MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 8, 2011
Prof. Alan Guth

QUIZ 3

Reformatted to Remove Blank Pages

A SUMMARY OF USEFUL INFORMATION IS AT THE END OF THE EXAM.

Problem	Maximum	Score
1	20	
2	40	
3	30	
4	10	
TOTAL	100	

____

Your Name

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(4 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (2 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

(b) *(6 points)* Consider a star in a circular orbit of radius r in a galaxy. What is the velocity of the star, in terms of the total mass $M(r)$ contained within a sphere of radius r? (For simplicity, assume that the mass distribution is spherical.)

(c) *(2 points)* Observations of galaxies show that the surface brightness of spiral galaxies is highly concentrated near the center of the spiral galaxy disk. Therefore, for $r > 5$ kpc or so, the density of stars is approximately zero. What does your answer for $v(r)$ above predict for the dependence of $v(r)$ on r at large radii, if all of the mass in the galaxy is due to stars?

— Problem 1 continues on next page —

Problem 1, continued:

(d) *(3 points)* How do observed galactic rotation curves $v(r)$ actually behave for large r?

The following problem was part of Problem 1 of the Review Problems for Quiz 3.

(e) *(5 points)* Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

PROBLEM 2: THE SLOAN DIGITAL SKY SURVEY z = 5.82 QUASAR *(40 points)*

This problem was Problem 11 of the Review Problems for Quiz 3, and was originally Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\text{phys},0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\text{phys,e}}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

PROBLEM 3: NUCLEOSYNTHESIS AFTER THE DEUTERIUM BOTTLENECK *(30 points)*

This problem will be a continuation of the deuterium bottleneck problem from Problem Set 8. In that problem we examined the equilibrium for the reaction

$$n + p \longleftrightarrow D \,, \tag{3.1}$$

as the universe cooled. We found that the equilibrium would reach a point where half of the neutrons would be bound into deuterium at a temperature of

$$T_D = 7.64 \times 10^8 \text{ K} \,, \tag{3.2}$$

where I will use the subscript "D" for deuterium. It will also be useful to know that

$$kT_D = .0659 \text{ MeV} \qquad \text{and} \qquad \left(\frac{m_p k T_D}{2\pi\hbar^2}\right)^{3/2} = 4.01 \times 10^{39} \text{ m}^{-3} \,. \tag{3.3}$$

At this time, the total baryon number density of the universe would have been

$$n_b = 5.54 \times 10^{24} \text{ m}^{-3} \,. \tag{3.4}$$

The fraction f of these baryons that are neutrons is 0.14, so a fraction $1 - f = 0.86$ are protons. (Protons and neutrons can interconvert, but that process is very slow compared to the processes discussed in this problem.)

(a) *(5 points)* Once the deuterium bottleneck breaks, we assume that essentially all the neutrons in the universe become rapidly bound into He^4 nuclei, which are each composed of 2 protons and 2 neutrons. The left-over protons remain free, later forming hydrogen atoms. After this process is completed, what is the number density n_{He^4} of He^4 nuclei, and the number density n_p of free protons? Be sure to make use of the numbers given in the preamble. Give an answer in terms of symbols, and then evaluate it numerically.

(b) *(5 points)* To understand why helium forms so readily once the deuterium bottleneck breaks, we can look at the equilibrium for the reaction

$$p + p + n + n \longleftrightarrow \text{He}^4 \,. \tag{3.5}$$

While this reaction normally occurs through a chain involving deuterium and either He^3 or tritium (H^3), the multistep nature of the process does not affect the equilibrium abundances. Given that the above reaction is possible, express the chemical potential μ_{He^4} for He^4 nuclei in terms of the chemical potentials for protons and neutrons, μ_p and μ_n.

— *Problem 3 continues on the next page* —

Problem 3, continued:

(c) *(7 points)* Let B_{He^4} denote the binding energy of He^4, which has a value of 28.30 MeV. In terms of B_{He^4}, the temperature T, and the fundamental constants m_p (proton mass), k (Boltzmann constant), and $\hbar$ (Planck's constant, $h/2\pi$), write the relation between the number densities n_p, n_n, and n_{He^4} that will hold in thermal equilibrium. Note that He^4 is spinless, and so has only one spin state. Except for the binding energy, which has already been calculated as B_{He^4}, you may approximate $m_p = m_n = \frac{1}{4} m_{He^4}$.

(d) *(6 points)* Assume that the universe rapidly reaches thermal equilibrium after the deuterium bottleneck breaks. In that case it will never be the case that all neutrons will become bound in He^4, since the equilibrium for the reaction of Eq. (3.5) will not allow it. What will be the number density n_n of neutrons immediately after thermal equilibrium is reached? Give an answer in terms of symbols, and evaluate its order of magnitude numerically. It may be useful to know that $e^{B_{He^4}/kT_D} \approx 3 \times 10^{186}$.

(e) *(7 points)* Finally, we might wonder whether significant amounts of lithium will be produced. Consider the Li^7 nucleus, composed of 3 protons and 4 neutrons, with a binding energy $B_{Li^7} = 39.24$ MeV. Assuming thermal equilibrium, express the number density of Li^7 nuclei in terms of n_n, n_p, the temperature T_D, and fundamental constants. Li^7 has spin $s = 3/2$, so there are $2s + 1 = 4$ spin states. Again, give an answer in terms of symbols, and then evaluate its order of magnitude numerically. Note: $e^{B_{Li^7}/kT_D} \approx 4 \times 10^{258}$.

PROBLEM 4: DOUBLING OF ELECTRONS *(10 points)*

Suppose that instead of one species of electrons and their antiparticles, suppose there was also another species of electron-like and positron-like particles. Suppose that the new species has the same mass and other properties as the electrons and positrons. If this were the case, what would be the ratio T_ν/T_γ of the temperature today of the neutrinos to the temperature of the CMB photons.

USEFUL INFORMATION:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\rm coord} = \frac{c}{a(t)} .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{a(t_{\rm observed})}{a(t_{\rm emitted})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) , \quad \vec{p} = \gamma m_0 \vec{v} , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \quad \ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho_m(t_i) \text{ (matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\, \rho_r(t_i) \text{ (radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) , \quad \Omega \equiv \rho/\rho_c , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} .$$

Flat ($k = 0$): $a(t) \propto t^{2/3}$ (matter-dominated) ,
$a(t) \propto t^{1/2}$ (radiation-dominated) ,
$\Omega = 1$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$$\Omega = \frac{2}{1 + \cos\theta} > 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{k}}\right)^3 .$$

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)$, $\dfrac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)$,

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{\kappa}}\right)^3 ,$$

$$\kappa \equiv -k > 0 .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\,\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\;, \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\;, \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\;.$$

$$g_\gamma = g^*_\gamma = 2\;,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\;,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\;,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}\;,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3\;.$$

CHEMICAL EQUILIBRIUM:

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i\frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3}\,e^{(\mu_i - m_i c^2)/kT}\;.$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4}\frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860\text{ MeV}}{\sqrt{t\text{ (in sec)}}}\left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt'$$
$$= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}}c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2} ,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z} ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} .$$

Age of universe:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}}$$
$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

Look-back time:

$$t_{\text{look-back}}(z) =$$
$$\frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}} .$$

PHYSICAL CONSTANTS:

$$G = 6.674\times10^{-11}\ \text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2} = 6.674\times10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}$$
$$k = \text{Boltzmann's constant} = 1.381\times10^{-23}\,\text{joule/K}$$
$$= 1.381\times10^{-16}\,\text{erg/K}$$
$$= 8.617\times10^{-5}\,\text{eV/K}$$

$$\hbar = \frac{h}{2\pi} = 1.055\times10^{-34}\ \text{joule}\cdot\text{s}$$
$$= 1.055\times10^{-27}\ \text{erg}\cdot\text{s}$$
$$= 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$$

$$c = 2.998\times10^{8}\ \text{m/s}$$
$$= 2.998\times10^{10}\ \text{cm/s}$$

$\hbar c = 197.3$ MeV-fm, $\quad 1\ \text{fm} = 10^{-15}$ m

$1\ \text{yr} = 3.156\times10^{7}$ s

$1\ \text{eV} = 1.602\times10^{-19}\ \text{joule} = 1.602\times10^{-12}\ \text{erg}$

$1\ \text{GeV} = 10^9\ \text{eV} = 1.783\times10^{-27}$ kg (where $c\equiv1$)

$\qquad = 1.783\times10^{-24}$ g .

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.616\times10^{-35}\ \text{m} ,$$
$$= 1.616\times10^{-33}\ \text{cm} ,$$
$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391\times10^{-44}\ \text{s} ,$$
$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.177\times10^{-8}\ \text{kg} ,$$
$$= 2.177\times10^{-5}\ \text{g} ,$$
$$E_P = \sqrt{\frac{\hbar c^5}{G}} = 1.221\times10^{19}\ \text{GeV} .$$