PROBLEM 1: DID YOU DO THE READING? (20 points)

(a) (8 points) During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.

(i) In The First Three Minutes, Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium, \( \text{He}^4 \). Describe at least one of these two chains.

(ii) Explain briefly what is the deuterium bottleneck, and what is its role during nucleosynthesis.

(b) (12 points) In Chapter 4 of The First Three Minutes, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

"The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures."

In this part of the problem you will explore more quantitatively this statement.

(i) For a radiation-dominated universe the scale-factor \( a(t) \propto t^{1/2} \). Find the cosmic time \( t \) as a function of the Hubble expansion rate \( H \).

(ii) The mass density stored in radiation \( \rho_r \) is proportional to the temperature \( T \) to the fourth power: i.e., \( \rho_r \propto T^4 \). For a certain constant \( \alpha \), the standard formula of temperature is measured in degrees Kelvin \( (K) \). Then \( \rho_r \) has the standard units of density in kilograms per cubic meter. For a specific constant \( \alpha \), let the temperature \( T \) to the fourth power be \( \rho_r \). For some constant \( \alpha \), let \( T \) to the fourth power be \( \rho_r \). For some constant \( \alpha \), let \( T \) to the fourth power be \( \rho_r \).

(iii) Using the results in (i) and (ii), express the cosmic time \( t \) as a function of the temperature. Your result should agree with Weinberg's claim above.

What is the cosmic time, in seconds, when the temperature \( T = 10^9 \text{K} \)?
Problem 2: Tracing Light Rays in a Closed Universe

Consider a radial light ray moving through a matter-dominated universe. Suppose that a photon leaves the origin of the coordinate system at time $t = 0$ and $r = 0$. How long will it take for the photon to return to the starting point? Express your answer as a fraction of the full lifetime of the universe, $H^{-1}$.

The geodesics are described by functions $\psi(t)$ and $\theta(t)$, where $s$ is the arc length along the path. Write an expression for the physical horizon distance, $\ell_{\text{PH}}$, where

$$\ell_{\text{PH}} \equiv \int_0^\infty \sqrt{g_{rr}} \, ds$$

Parameter equations: $r(\theta) = \frac{\alpha}{\cos \theta} + \frac{\beta}{\sin \theta}$, where $\alpha$ and $\beta$ are constants, and $s(\theta) = 0$ (for the space described in part (b)), where $s$ is a periodic (angular) variable, with a range of $0$ to $\pi$.

Consider the curve described by $\psi(t)$ and $\theta(t)$, where $t$ runs from $0$ to $T$. Write an expression, in terms of $\theta$, for the physical horizon distance as a definite integral, making sure to specify the limits of integration.

Again for the space described in part (b), consider a geodesic described by functions $\theta(\tau)$ and $s(\tau)$, where $s$ is a periodic (angular) variable, with a range of $0$ to $\pi$.

Consider the geodesic equation, $d^2s/d\tau^2 = 0$, along the path for which $\theta(\psi) = \phi$ and $r(\psi) = \rho$. Write an expression for $d\psi/d\tau$, where $d\psi/d\tau = c$. Find an expression for $\phi$, where $\phi$ is the angle of the photon from the origin to the point of the photon's return.
PROBLEM 4: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

(20 points)
The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

\[ ds^2 = a^2(t) \left( dr^2 + \frac{1}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

Calculate the volume \( V(\text{max}) \) of the sphere described by \( r \leq \text{max} \).

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

### USEFUL INFORMATION:

**Speed of Light in Comoving Coordinates:**

- \( \frac{c}{a} = \frac{c}{a(t)} \)
- \( \frac{c}{a} = \gamma \)

**Cosmological Redshift:**

\( \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \left( \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} \right)^{1+z} \)

**Doppler Shift (for motion along a line):**

\( \frac{\lambda}{\lambda_0} = \left( 1 + \frac{v}{c} \right) \)

**Special Relativity:**

- Time Dilation Factor:
  \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \)
- Lorentz-Fitzgibbon Contraction Factor:
  \( \gamma_R \equiv \frac{1}{\sqrt{1 - \beta^2}} \)
- Relativity of Simultaneity:
  \( \beta \equiv \frac{v}{c} \)

### EVOLUTION OF A MATTER-DOMINATED UNIVERSE

1. **Hubble Law:**
   \( H = \frac{1}{t} \frac{\dot{a}}{a} \)
   \( H^2 = \frac{8\pi G}{3} \rho - \frac{k c}{a^2} \)
   \( \ddot{a} = -\frac{4\pi G}{3} \rho a \)

2. **手续费**:  The following problem was Problem 1, Quiz 3, 1990.
GEODESIC EQUATION:

\[ \varepsilon^{d}p \theta \varepsilon^m \varepsilon^{t} + \varepsilon^{d}p \varepsilon^{t} + \varepsilon^{d}p \varepsilon^{t} - \varepsilon^{d}p \varepsilon^{t} = \varepsilon^{d}p \]

SCHWARZSCHILD METRIC:

\[ \left\{ \left( \varepsilon^{d}p \theta \varepsilon^m \varepsilon^{t} + \varepsilon^{d}p \right) \varepsilon^{t} + \frac{\varepsilon^{d}p - 1}{\varepsilon^{d}p} \right\} \left( 0 \right) \varepsilon^{d}p \varepsilon^{t} = \varepsilon^{d}p \varepsilon^{t} = \varepsilon^{d}p \]

ROBERTSON-WALKER METRIC:

\[ 0 < q - \equiv q \]

\[ \left( \frac{y^2}{\nu} \right) \frac{\nu^2}{d} \frac{\nu}{\theta} \equiv \theta \text{ where} \]

\[ 1 > \frac{\theta \cos \theta + 1}{\nu} = \Omega \]

\[ \left( 1 - \theta \right) \cos \theta = \frac{y^2}{\nu} \text{ and } \left( \theta - \theta \sin \theta \right) \sin \theta = \nu \text{ : } \left( 0 > q \right) \text{ Open} \]

\[ \left( \frac{y^2}{\nu} \right) \frac{\nu^2}{d} \frac{\nu}{\theta} \equiv \theta \text{ where} \]

\[ 1 > \frac{\theta \cos \theta + 1}{\nu} = \Omega \]

\[ \left( \theta \cos \theta - 1 \right) \cos \theta = \frac{y^2}{\nu} \text{ and } \left( \theta \sin \theta - \theta \right) \sin \theta = \nu \text{ : } \left( 0 < q \right) \text{ Closed} \]

\[ I = \Omega \]

\[ \varepsilon^{d}p \varepsilon^{t} \circ \left( \theta \right) = \varepsilon^{d}p \varepsilon^{t} \circ \left( 0 = q \right) \text{ Flat} \]

\[ 8.286 \text{ QUIZ 2 FALL 2011} \]