MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 9, 2011
Prof. Alan Guth

QUIZ 2 SOLUTIONS

Quiz Date: November 3, 2011

PROBLEM 1: DID YOU DO THE READING? *(20 points)*†

(a) *(8 points)*

(i) *(4 points)* We will use the notation X^A to indicate a nucleus,* where X is the symbol for the element which indicates the number of protons, while A is the mass number, namely the total number of protons and neutrons. With this notation H^1, H^2, H^3, He^3 and He^4 stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + p \to He^4 + \gamma,}$$

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + n \to He^4 + \gamma.}$$

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place (in general with different probabilities). For example:

$$p + n \to H^2 + \gamma \qquad H^2 + H^2 \to He^4 + \gamma,$$

$$p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + H^2 \to He^4 + n,$$

$$p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + H^2 \to He^4 + p,$$

$$\ldots$$

Students who described chains different from those of Weinberg, but that can still take place, got full credit for this part. Also, notice that photons in the reactions above carry the additional energy released. However, since the main point was to describe the nuclear reactions, students who didn't include the photons still received full credit.

(ii) *(4 points)* The *deuterium bottleneck* is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium (H^2) plays a crucial role in nucleosynthesis, since it is the starting point for all the chains. However, the deuterium nucleus is extremely loosely bound compared to H^3, He^3, or especially He^4. So, there will be a range of temperatures which are low enough for H^3, He^3, and He^4 nuclei to be bound, but too high to allow the deuterium nucleus to be stable. This is the temperature range where the *deuterium bottleneck* is in action: even if H^3, He^3, and He^4 nuclei could in principle be stable at those temperatures, they do not form because deuterium, which is the starting point for their formation, cannot be formed yet. Nucleosynthesis cannot proceed at a significant rate until the temperature is low enough so that deuterium nuclei are stable; at this point the deuterium bottleneck has been passed.

(b) *(12 points)*

(i) *(3 points)* If we take $a(t) = bt^{1/2}$, for some constant b, we get for the Hubble expansion rate:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \quad \Longrightarrow \quad \boxed{t = \frac{1}{2H}.}$$

(ii) *(6 points)* By using the Friedmann equation with $k = 0$ and $\rho = \rho_r = \alpha T^4$, we find:

$$H^2 = \frac{8\pi}{3} G\rho_r = \frac{8\pi}{3} G\alpha T^4 \quad \Longrightarrow \quad \boxed{H = T^2 \sqrt{\frac{8\pi}{3} G\alpha}\ .}$$

If we substitute the given numerical values $G \simeq 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$ and $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg \cdot m^{-3} \cdot K^{-4}}$ we get:

$$\boxed{H \simeq T^2 \times 5.03 \times 10^{-21}\,\mathrm{s^{-1} \cdot K^{-2}}\ .}$$

Notice that the units correctly combine to give H in units of $\mathrm{s^{-1}}$ if the temperature is expressed in degrees Kelvin (K). In detail, we see:

$$[G\alpha]^{1/2} = (\mathrm{N \cdot m^2 \cdot kg^{-2} \cdot kg \cdot m^{-3} \cdot K^{-4}})^{1/2} = \mathrm{s^{-1} \cdot K^{-2}}\ ,$$

* Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.

where we used the fact that 1 N = 1 kg · m · s^{-2}. At $T = T_{\rm nucl} \simeq 0.9{\times}10^9$K we get:

$$\boxed{H \simeq 4.07 \times 10^{-3}{\rm s}^{-1}.}$$

(iii) *(3 points)* Using the results in parts (i) and (ii), we get

$$\boxed{t = \frac{1}{2H} \simeq \left(\frac{9.95 \times 10^{19}}{T^2}\right) {\rm s} \cdot {\rm K}^2 \ .}$$

To good accuracy, the numerator in the expression above can be rounded to 10^{20}. The above equation agrees with Weinberg's claim that, for a radiation dominated universe, time is proportional to the inverse square of the temperature. In particular for $T = T_{\rm nucl}$ we get:

$$\boxed{t_{\rm nucl} \simeq 123 \text{ s} \approx 2 \text{ min.}}$$

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)**

(a) *(7 points)* Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\, dt^2 + a^2(t) d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{a^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm \frac{c}{a(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) *(8 points)* The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{a(t')} dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} a(t)\, d\psi = \boxed{a(t) \int_0^t \frac{c}{a(t')} dt' \ .}$$

(c) *(10 points)* From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{a(t)} \ .$$

Then using $a = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \ .}$$

(d) *(5 points)* This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $a = \alpha(1 - \cos\theta)$, one can see that a returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 5. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 \ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point

$(0,0,0,1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0,0,0,1)$ can be called the north pole. Each point (x,y,z,w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x,y,z,w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0,0,0,-1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 3: AN EXERCISE IN TWO-DIMENSIONAL METRICS
*(30 points)**

(a) Since

$$r(\theta) = (1 + \epsilon \cos^2\theta)\, r_0 \ ,$$

as the angular coordinate θ changes by $\mathrm{d}\theta$, r changes by

$$\mathrm{d}r = \frac{\mathrm{d}r}{\mathrm{d}\theta}\,\mathrm{d}\theta = -2\epsilon r_0 \cos\theta \sin\theta \,\mathrm{d}\theta \ .$$

$\mathrm{d}s^2$ is then given by

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \\ &= 4\epsilon^2 r_0^2 \cos^2\theta \sin^2\theta \,\mathrm{d}\theta^2 + (1+\epsilon\cos^2\theta)^2\, r_0^2 \,\mathrm{d}\theta^2 \\ &= \left[4\epsilon^2 \cos^2\theta \sin^2\theta + (1+\epsilon\cos^2\theta)^2\right] r_0^2 \,\mathrm{d}\theta^2 \ ,\end{aligned}$$

so

$$\mathrm{d}s = r_0 \sqrt{4\epsilon^2 \cos^2\theta \sin^2\theta + (1+\epsilon\cos^2\theta)^2}\,\mathrm{d}\theta \ .$$

Since θ runs from θ_1 to θ_2 as the curve is swept out,

$$\boxed{S = r_0 \int_{\theta_1}^{\theta_2} \sqrt{4\epsilon^2 \cos^2\theta \sin^2\theta + (1+\epsilon\cos^2\theta)^2}\, d\theta \ .}$$

(b) Since θ does not vary along this path,

$$\mathrm{d}s = \sqrt{1 + \frac{r}{a}}\,\mathrm{d}r \ ,$$

and so

$$\boxed{R = \int_0^{r_0} \sqrt{1 + \frac{r}{a}}\,\mathrm{d}r \ .}$$

(c) Since the metric does not contain a term in $\mathrm{d}r\,\mathrm{d}\theta$, the r and θ directions are orthogonal. Thus, if one considers a small region in which r is in the interval r' to $r' + \mathrm{d}r'$, and θ is in the interval θ' to $\theta' + \mathrm{d}\theta'$, then the region can be treated as a rectangle. The side along which r varies has length $\mathrm{d}s_r = \sqrt{1 + (r'/a)}\,\mathrm{d}r'$, while the side along which θ varies has length $\mathrm{d}s_\theta = r'\,\mathrm{d}\theta'$. The area is then

$$\mathrm{d}A = \mathrm{d}s_r\,\mathrm{d}s_\theta = r' \sqrt{1 + (r'/a)}\,\mathrm{d}r'\,\mathrm{d}\theta' \ .$$

To cover the area for which $r < r_0$, r' must be integrated from 0 to r_0, and θ' must be integrated from 0 to 2π:

$$A = \int_0^{r_0} \mathrm{d}r' \int_0^{2\pi} \mathrm{d}\theta'\, r' \sqrt{1 + (r'/a)} \ .$$

But

$$\int_0^{2\pi} \mathrm{d}\theta' = 2\pi \ ,$$

so

$$\boxed{A = 2\pi \int_0^{r_0} \mathrm{d}r'\, r' \sqrt{1 + (r'/a)} \ .}$$

You were not asked to carry out the integration, but it can be done by using the substitution $u = 1 + (r'/a)$, so $\mathrm{d}u = (1/a)\,\mathrm{d}r'$, and $r' = a(u-1)$. The result is

$$A = \frac{4\pi a^2}{15}\left[2 + \left(\frac{3r_0^2}{a^2} + \frac{r_0}{a} - 2\right)\sqrt{1 + \frac{r_0}{a}}\,\right] \ .$$

(d) The nonzero metric coefficients are given by

$$g_{rr} = 1 + \frac{r}{a} \ , \qquad g_{\theta\theta} = r^2 \ ,$$

so the metric is diagonal. For $i = 1 = r$, the geodesic equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{rr}\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r}\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} \ ,$$

so if we substitute the values from above, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial}{\partial r}\left(1 + \frac{r}{a}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial r^2}{\partial r}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \ .$$

Simplifying slightly,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \ .}$$

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \left(1 + \frac{r}{a}\right)\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} \ .$$

Inserting this expansion into the boxed equation above, the first term can be brought to the right-hand side, giving

$$\boxed{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = -\frac{1}{2a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \ .}$$

The $i = 2 = \theta$ equation is simpler, because none of the g_{ij} coefficients depend on θ, so the right-hand side of the geodesic equation vanishes. One has simply

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = 0 \ .}$$

For most purposes this is the best way to write the equation, since it leads immediately to $r^2(\mathrm{d}\theta/\mathrm{d}s) = const$. However, it is possible to expand the derivative, giving the alternative form

$$r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0 \ .$$

PROBLEM 4: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

*(20 points)**

The absence of off-diagonal terms in the metric means that the three directions found by varying r, θ, and ϕ, one at a time, are mutually orthogonal. Thus the region defined by varying r by $\mathrm{d}r$, θ by $\mathrm{d}\theta$, and ϕ by $\mathrm{d}\phi$ is an infinitesimal rectangular solid, the volume of which is the product of the lengths of the three sides. Thus,

$$dV = a(t)\frac{dr}{\sqrt{1-kr^2}} \times a(t)r\,d\theta \times a(t)r\sin\theta\, d\phi$$

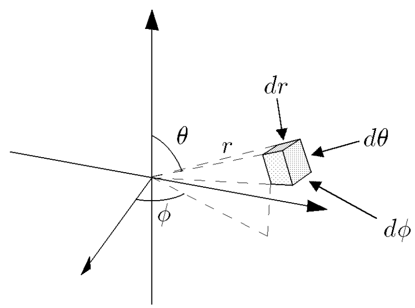
The total volume is then

$$V = \int dV = a^3(t)\int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2\sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi a^3(t)\int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}} \ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r + dr$, between θ and $\theta + d\theta$, and between ϕ and $\phi + d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$ds_1 = a(t)\frac{dr}{\sqrt{1 - kr^2}}$$

$$ds_2 = a(t) r\, d\theta$$

$$ds_3 = a(t) r \sin\theta\, d\theta\ .$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\, d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\sqrt{k}\, r = \sin\psi \quad (\text{if } k > 0)$$

$$\sqrt{-k}\, r = \sinh\psi \quad (\text{if } k > 0).$$

The answer is

$$V = \begin{cases} 2\pi a^3(t)\left[\dfrac{\sin^{-1}\left(\sqrt{k}\, r_{\max}\right)}{k^{3/2}} - \dfrac{\sqrt{1 - kr_{\max}^2}}{k}\right] & (\text{if } k > 0) \\ 2\pi a^3(t)\left[\dfrac{\sqrt{1 - kr_{\max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\, r_{\max}\right)}{(-k)^{3/2}}\right] & (\text{if } k < 0) \quad .] \end{cases}$$

---

†Solution written by Daniele Bertolini.

*Solution written by Alan Guth.