MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 18, 2011
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QUIZ 3 SOLUTIONS

Quiz Date: December 8, 2011

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(4 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (2 points for each right answer; circle at most 2.)

- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
- **[(ii)]** In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
- (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
- **[(iv)]** In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
- (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

Comment: The printed quiz erroneously said that you would receive 3 points for each right answer, but the problem is only worth 4 points total. It was intended to be 2 points for each right answer.

Explanation of incorrect answers: (i) The neutron decays to a proton, electron, and an anti-electron-neutrino, with a mean life of 881.5 ± 1.5 *s (about 15 minutes), according to the Particle Data Group, http://pdg.lbl.gov/. Alpher and*

Herman knew about this decay, although at that time the lifetime was believed to be somewhat longer than the presently accepted value. (ii) In the current theory, alpha particles do not form in significant numbers until about $3\frac{3}{4}$ *minutes after the big bang. When the quarks first combine, they form a gas of nearly equal numbers of protons and neutrons. (v) The ratio of photons to nuclear particles (baryons) in the universe is still believed to be about* 10^9*. This number is valid from a time much less than a second, when baryogenesis took place, up to the present day.*

(b) *(6 points)* Consider a star in a circular orbit of radius r in a galaxy. What is the velocity of the star, in terms of the total mass $M(r)$ contained within a sphere of radius r? (For simplicity, assume that the mass distribution is spherical.)

Answer: The gravitational acceleration due to the galaxy must provide the centripetal acceleration required to keep the star moving in a circular orbit. Thus, assuming a spherical distribution of matter,

$$\frac{v^2}{r} = \frac{GM(r)}{r^2} ,$$

where $M(r)$ *is the total mass of the galaxy contained within a sphere of radius* r*. Solving for* v *gives*

$$\boxed{v(r) = \sqrt{\frac{GM(r)}{r}} \ .}$$

(c) *(2 points)* Observations of galaxies show that the surface brightness of spiral galaxies is highly concentrated near the center of the spiral galaxy disk. Therefore, for $r > 5$ kpc or so, the density of stars is approximately zero. What does your answer for $v(r)$ above predict for the dependence of $v(r)$ on r at large radii, if all of the mass in the galaxy is due to stars?

Answer: For $r > 5$ *kpc or so, we can approximate the density of stars to be zero, and therefore for such radii* $M_*(r) \simeq$ *const., where* $M_*(r)$ *is the total mass of stars within a sphere of radius* r*. From the answer to part (b), then,*

$$\boxed{v(r) \propto r^{-1/2}}$$

if all of the mass in the galaxy is due to stars.

(d) *(3 points)* How do observed galactic rotation curves $v(r)$ actually behave for large r?

Answer: Observed galactic rotation curves plateau for large radii, as observed in M31, or even increase slightly, as observed in our own galaxy. In particular, they do not fall off appreciably with radius, as they would have to if stars were the only gravitationally interacting matter present.

(e) *(5 points)* Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

[(ii)] After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Comment: The root mean squared temperature fluctuation is actually 1.1×10^{-5}.

PROBLEM 2: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) *(15 points)* Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/a^3(t)$ and $\rho_\Lambda =$ const, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

one has

$$\begin{aligned}\left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3} G \left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\ &= \frac{8\pi}{3} G \rho_{c,0} \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\ &= H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) .\end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}} ; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} ,$$

and

$$H_0^2 = \frac{8\pi}{3} G \rho_{c,0} .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0 \, x \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} ,$$

which in turn implies that

$$dt = \frac{1}{H_0} \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}}} \ .$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}}} \ ,$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} \ ; \qquad \mathrm{d}x = -\frac{\mathrm{d}z}{(1+z)^2} \ .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) \ .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0} \ \mathrm{yr} \ ,$$

where $H_0 = 100\,h_0\,\text{km-sec}^{-1}\text{-Mpc}^{-1}$, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9 \ \mathrm{yr} \ .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \ \mathrm{yr} \ .}$$

So the SDSS people were right on target.

(b) *(5 points)* Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} \ ,$$

since the cosmological redshift is given by

$$1 + z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} \ .$$

Using the expression for $\mathrm{d}t$ from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}}} \ .$$

Again one could write the answer other ways, including

$$\boxed{t_e = \frac{1}{H_0}\int_z^\infty \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}}} \ .$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}x_{\max}^3} + \sqrt{\Omega_{\Lambda,0}}\,x_{\max}^{3/2}}{\sqrt{\Omega_m}}\right) \ .$$

Using $x_{\max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr} .}$$

So again the SDSS people were right.

(c) *(10 points)* To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $a(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\} ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{a(t)} .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t .$$

This is not the final answer, however, because we don't explicitly know $a(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}} .$$

So

$$\ell_c = \frac{c}{a(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 a(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Finally, then

$$\boxed{\ell_{\text{phys},0} = a(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}} ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{0}^{z} \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{\mathrm{d}z'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099 ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\text{phys},0} = 27.23 \text{ light-yr} .}$$

Right again.

(d) *(5 points)* $\ell_{\text{phys},e} = a(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{a(t_e)}{a(t_0)}\,\ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z} .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr} .}$$

The SDSS announcement is still okay.

(e) *(5 points)* The speed defined in this way obeys the Hubble law exactly, so

$$\boxed{v = H_0\,\ell_{\text{phys},0} ,}$$

where $\ell_{\rm phys,0}$ is the answer to part (c). The answer can also be written by using this answer, so

$$\boxed{v = c\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}}}$$

or

$$\boxed{v = c\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Numerically, we have already found that this integral has the value

$$\boxed{v = 1.8099\,c\ .}$$

The SDSS people get an A.

PROBLEM 3: NUCLEOSYNTHESIS AFTER THE DEUTERIUM BOTTLENECK *(30 points)*

(a) *(5 points)* We are given n_b the number density of baryons, and we are also given the fraction f which are neutrons. If all the neutrons become bound into He^4, then the number density of He^4 nuclei is

$$\boxed{n_{\rm He^4} = \frac{1}{2} f n_b\ ,}$$

since the total (free plus bound) number density of neutrons is fn_b, and it takes two neutrons to make a He^4 nucleus. Since each He^4 nucleus has the same number of protons and neutrons, the density of bound protons will be fn_b, while the total number density of protons is $(1-f)n_b$. Thus the number density of free protons will be

$$\boxed{n_p = (1-f)n_b - fn_b = (1-2f)n_b\ .}$$

Numerically,

$$\boxed{\begin{aligned} n_{\rm He^4} &= \frac{1}{2}(0.14)\times 5.54\times 10^{24}\ \mathrm{m}^{-3} = 3.88\times 10^{23}\ \mathrm{m}^{-3}\ ,\\ n_p &= (1-2f)n_b = 0.72\times 5.54\times 10^{24}\ \mathrm{m}^{-3} = 3.99\times 10^{24}\ \mathrm{m}^{-3}\ .\end{aligned}}$$

(b) *(5 points)* The sums of the chemical potentials on the two sides of the reaction must be equal, so

$$\boxed{\mu_{\rm He^4} = 2(\mu_p + \mu_n)\ .}$$

(c) *(7 points)* In thermal equilibrium, the number densities for neutrons, protons, and He^4 are all given by the formula in the formula sheet. Using $g_n = g_p = \frac{1}{2}$, and $g_{\rm He^4} = 1$, the formula implies

$$n_n = 2\left(\frac{m_n kT_D}{2\pi\hbar^2}\right)^{3/2} e^{(\mu_n - m_n c^2)/kT_D}\ ,$$

$$n_p = 2\left(\frac{m_p kT_D}{2\pi\hbar^2}\right)^{3/2} e^{(\mu_p - m_p c^2)/kT_D}\ ,$$

$$n_{\rm He^4} = \left(\frac{m_{\rm He^4} kT_D}{2\pi\hbar^2}\right)^{3/2} e^{(\mu_{\rm He^4} - m_{\rm He^4} c^2)/kT_D}\ ,$$

where I have used the algebraic identity

$$\frac{(2\pi mkT_D)^{3/2}}{(2\pi\hbar)^3} = \left(\frac{mkT_D}{2\pi\hbar^2}\right)^{3/2}\ .$$

Given the answer to part (b), the chemical potentials will cancel out if we calculate the ratio of $n_{\rm He^4}/(n_n^2 n_p^2)$. If we use the definition $B_{\rm He^4} = [2(m_n + m_p) - m_{\rm He^4}]\,c^2$ and the approximation $m_p = m_n = \frac{1}{4}m_{\rm He^4}$, the ratio simplifies to

$$\boxed{\frac{n_{\rm He^4}}{n_n^2 n_p^2} = \frac{1}{2}\left(\frac{m_p kT_D}{2\pi\hbar^2}\right)^{-9/2} e^{B_{\rm He^4}/kT_D}\ ,}$$

where I used

$$\frac{4^{3/2}}{(2\cdot 2)^2} = \frac{1}{2}\ .$$

(d) *(6 points)* The approximations used to find n_p and $n_{\rm He^4}$ are still valid, since a trace amount of free neutrons will not significantly affect these answers. Thus we know the values of n_p and $n_{\rm He^4}$, so we can use the answer to part (c) to find n_n. With a little algebra, one finds

$$n_n^2 = \frac{2n_{\rm He^4}}{n_p^2}\left(\frac{m_p kT_D}{2\pi\hbar^2}\right)^{9/2} e^{-B_{\rm He^4}/kT_D}\ ,$$

so

$$\boxed{n_n = \sqrt{\frac{f}{(1-2f)^2 n_b}} \left(\frac{m_p k T_D}{2\pi\hbar^2}\right)^{9/4} e^{-\frac{1}{2}B_{\mathrm{He}^4}/kT_D} \ .}$$

Numerically,

$$n_n \approx \sqrt{\frac{0.14}{(0.72)^2}\frac{1}{\sqrt{5.54\times 10^{24}\ \mathrm{m}^{-3}}}} \left(4.01\times 10^{39}\ \mathrm{m}^{-3}\right)^{3/2} \left(3\times 10^{186}\right)^{-1/2}$$

$$\boxed{\approx 3\times 10^{-47}\ \mathrm{m}^{-3} \ .}$$

(e) *(7 points)* Again we can write a ratio of densities for which the chemical potentials cancel out:

$$\frac{n_{\mathrm{Li}^7}}{n_p^3 n_n^4} = \frac{4\cdot 7^{3/2}}{2^7}\left(\frac{m_p k T_D}{2\pi\hbar^2}\right)^{-9} e^{B_{\mathrm{Li}^7}/kT_D} \ .$$

So

$$n_{\mathrm{Li}^7} = \boxed{\frac{7^{3/2}}{32} n_p^3 n_n^4 \left(\frac{m_p k T_D}{2\pi\hbar^2}\right)^{-9} e^{B_{\mathrm{Li}^7}/kT_D}}$$

$$= \frac{7^{3/2}}{32}(0.72)^3(5.54\times 10^{24}\ \mathrm{m}^{-3})^3(3\times 10^{-47}\ \mathrm{m}^{-3})^4$$

$$\times (4.01\times 10^{39}\ \mathrm{m}^{-3})^{-6}(4\times 10^{258})$$

$$\approx \boxed{3\times 10^{-92}\ \mathrm{m}^{-3} \ .}$$

PROBLEM 4: DOUBLING OF ELECTRONS *(10 points)*

The entropy density of black-body radiation is given by

$$s = g\left[\frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3}\right]T^3$$

$$= g\, C\, T^3 \ ,$$

where C is a constant. At the time when the electron-positron pairs disappear, the neutrinos are decoupled, so their entropy is conserved. All of the entropy

from electron-positron pairs is given to the photons, and none to the neutrinos. The same will be true here, for both species of electron-positron pairs.

The conserved neutrino entropy can be described by $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation n^- and n^+ for the new electron-like and positron-like particles, and also the convention that

Primed quantities: values after $e^+e^-n^+n^-$ annihilation

Unprimed quantities: values before $e^+e^-n^+n^-$ annihilation.

For the neutrinos,

$$S'_\nu = S_\nu \quad\Longrightarrow\quad g_\nu C\left(a'T'_\nu\right)^3 = g_\nu C\left(aT_\nu\right)^3 \quad\Longrightarrow$$

$$\boxed{a'T'_\nu = aT_\nu \ .}$$

For the photons, before $e^+e^-n^+n^-$ annihilation we have

$$T_\gamma = T_{e^+e^-n^+n^-} = T_\nu \ ; \qquad g_\gamma = 2,\ g_{e^+e^-} = g_{n^+n^-} = 7/2 \ .$$

When the e^+e^- and n^+n^- pairs annihilate, their entropy is added to the photons:

$$S'_\gamma = S_{e^+e^-} + S_{n^+n^-} + S_\gamma \quad\Longrightarrow\quad 2C\left(a'T'_\gamma\right)^3 = \left(2 + 2\cdot\frac{7}{2}\right) C\left(aT_\gamma\right)^3 \quad\Longrightarrow$$

$$\boxed{a'T'_\gamma = \left(\frac{9}{2}\right)^{1/3} aT_\gamma \ ,}$$

so aT_γ increases by a factor of $(9/2)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{2}{9}\right)^{1/3} T'_\gamma \ .}$$

This ratio would remain unchanged until the present day.