MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 6, 2011
Prof. Alan Guth

QUIZ 1

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A SUMMARY OF USEFUL INFORMATION IS AT THE END OF THE EXAM.

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Your Name

Problem	Maximum	Score
1	25	
2	30	
3	20	
4	25	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(10 points)* Hubble's law relates the distance of galaxies to their velocity. The Doppler effect provides an accurate tool to measure velocity, while the measure of cosmic distances is more problematic. Explain briefly the method that Hubble used to estimate the distance of galaxies in deriving his law.

(b) *(5 points)* One expects Hubble's law to hold as a consequence of the Cosmological Principle. What does the Cosmological Principle state?

(c) *(10 points)* Give a brief definition for the words *homogeneity* and *isotropy*. Then say for each of the following two statements whether it is true or false. If true explain briefly why. If false give a counter-example. You should assume Euclidean geometry (which Weinberg implicitly assumed in his discussion).

(i) If the universe is isotropic around one point then it has to be homogeneous.

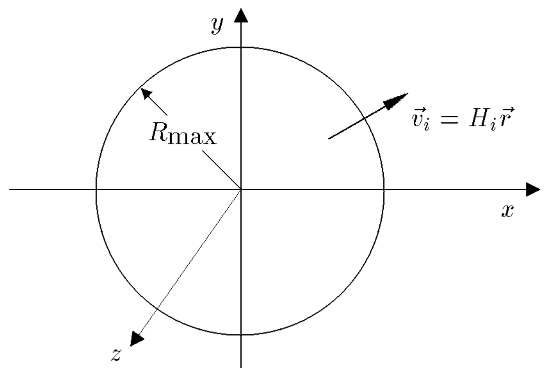
(ii) If the universe is isotropic around two or more distinct points then it has to be homogeneous.

(d) Bonus question: *(2 points extra credit)* If we allow curved (i.e., non-Euclidean) spaces, is it true that a universe which is isotropic around two distinct points has to be homogeneous? If true explain briefly why, and otherwise give a counter-example.

PROBLEM 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(30 points)*

The following problem was Problem 2, Quiz 2, 2000. It also appeared as Problem 14 on this year's Quiz 1 Review Problems.

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i\vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} \ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} + \gamma r^n(r_i,t)\,\hat{r} \ ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i,t)} + \gamma r^n(r_i,t) \ .$$

— *Problem 2 continues on the following page* —

Problem 2, Continued. Please start your answer on the blank page to the right.

a) *(5 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u. *(Hint: be sure that u is the only time-dependent quantity in your equation; r, ρ, etc. must be rewritten in terms of u, ρ_i, etc.)*

b) *(5 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

c) *(6 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

d) *(14 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$a(t) \equiv u(r_i, t) \ .$$

Show, for this value of n, that the differential equation for a can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on a and $\dot{a}$, but not $\ddot{a}$ or any higher derivatives.

If all has not gone so well, you may not know what differential equation $a(t)$ obeys. If that is the case, then for a maximum of 12 points you can consider the generic equation

$$\ddot{a} + \frac{A}{a^p} + Ba^q = 0 \ ,$$

where A and B are arbitrary constants, and p and q are positive integers, with $p > 1$. Show that this equation can be integrated once, as described in the previous paragraph.

PROBLEM 3: COSMOLOGICAL VS. SPECIAL RELATIVISTIC REDSHIFT *(20 points)*

The cosmological redshift is caused by a combination of a velocity redshift and a gravitational redshift, although it is easier to calculate than these words would suggest. It is given by the equation on the formula sheet, where the effects of gravity appear through the effect that gravity has in determining $a(t)$. However, when the distances are small and the velocities are low, the effects of gravity are small. For such cases the special relativity redshift calculation and the cosmological redshift calculation should be very close to each other. In this problem we will compare the two calculations for a specific numerical example.

Consider a flat matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Take the age of the universe as $t_0 = 13.7$ Gyr (1 Gyr $= 10^9$ yr). Suppose that we observe a galaxy X, which moves with the Hubble expansion, for which the light that we now observe left the galaxy at $t_e = 13.5$ Gyr. For parts (a)-(c) below, give your answer both as a general formula in terms of t_0, t_e, and (possibly) c (the speed of light), and also as a numerical value. Treat the input numbers as if they were exact, and give your numerical answers to at least six significant figures.

(a) *(5 points)* What is the redshift z of the light?

(b) *(5 points)* What is the current physical distance ℓ_p to the galaxy X. The numerical value should be expressed in light-years.

(c) *(5 points)* What is the current velocity v_p (i.e., the rate of change of the current physical distance) of the galaxy X, relative to us? The numerical value should be expressed as a fraction of c.

(d) *(5 points)* Using the velocity that you found in part (c), what redshift z would you calculate using the special relativity Doppler shift formula? For this part you can give *only* a numerical answer, with no general expression.

PROBLEM 4: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)*

Consider again a flat matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Let t_0 denote the current time.

(a) *(5 points)* What is the current value of the physical horizon distance $\ell_{p,\mathrm{horizon}}(t_0)$? That is, what is the present distance of the most distant matter that can be seen, limited only by the speed of light. Useful formula:

$$\ell_{p,\mathrm{horizon}}(t) = a(t) \int_0^t \frac{c\,dt'}{a(t')} .$$

(b) *(5 points)* Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a **comoving** coordinate system with us at the origin, and the source of the photon along the positive x-axis. What is the coordinate x_0 of the photon at $t = 0$?

(c) *(5 points)* As the photon travels from the source to us, what is its coordinate $x(t)$ as a function of time?

(d) *(5 points)* What is the physical distance $\ell_p(t)$ between the photon and us as a function of time?

(e) *(5 points)* What is the maximum physical distance $\ell_{p,\max}$ between the photon and us, and at what time $t_{\max}$ does it occur?

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QUIZ 1 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3} G\rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

$$\text{Flat } (k=0): \qquad a(t) \propto t^{2/3} \ , \qquad \Omega = 1 \ .$$