Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 3, 2013

Prof. Alan Guth

QUIZ 1

Reformatted to Remove Blank Pages*

A FORMULA SHEET IS AT THE END OF THE EXAM.

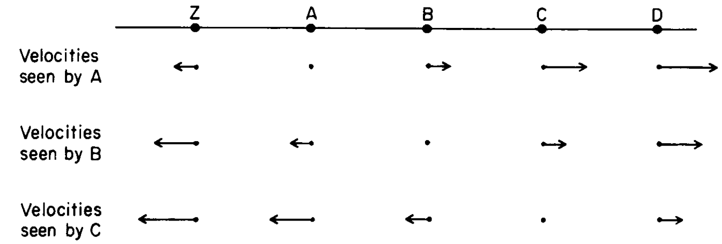
You may rip off and keep the formula sheet.

Please answer all questions in this stapled booklet.

* A few corrections announced at the quiz have been incorporated.

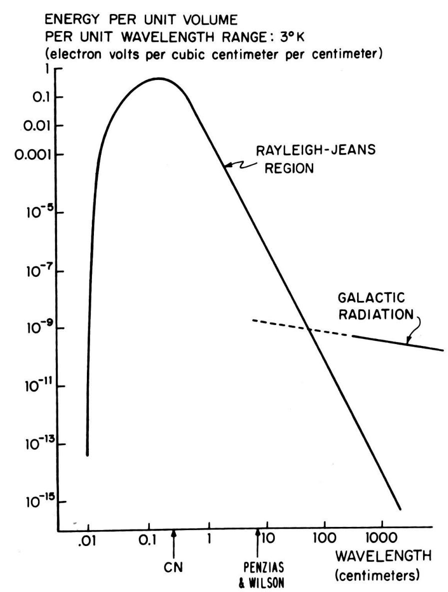
PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(5 points)*



Weinberg used the diagram above to explain an important property of the Hubble expansion law. What property was it. Explain in one or a few sentences how this diagram illustrates the property in question.

(b) *(5 points)* The diagram on the right shows the Planck spectrum for black-body radiation at 3.0 K, as used in Weinberg's book. Weinberg explained that the intensity of black-body radiation falls off at long wavelengths because "it is hard to fit radiation into any volume whose dimensions are smaller than the wavelength." In one or a few sentences, what is is reason for the suppression in intensity at short wavelengths?

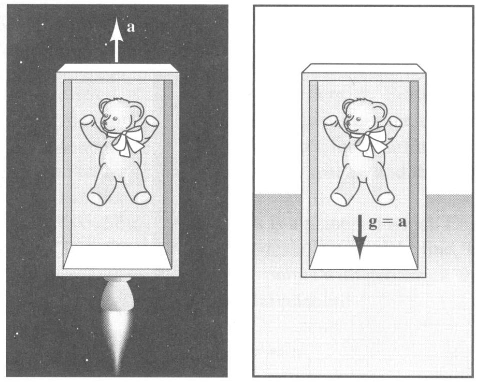


(c) *(5 points)* The label CN on the diagram at the right refers to cyanogen, a radical consisting of one carbon and one nitrogen atom. In one or a few sentences, what does cyanogen have to do with black-body radiation?

(d) *(5 points)* Another label in the diagram above is "Penzias & Wilson." What discovery did they make *(3 points)*, and where were they employed *(2 points)*?

— *Problem 1 continues on next page* —

(e) *(5 points)* Ryden used the teddy-bear diagram at the right to illustrate an important property of general relativity. What property was it? Explain in one or a few sentences how this diagram illustrates the property in question.



PROBLEM 2: AN EXPONENTIALLY EXPANDING FLAT UNIVERSE *(30 points)*

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$a(t) = a_0 e^{\chi t} ,$$

where a_0 and χ are constants.

(a) *(5 points)* Consider two galaxies which at some time t_1 are separated by a physical distance ℓ_p. At this time one galaxy emits a pulse of light in the direction of the other. If the light pulse is received at the second galaxy at time t_2, what is the redshift z? (Recall that $1 + z$ is the factor by which the wavelength or the period of the light wave is increased.) Your answer can depend on any or all of the quantities a_0, χ, c, t_1 t_2, and ℓ_p.

(b) *(10 points)* At what time t_2 does the second galaxy receive the light pulse? Your answer should depend only on one or more of the quantities c, a_0, χ, ℓ_p, and t_1, but not on z.

(c) *(5 points)* Is it possible for ℓ_p to be so large that the light pulse is never received by the second galaxy? If so, how large must ℓ_p be for this to happen?

(d) *(5 points)* At what time $t_{\rm eq}$ is the light ray equidistant between the two galaxies?

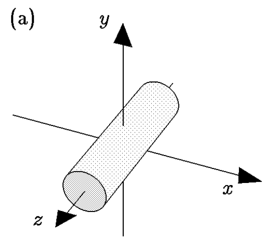
(e) *(5 points)* If the light pulse has duration Δt when it is emitted, as measured by observers on the emitting galaxy, what is the duration measured by observers on the receiving galaxy? (Assume that Δt is small compared to cosmological time scales, such as $t_2 - t_1$, or χ^{-1}.) The answer can depend on any or all of a_0, χ, c, t_1, t_2, ℓ_p, z, or Δt.

PROBLEM 3: A CYLINDRICAL UNIVERSE *(25 points)*

The following problem was on Problem Set 3 this year.

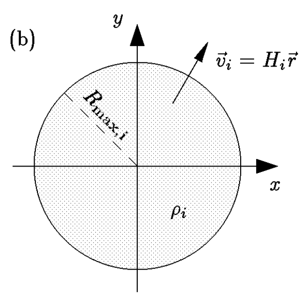
The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:

(a)



3D Perspective

(b)



2D Cross Section
(Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}\ ; \qquad \hat{r} = \frac{\vec{r}}{r}\ ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i\vec{r}\ .$$

— Problem 3 continues on next page —

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r}\ ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

(c) *(5 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i}\ ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

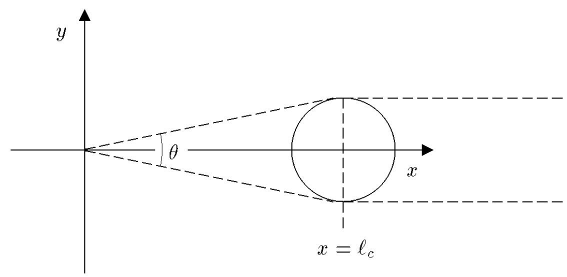
(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

(e) *(5 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a)\ .$$

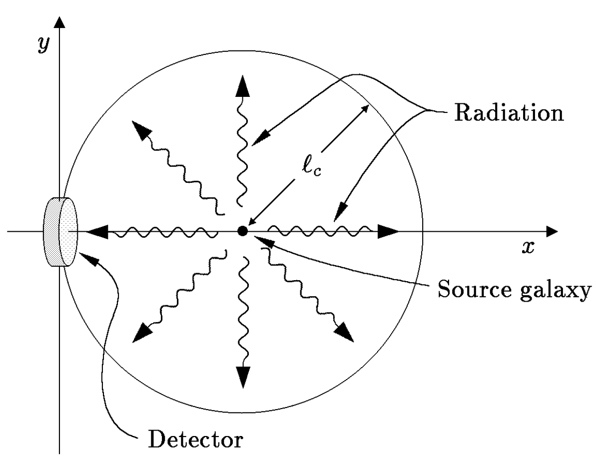
What is $V(a)$? Will this universe expand forever, or will it collapse?

PROBLEM 4: ANGULAR SIZE AND RADIATION FLUX RECEIVED FROM A DISTANT GALAXY *(25 points)*



(a) *(10 points)* Suppose that we observe a distant galaxy that currently has a physical (proper) distance $\ell_p = a(t_0)\ell_c$, where t_0 as usual denotes the current time, and $a(t)$ is the scale factor. Treat the galaxy as a sphere, which had (physical) radius R_1 at the time of emission, and R_0 today. Suppose that we do not know the form of the function $a(t)$, but we do know the redshift z with which the radiation is received. In terms of some or all of the quantities c, t_0, R_0, R_1, ℓ_p, and z, what is the angle θ that the distant galaxy subtends in our view. You should assume that $\theta \ll 1$.

(b) *(15 points)* Suppose that the galaxy had power output P (measured, say, in joules per second, also called watts) at the time of emission, and assume that the power was radiated uniformly in all directions. What is the radiation energy flux J from this galaxy at the Earth today? Energy flux (which might be measured in watts per square meter) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow. Express your answer in terms of some or all of the quantities c, t_0, R_0, R_1, ℓ_p, z, and P.



Problem	Maximum	Score
1	25	
2	30	
3	25	
4	25	
TOTAL	105	

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QUIZ 1 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3} G \rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)} \, \rho(t_i)$$

$$\Omega \equiv \rho / \rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

$$\text{Flat } (k=0): \qquad a(t) \propto t^{2/3} \ ,$$

$$\Omega = 1 \ .$$