PROBLEM 1: DID YOU DO THE READING?

(a) (10 points) Hubble's law relates the distance of galaxies to their velocity. The Doppler effect provides an accurate tool to measure velocity, while the measurement of cosmic distances is more problematic. Explain briefly the method that Hubble used to estimate the distance of galaxies in deriving his law.

(b) (5 points) One expects Hubble's law to hold as a consequence of the Cosmological Principle. What does the Cosmological Principle state?

(c) (10 points) Give a brief definition for the words **homogeneity** and **isotropy**. Then say for each of the following two statements whether it is true or false. If true explain briefly why. If false give a counter-example. You should assume Euclidean geometry (which Weinberg implicitly assumed in his discussion).

(i) If the universe is isotropic around one point then it has to be homogeneous.

(ii) If the universe is isotropic around two or more distinct points then it has to be homogeneous.

(d) Bonus question: (2 points extra credit) If we allow curved (i.e., non-Euclidean) spaces, is it true that a universe which is isotropic around two distinct points has to be homogeneous?

(A Summary of Useful Information is at the end of the exam.)
PROBLEM 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW (30 points)

The following problem was Problem 2, Quiz 2, 2000. It also appeared as Problem 1 on this year’s Quiz 1 Review Problems.

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density \( \rho_i \) and an initial pattern of velocities corresponding to Hubble expansion:

\[ \vec{v}_i = H_i \vec{r} \]

We denoted the radius at time \( t \) of a particle which started at radius \( r_i \) by the function

\[ r(r_i, t) \]

Assuming Newton’s law of gravity, we concluded that each particle would experience an acceleration given by

\[ \vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)} \hat{r} \]

where \( M(r_i) \) denotes the total mass contained initially in the region \( r < r_i \), given by

\[ M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \]

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the \( n \)th power of the distance, with a strength that is independent of the mass. That is, suppose \( \vec{g} \) is given by

\[ \vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)} \hat{r} + \gamma r_n(r_i, t) \hat{r} \]

where \( \gamma \) is a constant. The function \( r(r_i, t) \) then obeys the differential equation

\[ \ddot{r} = -\frac{GM(r_i)}{r^2(r_i, t)} + \gamma r_n(r_i, t) \]

— Problem 2 continues on the following page —

a) (5 points)

As done in the lecture notes, we define

\[ u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \]

Write the differential equation obeyed by \( u \).

(b) (5 points)

For what value of the power \( n \) is the differential equation found in part (a) independent of \( r_i \)?

(c) (6 points)

Write the initial conditions for \( u \) which, when combined with the differential equation found in (a), uniquely determine the function \( u \).

(d) (14 points)

If all is going well, then you have learned that for a certain value of \( n \), the function \( u(r_i, t) \) will in fact not depend on \( r_i \), so we can define

\[ a(t) \equiv u(r_i, t) \]

Show, for this value of \( n \), that the differential equation for \( a \) can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on \( a \) and \( \dot{a} \), but not \( \ddot{a} \) or any higher derivatives.

If all has not gone so well, you may not know what differential equation you can consider to obtain an equation related to the conservation of energy. The desired differential equation should include terms depending on \( a \) and \( \dot{a} \), but not \( \ddot{a} \) or any higher derivative.

Would experience an acceleration given by

\[ (1.7) \]

Show that this equation can be integrated once, as described in the previous paragraph.

In Lecture Notes 3 we developed a Newtonian model of cosmology by considering a uniform sphere of mass centered at the origin, with initial mass density \( \rho_i \) and an initial pattern of velocities corresponding to Hubble expansion:

\[ \vec{v}_i = H_i \vec{r} \]

where \( H \) is a constant. The function \( r(t) \) then obeys the differential equation

\[ \ddot{r} + \frac{1}{r} \dot{r} = \frac{\rho_i}{M} \]

where \( M \) denotes the total mass contained initially in the region \( r < r_i \), given by

\[ M = \frac{4\pi}{3} r_i^3 \rho_i \]

Now let's consider the following problem:

**Problem 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW**
PROBLEM 3: COSMOLOGICAL VS. SPECIAL RELATIVISTIC RED-SHIFT

(20 points)

The cosmological redshift is caused by a combination of a velocity redshift and a gravitational redshift, although it is easier to calculate than these words would suggest. It is given by the equation on the formula sheet, where the effects of gravity appear through the effect that gravity has in determining \( a(t) \). However, when the distances are small and the velocities are low, the effects of gravity are small. For such cases the special relativity redshift calculation and the cosmological redshift calculation should be very close to each other. The numerical value should be expressed in light-years.

Consider a flat matter-dominated universe, with a scale factor given by

\[
a(t) = \frac{bt^2}{3},
\]

where \( b \) is a constant. Take the age of the universe as \( t_0 = 13.7 \text{ Gyr} \). Suppose that we observe a galaxy \( X \), which moves with the Hubble expansion, for which the light that we now observe left the galaxy at \( t_e = 13.5 \text{ Gyr} \).

For parts (a)-(c) below, give your answer both as a general formula in terms of \( t_0 \), \( t_e \), and \( c \) (the speed of light), and also as a numerical value. Treat the input numbers as if they were exact, and give your numerical answers to at least six significant figures.

(a) (5 points)

What is the redshift \( z \) of the light?

(b) (5 points)

What is the current physical distance \( \ell_p \) to the galaxy \( X \)? The numerical value should be expressed in light-years.

(c) (5 points)

What is the current velocity \( v_p \) (i.e., the rate of change of the current physical distance) of the galaxy \( X \), relative to us? The numerical value should be expressed as a fraction of \( c \).

(d) (5 points)

Using the velocity that you found in part (c), what redshift \( z \) would you calculate using the special relativity Doppler shift formula? For this part you can give only a numerical answer, with no general expression.

PROBLEM 4: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON

(25 points)

Consider again a flat matter-dominated universe, with a scale factor given by

\[
a(t) = \frac{bt^2}{3},
\]

where \( b \) is a constant. Let \( t_0 \) denote the current time.

(a) (5 points)

What is the current value of the physical horizon distance \( \ell_{p, \text{horizon}}(t_0) \)? That is, what is the present distance of the most distant matter that can be seen, limited only by the speed of light. Useful formula:

\[
\ell_{p, \text{horizon}}(t) = \int_0^t \frac{da(t')}{a(t')}.
\]

(b) (5 points)

Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this photon. Suppose that we set up a comoving coordinate system with us at the origin, and the source of the photon along the positive \( x \)-axis. What is the coordinate \( x_0 \) of the photon at \( t = 0 \)?

(c) (5 points)

As the photon travels from the source to us, what is its coordinate \( x(t) \) as a function of time?

(d) (5 points)

What is the physical distance \( \ell_p(t) \) between the photon and us as a function of time?

(e) (5 points)

What is the maximum physical distance \( \ell_{p, \text{max}} \) between the photon and us, and at what time \( t_{\text{max}} \) does it occur?
DOPPLER SHIFT (For motion along a line):
\[ z = \frac{v}{c} \] (nonrelativistic, source moving)
\[ z = \frac{v}{c} - \frac{v}{c} \] (nonrelativistic, observer moving)
\[ z = \sqrt{1 + \beta^2} - \frac{\beta}{c} \] (special relativity, with \( \beta = \frac{v}{c} \))

COSMOLOGICAL REDSHIFT:
\[ 1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = a(t_{\text{observed}}) \left/ a(t_{\text{emitted}}) \right. \]

SPECIAL RELATIVITY:
Time Dilation Factor:
\[ \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \]
\( \beta \equiv \frac{v}{c} \)
Lorentz-Fitzgerald Contraction Factor:
\[ \gamma \]
Relativity of Simultaneity:
Trailing clock reads later by an amount \( \frac{\beta}{c} \).

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} \]
\[ \ddot{a} = -\frac{4\pi G}{3} \rho a \]
\( \rho(t) = \frac{a^3(t)\rho_i}{a^3(t_i)\rho_i} \)
\( \Omega = \frac{\rho}{\rho_c} \)
where \( \rho_c = \frac{3H^2}{8\pi G} \).

Flat (\( k = 0 \)):
\( a(t) \propto t^{2/3} \)
\( \Omega = 1 \).

QUIZ FORMULA SHEET