have a chance to invent a better proof.)

(d) *(2 points extra credit)* $\boxed{\text{False.}}$ If we relax the hypothesis of Euclidean geometry, then isotropy around two points does not necessarily imply homogeneity. A counter-example we mentioned in class is a two-dimensional universe consisting of the surface of a sphere. Think of the sphere in three Euclidean dimensions, but the model "universe" consists only of its two-dimensional surface. Imagine latitude and longitude lines to give coordinates to the surface, and imagine a matter distribution that depends only on latitude. This would not be homogeneous, but it would look isotropic to observers at both the north and south poles. While this example describes a two-dimensional universe, which therefore cannot be our universe, we will learn shortly how to construct a three-dimensional non-Euclidean universe with these same properties.

PROBLEM 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(30 points)**

(a) Substituting the equation for $M(r_i)$, given on the quiz, into the differential equation for r, also given on the quiz, one finds:

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} + \gamma r^n .$$

Dividing both sides of the equation by r_i, one has

$$\frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G r_i^2 \rho_i}{r^2} + \gamma \frac{r^n}{r_i} .$$

Substituting $u = r/r_i$, this becomes

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} + \gamma u^n r_i^{n-1} .}$$

(b) The only dependence on r_i occurs in the last term, which is proportional to r_i^{n-1}. This dependence disappears if $\boxed{n = 1}$, since the zeroth power of any positive number is 1.

(c) This is exactly the same as the case discussed in the lecture notes, since the initial conditions do not depend on the differential equation. At $t = 0$,

$$r(r_i, 0) = r_i \qquad \text{(definition of } r_i\text{)}$$
$$\dot{r}(r_i, 0) = H_i r_i \qquad \text{(since } \vec{v}_i = H_i \vec{r}\text{)}.$$

Dividing these equations by r_i one has the intial conditions

$$\boxed{\begin{aligned} u &= 1 \\ \dot{u} &= H_i \ . \end{aligned}}$$

(d) $a(t)$ should obey the differential equation obtained in part (a) for u, for the value of n that was obtained in part (b): $n = 1$. So,

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a .$$

Multiplying the equation by $\dot{a} \equiv da/dt$, one finds

$$\frac{d^2a}{dt^2}\frac{da}{dt} = \left\{ -\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a \right\} \frac{da}{dt} ,$$

which can be rewritten as

$$\frac{d}{dt}\left\{ \frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 \right\} = 0 .$$

Thus the quantity inside the curly brackets must be constant. Following the lecture notes, I will call this constant E:

$$\boxed{\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 = E .}$$

Or one can use the conventionally defined quantity

$$k = -\frac{2E}{c^2} ,$$

in which case the equation can be written

$$\left\{ \frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 \right\} = -\frac{kc^2}{2} .$$

One can then rewrite the equation in the more standard form

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho + \gamma - \frac{kc^2}{a^2}\ ,}$$

where I have used $\rho(t) = \rho_i/a^3(t)$.

Additional Note: Historically, the constant γ corresponds to the "cosmological constant" which was introduced by Albert Einstein in 1917 in an effort to build a static model of the universe. The cosmological constant Λ, as defined by Einstein, is related to γ by

$$\gamma = \frac{1}{3}\Lambda c^2\ .$$

The differential equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{a^2}\ ,$$

which shows that the cosmological constant contributes like a constant addition to the mass density. Modern physicists interpret the cosmological constant as a manifestation of the mass density of the vacuum. From the above equation, we can see that the mass density of the vacuum is related to Einstein's cosmological constant Λ by

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}\ .$$

Alternative Question:

Some of you answered the alternative question, seeking an integral of the equation

$$\ddot{a} + \frac{A}{a^p} + Ba^q = 0\ .$$

The technique is the same as above. One must first multiply the equation by $\dot{a}$, and then it becomes integrable:

$$0 = \dot{a}\left(\ddot{a} + \frac{A}{a^p} + Ba^q\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\dot{a}^2 - \frac{1}{p-1}\frac{A}{a^{p-1}} + \frac{1}{q+1}Ba^{q+1}\right\}\ .$$

Since the time derivative of the quantity in curly brackets is zero, the quantity must be a constant, which we can call E:

$$\boxed{E = \frac{1}{2}\dot{a}^2 - \frac{1}{p-1}\frac{A}{a^{p-1}} + \frac{1}{q+1}Ba^{q+1}\ .}$$

PROBLEM 3: COSMOLOGICAL VS. SPECIAL RELATIVISTIC REDSHIFT *(20 points)*†

(a) *(5 points)* Using the expression for the cosmological redshift we get:

$$\boxed{z = \frac{a(t_0)}{a(t_e)} - 1 = \left(\frac{t_0}{t_e}\right)^{2/3} - 1\ .}$$

Inserting $t_0 = 13.7$ Gyr and $t_e = 13.5$ Gyr in the above expression we get:

$$\boxed{z \simeq 0.00985232\ .}$$

(b) *(5 points)* First we need to find the coordinate distance to the galaxy X:

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}dt = \frac{3c}{b}\left(t_0^{1/3} - t_e^{1/3}\right)\ .$$

Then the current physical distance is found by multiplying the coordinate distance by the present value of the scale factor:

$$\boxed{\ell_p(t_0) = a(t_0)\ell_c = 3ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right]\ .}$$

We want to evaluate the physical distance in light years (ly). Notice that:

$$c = \frac{1\ \text{ly}}{1\ \text{yr}}\ ,$$

so that:

$$\boxed{\ell_p(13.7\ \text{Gyr}) = 3 \times 13.7 \times 10^9\left[1 - \left(\frac{13.5}{13.7}\right)^{1/3}\right]\ \text{ly} \simeq 2.00981 \times 10^8\ \text{ly}.}$$

(c) *(5 points)* At any time the physical distance can be written as:

$$\ell_p(t) = a(t)\ell_c\ .$$

For objects moving with Hubble's flow the coordinate distance ℓ_c is independent of time, so that:

$$v_p(t) = \frac{d\ell_p(t)}{dt} = \dot{a}(t)\ell_c = \frac{\dot{a}(t)}{a(t)}a(t)\ell_c \equiv H(t)\ell_p(t) \ .$$

This is exactly Hubble's law. To obtain the current velocity we need the current value of the Hubble's parameter $H(t_0)$:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{2}{3t} \quad \Longrightarrow \quad H(t_0) = \frac{2}{3t_0} \ .$$

Then, using the expression for $\ell_p(t_0)$ we found in part (*b*):

$$\boxed{v_p(t_0) = 2c\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] \ ,}$$

$$\boxed{v_p(13.7 \text{ Gyr}) \simeq 0.00978011c \ .}$$

Note: A common mistake was to start with the answer from part (b), $\ell_p(t_0) = 3ct_0\left[1 - (t_e/t_0)^{1/3}\right]$, and differentiate with respect to t_0. The problem with this method is that it holds t_e fixed, instead of ℓ_c. We are interested in how the distance to a specific distant galaxy changes with time. The galaxy has a fixed coordinate distance ℓ_c, but the light that we see later will not have the same time of emission t_e.

(d) *(5 points)* The Doppler shift we would get from special relativity is given by:

$$z_{sr} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \ .$$

Using

$$\beta = \frac{v_p(13.7 \text{ Gyr})}{c} \simeq 0.00978011 \ ,$$

we get

$$\boxed{z_{sr} \simeq 0.00982840 \ .}$$

Notice that the difference with the result found in part (*a*) is less than 1%:

$$\frac{z - z_{sr}}{z} \simeq 0.2\% \ .$$

PROBLEM 4: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)**

(a) They key idea is that the coordinate speed of light is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} \ ,$$

so the coordinate distance (in notches) that light can travel between $t = 0$ and now ($t = t_0$) is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} \ .$$

The corresponding physical distance is the horizon distance:

$$\ell_{p,\text{horizon}}(t_0) = a(t_0)\int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} \ .$$

Evaluating,

$$\ell_{p,\text{horizon}}(t_0) = bt_0^{2/3}\int_0^{t_0} \frac{c\,\mathrm{d}t}{bt^{2/3}} = t_0^{2/3}\left[3ct_0^{1/3}\right] = \boxed{3ct_0 \ .}$$

(b) As stated in part (a), the coordinate distance that light can travel between $t = 0$ and $t = t_0$ is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{3ct_0^{1/3}}{b} \ .$$

Thus, if we are at the origin, at $t = 0$ the photon must have been at

$$\boxed{x_0 = \frac{3ct_0^{1/3}}{b} \ .}$$

(c) The photon starts at $x = x_0$ at $t = 0$, and then travels in the negative x-direction at speed $c/a(t)$. Thus, it's position at time t is given by

$$x(t) = x_0 - \int_0^t \frac{c\,\mathrm{d}t'}{a(t')} = \frac{3ct_0^{1/3}}{b} - \frac{3ct^{1/3}}{b} = \boxed{\frac{3c}{b}\left(t_0^{1/3} - t^{1/3}\right) \ .}$$

(d) Since the coordinate distance between us and the photon is $x(t)$, measured in notches, the physical distance (in, for example, meters) is just $a(t)$ times $x(t)$. Thus.

$$\ell_p(t) = a(t)x(t) = \boxed{3ct^{2/3}\left(t_0^{1/3} - t^{1/3}\right) .}$$

(e) To find the maximum of $\ell_p(t)$, we set the derivative equal to zero:

$$\frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[3c\left(t^{2/3}t_0^{1/3} - t\right)\right] = 3c\left[\frac{2}{3}\left(\frac{t_0}{t}\right)^{1/3} - 1\right] = 0 ,$$

so

$$\left(\frac{t_0}{t_{\max}}\right)^{1/3} = \frac{3}{2} \quad\Longrightarrow\quad \boxed{t_{\max} = \left(\frac{2}{3}\right)^3 t_0 = \frac{8}{27}\,t_0 .}$$

The maximum distance is then

$$\ell_{p,\max} = \ell_p(t_{\max}) = 3c\left(\frac{2}{3}\right)^2 t_0^{2/3}\left[t_0^{1/3} - \left(\frac{2}{3}\right)t_0^{1/3}\right] = 3c\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)t_0$$

$$= \boxed{\frac{4}{9}ct_0 .}$$

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†Solution written by Daniele Bertolini.

*Solution written by Alan Guth.