MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 3, 2011

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QUIZ 2

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A SUMMARY OF USEFUL INFORMATION IS AT THE END OF THE EXAM.

Problem	Maximum	Score
1	20	
2	30	
3	30	
4	20	
TOTAL	100	

Your Name ______________________

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(8 points)* During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.

(i) In *The First Three Minutes,* Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium, He^4. Describe at least one of these two chains.

(ii) Explain briefly what is the *deuterium bottleneck*, and what is its role during nucleosynthesis.

(b) *(12 points)* In Chapter 4 of *The First Three Minutes*, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures.

In this part of the problem you will explore more quantitatively this statement.

(i) For a radiation-dominated universe the scale-factor $a(t) \propto t^{1/2}$. Find the cosmic time t as a function of the Hubble expansion rate H.

(ii) The mass density stored in radiation ρ_r is proportional to the temperature T to the fourth power: i.e., $\rho_r \simeq \alpha T^4$, for some constant α. For a wide range of temperatures we can take $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg}\cdot\mathrm{m}^{-3}\cdot\mathrm{K}^{-4}$. If the temperature is measured in degrees Kelvin (K), then ρ_r has the standard SI units, $[\rho_r] = \mathrm{kg}\cdot\mathrm{m}^{-3}$. Use the Friedmann equation for a flat universe $(k = 0)$ with $\rho = \rho_r$ to express the Hubble expansion rate H in terms of the temperature T. You will need the SI value of the gravitational constant $G \simeq 6.67\times10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$. What is the Hubble expansion rate, in inverse seconds, at the start of nucleosynthesis, when $T = T_{\rm nucl} \simeq 0.9 \times 10^9\,\mathrm{K}$?

(iii) Using the results in (i) and (ii), express the cosmic time t as a function of the temperature. Your result should agree with Weinberg's claim above. What is the cosmic time, in seconds, when $T = T_{\rm nucl}$?

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 5 of the Review Problems for Quiz 2. (Originally it was from Quiz 2, 1998.)

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance $\ell_{\rm phys}$ at time t. You should leave your answer in the form of a definite integral.

The form of $a(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $a(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$

$$a = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3} \frac{G\rho a^3}{c^2} .$$

These equations are identical to those on the formula sheet, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 3: AN EXERCISE IN TWO-DIMENSIONAL METRICS *(30 points)*

(a) *(8 points)* Consider first a two-dimensional space with coordinates r and θ. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 .$$

Consider the curve described by

$$r(\theta) = (1 + \epsilon \cos^2\theta)\, r_0 ,$$

where ϵ and r_0 are constants, and θ runs from θ_1 to θ_2. Write an expression, in the form of a definite integral, for the length S of this curve.

(b) *(5 points)* Now consider a two-dimensional space with the same two coordinates r and θ, but this time the metric will be

$$\mathrm{d}s^2 = \left(1 + \frac{r}{a}\right) \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 ,$$

where a is a constant. θ is a periodic (angular) variable, with a range of 0 to 2π, with 2π identified with 0. What is the length R of the path from the origin ($r = 0$) to the point $r = r_0, \theta = 0$, along the path for which $\theta = 0$ everywhere along the path? You can leave your answer in the form of a definite integral. (Be sure, however, to specify the limits of integration.)

(c) *(7 points)* For the space described in part (b), what is the total area contained within the region $r < r_0$. Again you can leave your answer in the form of a definite integral, making sure to specify the limits of integration.

(d) *(10 points)* Again for the space described in part (b), consider a geodesic described by the usual geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell} \right) \frac{\mathrm{d}x^k}{\mathrm{d}s} \frac{\mathrm{d}x^\ell}{\mathrm{d}s} .$$

The geodesic is described by functions $r(s)$ and $\theta(s)$, where s is the arc length along the curve. Write explicitly both (i.e., for i=1=r and i=2=θ) geodesic equations.

PROBLEM 4: VOLUMES IN A ROBERTSON-WALKER UNIVERSE
(20 points)

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \quad .$$

Calculate the volume $V(r_{\rm max})$ of the sphere described by

$$r \le r_{\rm max} \ .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

USEFUL INFORMATION:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\rm coord} = \frac{c}{a(t)} \ .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{a(t_{\rm observed})}{a(t_{\rm emitted})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot a = -\frac{4\pi}{3} G\rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$$\Omega = 1 \; .$$

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)\;, \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)\;,$

$$\Omega = \frac{2}{1+\cos\theta} > 1\;,$$

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \; .$

Open ($k < 0$): $ct = \alpha\,(\sinh\theta - \theta)\;, \quad \frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\;,$

$$\Omega = \frac{2}{1+\cosh\theta} < 1\;,$$

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \;,$

$$\kappa \equiv -k > 0 \; .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \;,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$