MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 27, 2013
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Thursday, October 3, 2013, during the normal class time.

QUIZ COVERAGE: Lecture Notes 1, 2 and 3; Problem Sets 1, 2, and 3; Weinberg, Chapters 1-3, Ryden, Chapters 1, 2, and 3. (While all of Ryden's Chapter 3 has been assigned, questions on the quiz will be limited to Section 3.1. The material in Sections 3.2 and 3.3 will be discussed in lecture later in the course, and you will not be responsible for it until then. Section 3.4 (for the $\kappa = 0$ case) may help you understand the cosmological Doppler shift, also discussed in Lecture Notes 2, but there will be no questions specifically focussed on Ryden's discussion.) **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 2, 4, 7, 12, 15, 17, 19, and 22. The starred problems do not include any reading questions, but parts of the reading questions in these Review Problems may also recur on the upcoming quiz.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, and 2011. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. Since the schedule and the number of quizzes has varied over the years, the coverage of this quiz will not necessarily be the same as Quiz 1 from all previous years. In fact, however, the first quiz this year covers essentially the same material as the first quiz in either 2009 or 2011.

REVIEW SESSION: To help you study for the quiz, Tingtao Zhou will hold a review session on Monday, September 30, at 7:30 pm, in a room to be announced.

FUTURE QUIZZES: The other quiz dates this term will be Thursday November 7, and Thursday December 5, 2013.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the first quiz, this useful information will be the following:

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \;, \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \;, \quad \ddot a = -\frac{4\pi}{3} G \rho a \;,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \;, \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \;.$$

$$\text{Flat } (k=0): \qquad a(t) \propto t^{2/3} \;,$$

$$\Omega = 1 \;.$$

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

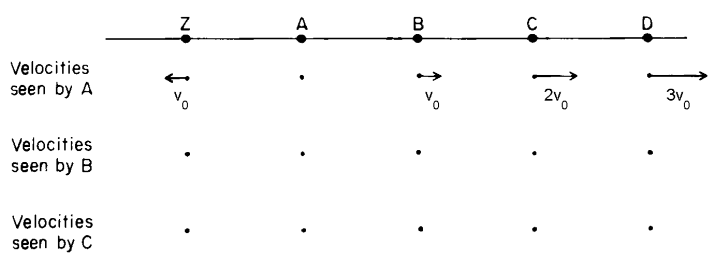
The following problem was Problem 1, Quiz 1, 2000. The parts were each worth 5 points.

a) The Doppler effect for both sound and light waves is named for Johann Christian Doppler, a professor of mathematics at the Realschule in Prague. He predicted the effect for both types of waves in xx42. What are the two digits xx?

b) When the sky is very clear (as it almost never is in Boston), one can see a band of light across the night sky that has been known since ancient times as the Milky Way. Explain in a sentence or two how this band of light is related to the shape of the galaxy in which we live, which is also called the Milky Way.

c) The statement that the distant galaxies are on average receding from us with a speed proportional to their distance was first published by Edwin Hubble in 1929, and has become known as Hubble's law. Was Hubble's original paper based on the study of 2, 18, 180, or 1,800 galaxies?

d) The following diagram, labeled *Homogeneity and the Hubble Law,* was used by Weinberg to explain how Hubble's law is consistent with the homogeneity of the universe:



The arrows and labels from the "Velocities seen by B" and the "Velocities seen by C" rows have been deleted from this copy of the figure, and it is your job to sketch the figure in your exam book with these arrows and labels included. (Actually, in Weinberg's diagram these arrows were not labeled, but the labels are required here so that the grader does not have to judge the precise length of hand-drawn arrows.)

e) The horizon is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe as a whole. During a time interval in which the linear size of the universe grows by 1%, does the horizon distance

(i) grow by more than 1%, or

(ii) grow by less than 1%, or

(iii) grow by the same 1%?

f) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

g) At a temperature of 3000 K, the nuclei and electrons that filled the universe combined to form neutral atoms, which interact very weakly with the photons of the background radiation. After this process, known as "recombination," the background radiation expanded freely. Since recombination, how have each of the following quantities varied as the size of the universe has changed? (Your answers should resemble statements such as "proportional to the size of the universe," or "inversely proportional to the square of the size of the universe". The word "size" will be interpreted to mean linear size, not volume.)

(i) the average distance between photons

(ii) the typical wavelength of the radiation

(iii) the number density of photons in the radiation

(iv) the energy density of the radiation

(v) the temperature of the radiation

***PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY** *(25 points)*

The following problem was Problem 2, Quiz 1, 2000.

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties were confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density does not fall. Like the conventional theory, the steady-state theory describes a homogeneous, isotropic, expanding universe, so the same comoving coordinate formulation can be used.

a) *(10 points)* The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $a(t)$ which is consistent with this hypothesis.

b) *(15 points)* Suppose that the mass density of the universe is ρ_0, which of course does not change with time. In terms of the general form for $a(t)$ that you found in part (a), calculate the rate at which new matter must be created for ρ_0 to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. *[If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $a(t)$.]*

PROBLEM 3: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1 on Quiz 1, 2007, where each of the 5 questions was worth 5 points:

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (*Bonus points*: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.

(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.

(iii) published a catalog, *Nebulae and Star Clusters*, listing 103 objects that astronomers should avoid when looking for comets.

(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.

(v) discovered that the orbital periods of the planets are proportional to the 3/2 power of the semi-major axis of their elliptical orbits.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were **not** part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory	(ii) MIT	(iii) Princeton University
(iv) pigeons	(v) ground hogs	(vi) Hubble's constant
(vii) liquid helium	(viii) 7.35 cm	

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.)

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus' lifetime.

(ii) approximately two and three decades after Copernicus' death, respectively.

(iii) about one hundred years after Copernicus' death.

(iv) approximately two and three centuries after Copernicus' death, respectively.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

(ii) 100 kpc (1 kpc = 1000 pc, 1 pc $= 3.086 \times 10^{16}$ m = 3.262 light-year).

(iii) 1 Mpc (1 Mpc $= 10^6$ pc).

(iv) 10 Mpc.

(v) 100 Mpc.

(vi) 1000 Mpc.

* PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE *(20 points)*

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems. The problem also appeared as Problem 2 on Quiz 1, 2007.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$a(t) = a_0 e^{\chi t} ,$$

where a_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift z. Express the time t_r at which we receive the pulse in terms of z, χ, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of z, χ, and any relevant physical constants.

PROBLEM 5: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift?

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, 10^4 light-years, 10^7 light years, or 10^{10} light-years? *[Note for 2011: this question was based on the reading from Joseph Silk's* **The Big Bang***, and therefore would be not be a fair question for this year.]*

(i) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

(j) At the time of the discovery of the cosmic microwave background, an active but independent effort was taking place elsewhere. P.J.E. Peebles had estimated that the universe must contain background radiation with a temperature of at least 10°K, and Robert H. Dicke, P.G. Roll, and D.T. Wilkinson had mounted an experiment to look for it. At what institution were these people working?

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The following problem was Problem 3, Quiz 2, 1988:

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as

$$a(t) = bt^{1/3} .$$

Here b denotes a constant.

(a) If a light pulse is emitted at time t_e and observed at time t_o, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that t_e and t_o are given, find the observed redshift z.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of c, t_o, and z (i.e., without t_e appearing).

(d) At an arbitrary time t in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of c, t, and t_o.

∗ PROBLEM 7: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

The following problem was Problem 3, Quiz 1, 2000.

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{\gamma} ,$$

where b and γ are constants. *[This universe differs from the matter-dominated universe described in the lecture notes in that ρ is not proportional to $1/a^3(t)$. Such behavior is possible when pressures are large, because a gas expanding under pressure can lose energy (and hence mass) during the expansion.]* For the following questions, any of the answers may depend on γ, whether it is mentioned explicitly or not.

a) *(5 points)* Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by a distant object. In terms of these quantities, find the value of the redshift parameter z with which the light is received.

b) *(5 points)* Find the "look-back" time as a function of z and t_0. The look-back time is defined as the length of the interval in cosmic time between the emission and observation of the light.

c) *(10 points)* Express the present value of the physical distance to the object as a function of H_0, z, and γ.

d) *(10 points)* At the time of emission, the distant object had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this object at the earth today? Express your answer in terms of P, H_0, z, and γ. *[Energy flux (which might be measured in erg-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow.]*

e) *(10 points)* Suppose that the distant object is a galaxy, moving with the Hubble expansion. Within the galaxy a supernova explosion has hurled a jet of material directly towards Earth with a speed v, measured relative to the galaxy, which is comparable to the speed of light c. Assume, however, that the distance the jet has traveled from the galaxy is so small that it can be neglected. With what redshift z_J would we observe the light coming from this jet? Express your answer in terms of all or some of the variables v, z (the redshift of the galaxy), t_0, H_0, and γ, and the constant c.

PROBLEM 8: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1996:

The following questions are worth 5 points each.

a) In 1814-1815, the Munich optician Joseph Frauenhofer allowed light from the sun to pass through a slit and then through a glass prism. The light was spread into a spectrum of colors, showing lines that could be identified with known elements — sodium, iron, magnesium, calcium, and chromium. Were these lines dark, or bright *(2 points)*? Why *(3 points)*?

b) The Andromeda Nebula was shown conclusively to lie outside our own galaxy when astronomers acquired telescopes powerful enough to resolve the individual stars of Andromeda. Was this feat accomplished by Galileo in 1609, by Immanuel Kant in 1755, by Henrietta Swan Leavitt in 1912, by Edwin Hubble in 1923, or by Walter Baade and Allan Sandage in the 1950s?

c) Some of the earliest measurements of the cosmic background radiation were made indirectly, by observing interstellar clouds of a molecule called cyanogen (CN). State whether each of the following statements is true or false *(1 point each)*:

(i) The first measurements of the temperature of the interstellar cyanogen were made over twenty years before the cosmic background radiation was directly observed.

(ii) Cyanogen helps to measure the cosmic background radiation by reflecting it toward the earth, so that it can be received with microwave detectors.

(iii) One reason why the cyanogen observations were important was that they gave the first measurements of the equivalent temperature of the cosmic background radiation at wavelengths shorter than the peak of the black-body spectrum.

(iv) By measuring the spectrum of visible starlight that passes through the cyanogen clouds, astronomers can infer the intensity of the microwave radiation that bathes the clouds.

(v) By observing chemical reactions in the cyanogen clouds, astronomers can infer the temperature of the microwave radiation in which they are bathed.

d) In about 280 B.C., a Greek philosopher proposed that the Earth and the other planets revolve around the sun. What was the name of this person? *[Note for 2011: this question was based on readings from Joseph Silk's* **The Big Bang**, *and therefore is not appropriate for Quiz 1 of this year.]*

e) In 1832 Heinrich Wilhelm Olbers presented what we now know as "Olbers' Paradox," although a similar argument had been discussed as early as 1610 by

Johannes Kepler. Olbers argued that if the universe were transparent, static, infinitely old, and was populated by a uniform density of stars similar to our sun, then one of the following consequences would result:

(i) The brightness of the night sky would be infinite.

(ii) Any patch of the night sky would look as bright as the surface of the sun.

(iii) The total energy flux from the night sky would be about equal to the total energy flux from the sun.

(iv) Any patch of the night sky would look as bright as the surface of the moon.

Which one of these statements is the correct statement of Olbers' paradox?

PROBLEM 9: A FLAT UNIVERSE WITH $a(t) \propto t^{3/5}$

The following problem was Problem 3, Quiz 1, 1996:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{3/5} ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(5 points)* What is the physical horizon distance at time t?

c) *(5 points)* Suppose a light pulse leaves galaxy A at time t_A and arrives at galaxy B at time t_B. What is the coordinate distance between these two galaxies?

d) *(5 points)* What is the physical separation between galaxy A and galaxy B at time t_A? At time t_B?

e) *(5 points)* At what time is the light pulse equidistant from the two galaxies?

f) *(5 points)* What is the speed of B relative to A at the time t_A? (By "speed," I mean the rate of change of the physical distance with respect to cosmic time, $d\ell_p/dt$.)

g) *(5 points)* For observations made at time t, what is the present value of the physical distance as a function of the redshift z (and the time t)? What physical distance corresponds to $z = \infty$? How does this compare with the horizon distance? (Note that this question does not refer to the galaxies A and B discussed in the earlier parts. In particular, you should not assume that the light pulse left its source at time t_A.)

h) *(5 points)* Returning to the discussion of the galaxies A and B which were considered in parts (c)-(f), suppose the radiation from galaxy A is emitted with total power P. What is the power per area received at galaxy B?

i) *(5 points)* When the light pulse is received by galaxy B, a pulse is immediately sent back toward galaxy A. At what time does this second pulse arrive at galaxy A?

PROBLEM 10: DID YOU DO THE READING? *(20 points)*

The following questions were taken from Problem 1, Quiz 1, 1998:

The following questions are worth 5 points each.

a) In 1917, Einstein introduced a model of the universe which was based on his newly developed general relativity, but which contained an extra term in the equations which he called the "cosmological term." (The coefficient of this term is called the "cosmological constant.") What was Einstein's motivation for introducing this term?

b) When the redshift of distant galaxies was first discovered, the earliest observations were analyzed according to a cosmological model invented by the Dutch astronomer W. de Sitter in 1917. At the time of its discovery, was this model thought to be static or expanding? From the modern perspective, is the model thought to be static or expanding?

c) The early universe is believed to have been filled with thermal, or black-body, radiation. For such radiation the number density of photons and the energy density are each proportional to powers of the absolute temperature T. Say

$$\text{Number density} \propto T^{n_1}$$

$$\text{Energy density} \propto T^{n_2}$$

Give the values of the exponents n_1 and n_2.

d) At about 3,000 K the matter in the universe underwent a certain chemical change in its form, a change that was necessary to allow the differentiation of matter into galaxies and stars. What was the nature of this change?

PROBLEM 11: ANOTHER FLAT UNIVERSE WITH $a(t) \propto t^{3/5}$ *(40 points)*

The following was Problem 3, Quiz 1, 1998:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{3/5} ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(10 points)* Suppose a message is transmitted by radio signal (traveling at the speed of light c) from galaxy A to galaxy B. The message is sent at cosmic time t_1, when the physical distance between the galaxies is ℓ_0. At what cosmic time t_2 is the message received at galaxy B? (Express your answer in terms of ℓ_0, t_1, and c.)

c) *(5 points)* Upon receipt of the message, the creatures on galaxy B immediately send back an acknowledgement, by radio signal, that the message has been received. At what cosmic time t_3 is the acknowledgment received on galaxy A? (Express your answer in terms of ℓ_0, t_1, t_2, and c.)

d) *(10 points)* The creatures on galaxy B spend some time trying to decode the message, finally deciding that it is an advertisement for Kellogg's Corn Flakes (whatever that is). At a time Δt after the receipt of the message, as measured on their clocks, they send back a response, requesting further explanation. At what cosmic time t_4 is the response received on galaxy A? In answering this part, you should not assume that Δt is necessarily small. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, Δt, and c.)

e) *(5 points)* When the response is received by galaxy A, the radio waves will be redshifted by a factor $1+z$. Give an expression for z. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.)

f) *(5 points; No partial credit)* If the time Δt introduced in part (d) is small, the time difference $t_4 - t_3$ can be expanded to first order in Δt. Calculate $t_4 - t_3$ to first order accuracy in Δt. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.) [Hint: while this part can be answered by using brute force to expand the answer in part (d), there is an easier way.]

∗PROBLEM 12: THE DECELERATION PARAMETER

The following problem was Problem 2, Quiz 2, 1992, where it counted 10 points out of 100.

Many standard references in cosmology define a quantity called the ***deceleration parameter*** q, which is a direct measure of the slowing down of the cosmic expansion. The parameter is defined by

$$q \equiv -\ddot{a}(t)\frac{a(t)}{\dot{a}^2(t)} .$$

Find the relationship between q and Ω for a matter-dominated universe. *[In case you have forgotten, Ω is defined by*

$$\Omega = \rho/\rho_c ,$$

where ρ is the mass density and ρ_c is the critical mass density (i.e., that mass density which corresponds to $k = 0$).]

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

We have learned that a matter-dominated homogeneous and isotropic universe can be described by a scale factor $a(t)$ obeying the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} .$$

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that the mass density of nonrelativistic matter falls off as $1/a^3(t)$ as the universe expands; the mass of each particle remains constant, and the density of particles falls off as $1/a^3(t)$ because the volume increases as $a^3(t)$. For the photon-dominated universe, the density of photons falls of as $1/a^3(t)$, but in addition the frequency (and hence the energy) of each photon redshifts in proportion to $1/a(t)$. Since mass and energy are equivalent, the mass density of the gas of photons falls off as $1/a^4(t)$.

For a flat (i.e., $k = 0$) matter-dominated universe we learned that the scale factor $a(t)$ is proportional to $t^{2/3}$. How does $a(t)$ behave for a photon-dominated universe?

PROBLEM 14: DID YOU DO THE READING?

The following problem was taken from Problem 1, Quiz 1, 2004, where each part counted 5 points, for a total of 25 points. The reading assignment included the first three chapters of Ryden, **Introduction to Cosmology***, and the first three chapters of Weinberg,* **The First Three Minutes**.

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(b) What is the value of the Newtonian gravitational constant G in Planck units? The Planck length is of the order of 10^{-35} m, 10^{-15} m, 10^{15} m, or 10^{35} m?

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (For the latter question, a simple "yes" or "no" will suffice.)

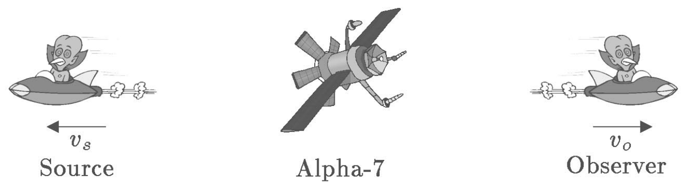
(d) In the "Standard Model" of the universe, when the universe cooled to about 3×10^a K, it became transparent to photons, and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^b K. What are the integers a and b?

(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

* PROBLEM 15: SPECIAL RELATIVITY DOPPLER SHIFT

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 20 points.

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with a speed v_s relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7 with speed v_o relative to Alpha-7.



(a) *(10 points)* Calculate the Doppler shift z of the radio wave as received by the observer. (Recall that radio waves are electromagnetic waves, just like light except that the wavelength is longer.)

(b) *(10 points)* Use the results of part (a) to determine v_{tot}, the velocity of the source spaceship as it would be measured by the observer spaceship. *(8 points will be given for the basic idea, whether or not you have the right answer for part (a), and 2 points will be given for the algebra.)*

PROBLEM 16: DID YOU DO THE READING?

The following question was taken from Problem 1, Quiz 1, 2005, where it counted 25 points.

(a) *(4 points)* What was the first external galaxy that was shown to be at a distance significantly greater than the most distant known objects in our galaxy? How was the distance estimated?

(b) *(5 points)* What is recombination? Did galaxies begin to form before or after recombination? Why?

(c) *(4 points)* In Chapter IV of his book, Weinberg develops a "recipe for a hot universe," in which the matter of the universe is described as a gas in thermal equilbrium at a very high temperature, in the vicinity of 10^9 K (several thousand million degrees Kelvin). Such a thermal equilibrium gas is completely described by specifying its temperature and the density of the conserved quantities. Which of the following is on this list of conserved quantities? Circle as many as apply.

(i) baryon number (ii) energy per particle (iii) proton number
(iv) electric charge (v) pressure

(d) *(4 points)* The wavelength corresponding to the mean energy of a CMB (cosmic microwave background) photon today is approximately equal to which of the following quantities? (You may wish to look up the values of various physical constants at the end of the quiz.)

(i) 2 fm (2×10^{-15} m)
(ii) 2 microns (2×10^{-6} m)
(iii) 2 mm (2×10^{-3} m)
(iv) 2 m.

(e) *(4 points)* What is the equivalence principle?

(f) *(4 points)* Why is it difficult for Earth-based experiments to look at the small wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?

* PROBLEM 17: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE

The following problem was taken from Problem 3, Quiz 1, 2005, where it counted 25 points.

Consider a flat universe that expands with a scale factor

$$a(t) = bt^{1/2} ,$$

where b is a constant. We will learn later that this is the behavior of the scale factor for a radiation-dominated universe.

(a) *(5 points)* At an arbitrary time $t = t_f$, what is the physical horizon distance? (By "physical," I mean as usual the distance in physical units, such as meters or centimeters, as measured by a sequence of rulers, each of which is at rest relative to the comoving matter in its vicinity.)

(b) *(3 points)* Suppose that a photon arrives at the origin, at time t_f, from a distant piece of matter that is precisely at the horizon distance at time t_f. What is the time t_e at which the photon was emitted?

(c) *(2 points)* What is the **coordinate** distance from the origin to the point from which the photon was emitted?

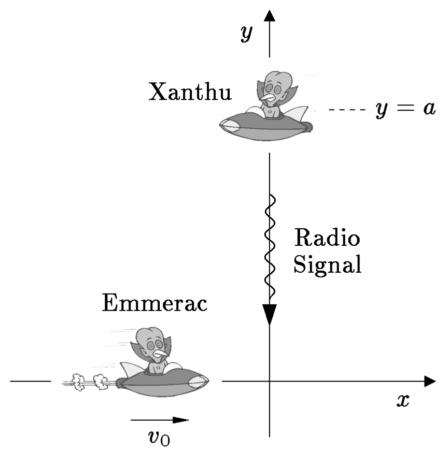
(d) *(10 points)* For an arbitrary time t in the interval $t_e \leq t \leq t_f$, while the photon is traveling, what is the **physical** distance $\ell_p(t)$ from the origin to the location of the photon?

(e) *(5 points)* At what time $t_{\max}$ is the physical distance of the photon from the origin at its largest value?

PROBLEM 18: TRANSVERSE DOPPLER SHIFTS

The following problem was taken from Problem 4, Quiz 1, 2005, where it counted 20 points.

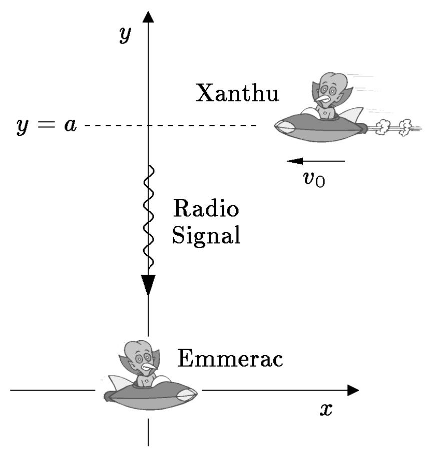
(a) *(8 points)* Suppose the spaceship Xanthu is at rest at location (x=0, y=a, z=0) in a Cartesian coordinate system. (We assume that the space is Euclidean, and that the distance scales in the problem are small enough so that the expansion of the universe can be neglected.) The spaceship Emmerac is moving at speed v_0 along the x-axis in the positive direction, as shown in the diagram, where v_0 is comparable to the speed of light. As the Emmerac crosses the origin, it receives a radio signal that had been sent some time earlier from the Xanthu. Is the radiation received redshifted or blueshifted? What is the redshift z (where negative values of z can be used to describe blueshifts)?



(b) *(7 points)* Now suppose that the Emmerac is at rest at the origin, while the Xanthu is moving in the negative x-direction, at $y = a$ and $z = 0$, as shown in the diagram. That is, the trajectory of the Xanthu can be taken as

$$(x{=}-v_0 t, y{=}a, z{=}0) \; .$$

At $t = 0$ the Xanthu crosses the y-axis, and at that instant it emits a radio signal along the y-axis, directed at the origin. The radiation is received some time later by the Emmerac. In this case, is the radiation received redshifted or blueshifted? What is the redshift z (where again negative values of z can be used to describe blueshifts)?

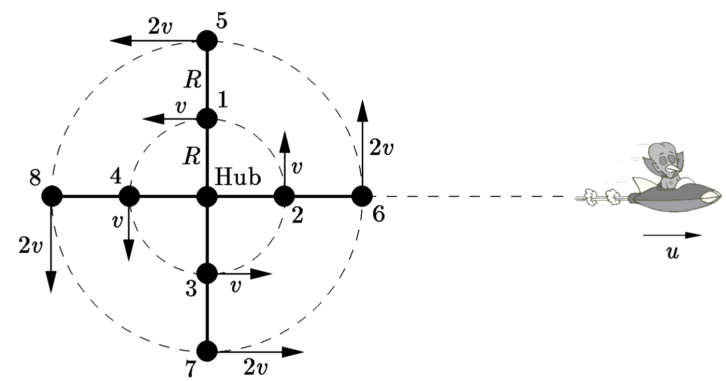


(c) *(5 points)* Is the sequence of events described in (b) physically distinct from the sequence described in (a), or is it really the same sequence of events described in a reference frame that is moving relative to the reference frame used in part (a)? Explain your reasoning in a sentence or two. *(Hint: note that there are three objects in the problem: Xanthu, Emmerac, and the photons of the radio signal.)*

∗PROBLEM 19: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND *(15 points)*

This problem was Problem 3 on Quiz 1, 2007.

Consider a high-speed merry-go-round which is similar to the one discussed in Problem 3 of Problem Set 1, but which has two levels. That is, there are four evenly spaced cars which travel around a central hub at speed v at a distance R from a central hub, and also another four cars that are attached to extensions of the four radial arms, each moving at a speed $2v$ at a distance $2R$ from the center. In this problem we will consider only light waves, not sound waves, and we will assume that v is not negligible compared to c, but that $2v < c$.



We learned in Problem Set 1 that there is no redshift when light from one car at radius R is received by an observer on another car at radius R.

(a) *(5 points)* Suppose that cars 5–8 are all emitting light waves in all directions. If an observer in car 1 receives light waves from each of these cars, what redshift z does she observe for each of the four signals?

(b) *(10 points)* Suppose that a spaceship is receding to the right at a relativistic speed u along a line through the hub, as shown in the diagram. Suppose that an observer in car 6 receives a radio signal from the spaceship, at the time when the car is in the position shown in the diagram. What redshift z is observed?

PROBLEM 20: SIGNAL PROPAGATION IN A FLAT MATTER-DOMINATED UNIVERSE *(55 points)*

The following problem was on Quiz 1, 2009.

Consider a flat, matter-dominated universe, with scale factor

$$a(t) = bt^{2/3} ,$$

where b is an arbitrary constant. For the following questions, the answer to any part may contain symbols representing the answers to previous parts, whether or not the previous part was answered correctly.

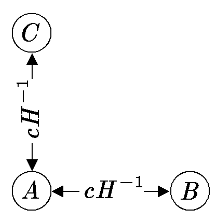
(a) *(10 points)* At time $t = t_1$, a light signal is sent from galaxy A. Let $\ell_{p,sA}(t)$ denote the physical distance of the signal from A at time t. (Note that $t = 0$ corresponds to the origin of the universe, not to the emission of the signal.) (i) Find the speed of separation of the light signal from A, defined as $\mathrm{d}\ell_{p,sA}/\mathrm{d}t$. What is the value of this speed (ii) at the time of emission, t_1, and (iii) what is its limiting value at arbitrarily late times?

(b) *(5 points)* Suppose that there is a second galaxy, galaxy B, that is located at a physical distance cH^{-1} from A at time t_1, where $H(t)$ denotes the Hubble expansion rate and c is the speed of light. (cH^{-1} is called the Hubble length.) Suppose that the light signal described above, which is emitted from galaxy A at time t_1, is directed toward galaxy B. At what time t_2 does it arrive at galaxy B?

(c) *(10 points)* Let $\ell_{p,sB}(t)$ denote the physical distance of the light signal from galaxy B at time t. (i) Find the speed of approach of the light signal towards B, defined as $-\mathrm{d}\ell_{p,sB}/\mathrm{d}t$. What is the value of this speed (ii) at the time of emission, t_1, and (iii) at the time of reception, t_2?

(d) *(10 points)* If an astronomer on galaxy A observes the light arriving from galaxy B at time t_1, what is its redshift z_{BA}?

(e) *(10 points)* Suppose that there is another galaxy, galaxy C, also located at a physical distance cH^{-1} from A at time t_1, but in a direction orthogonal to that of B. If galaxy B is observed from galaxy C at time t_1, what is the observed redshift z_{BC}? Recall that this universe is flat, so Euclidean geometry applies.



(f) *(10 points)* Suppose that galaxy A, at time t_1, emits electromagnetic radiation spherically symmetrically, with power output P. (P might be measured, for example, in watts, where 1 watt = 1 joule/second.) What is the radiation energy flux J that is received by galaxy B at time t_2, when the radiation reaches galaxy B? (J might be measured, for example, in watts per meter^2. Units are mentioned here only to help clarify the meaning of these quantities — your answer should have no explicit units, but should be expressed in terms of any or all of the given quantities t_1, P, and c, plus perhaps symbols representing the answers to previous parts.)

PROBLEM 21: DID YOU DO THE READING? *(25 points)*

The following problem appeared on Quiz 1 of 2011.

(a) *(10 points)* Hubble's law relates the distance of galaxies to their velocity. The Doppler effect provides an accurate tool to measure velocity, while the measure of cosmic distances is more problematic. Explain briefly the method that Hubble used to estimate the distance of galaxies in deriving his law.

(b) *(5 points)* One expects Hubble's law to hold as a consequence of the Cosmological Principle. What does the Cosmological Principle state?

(c) *(10 points)* Give a brief definition for the words *homogeneity* and *isotropy*. Then say for each of the following two statements whether it is true or false. If true explain briefly why. If false give a counter-example. You should assume Euclidean geometry (which Weinberg implicitly assumed in his discussion).

  (i) If the universe is isotropic around one point then it has to be homogeneous.

  (ii) If the universe is isotropic around two or more distinct points then it has to be homogeneous.

(d) Bonus question: *(2 points extra credit)* If we allow curved (i.e., non-Euclidean) spaces, is it true that a universe which is isotropic around two distinct points has to be homogeneous? If true explain briefly why, and otherwise give a counter-example.

*PROBLEM 22: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)*

The following problem appeared on Quiz 1 of 2011.

Consider again a flat matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Let t_0 denote the current time.

(a) *(5 points)* What is the current value of the physical horizon distance $\ell_{p,\mathrm{horizon}}(t_0)$? That is, what is the present distance of the most distant matter that can be seen, limited only by the speed of light.

(b) *(5 points)* Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a coordinate system with us at the origin, and the source of the photon along the positive x-axis. What is the coordinate x_0 of the photon at $t = 0$?

(c) *(5 points)* As the photon travels from the source to us, what is its coordinate $x(t)$ as a function of time?

(d) *(5 points)* What is the physical distance $\ell_p(t)$ between the photon and us as a function of time?

(e) *(5 points)* What is the maximum physical distance $\ell_{p,\max}(t)$ between the photon and us, and at what time $t_{\max}$ does it occur?

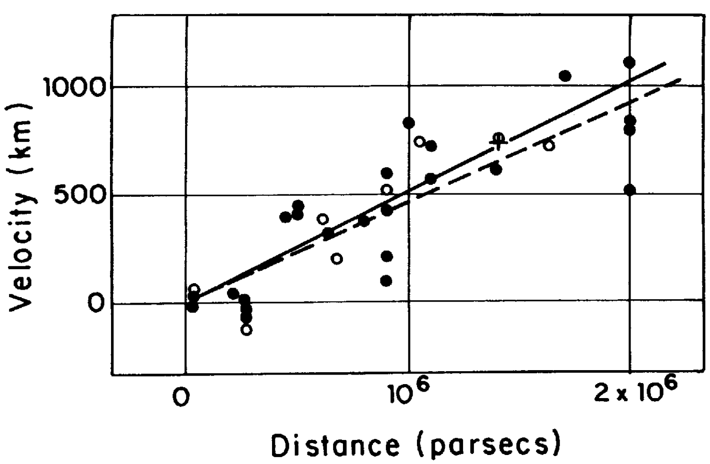
SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

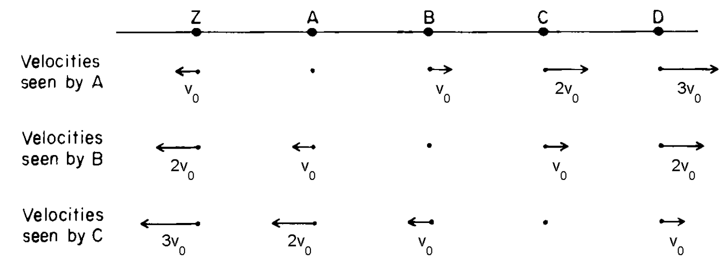
a) Doppler predicted the Doppler effect in $\boxed{\text{1842.}}$

b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look out from earth along the plane of the disk than when we look in any other direction.

c) Hubble's original paper on the expansion of the universe was based on a study of only $\boxed{\text{18 galaxies.}}$ Well, at least Weinberg's book says 18 galaxies. For my own book I made a copy of Hubble's original graph, which seems to show 24 black dots, each of which represents a galaxy, as reproduced below. The vertical axis shows the recession velocity, in kilometers per second. The solid line shows the best fit to the black dots, each of which represents a galaxy. Each open circle represents a group of the galaxies shown as black dots, selected by their proximity in direction and distance; the broken line is the best fit to these points. The cross shows a statistical analysis of 22 galaxies for which individual distance measurements were not available. I am not sure why Weinberg refers to 18 galaxies, but it is possible that the text of Hubble's article indicated that 18 of these galaxies were measured with more reliability than the rest.



d)



e) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by $\boxed{\text{more than 1\%.}}$ To see why, note that the horizon distance is equal to the scale factor times the comoving horizon distance. The scale factor grows by 1% during this time interval, but the comoving horizon distance also grows, since light from the distant galaxies has had more time to reach us.

f) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

g) (i) the average distance between photons: $\boxed{\text{proportional to the size of the universe}}$ (Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion. Since the universe expands uniformly, all distances grow by the same factor.)

(ii) the typical wavelength of the radiation: $\boxed{\text{proportional to the size of the universe}}$ (See Lecture Notes 3.)

(iii) the number density of photons in the radiation: $\boxed{\text{inversely proportional to the cube of the size of the universe}}$ (From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a cube is proportional to the cube of the length of a side, the average volume occupied by a photon grows as the cube of the size of the universe. The number density is the inverse of the average volume occupied by a photon.)

(iv) the energy density of the radiation: $\boxed{\text{inversely proportional to the fourth power of the size of the universe}}$ (The energy of each photon is proportional to its frequency, and hence inversely proportional to its wavelength. So from (ii) the energy of each photon is inversely proportional to the size of the universe, and from (iii) the number density is inversely proportional to the cube of the size.)

(v) the temperature of the radiation: $\boxed{\text{inversely proportional to the size of the universe}}$ (The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY *(25 points)*

a) *(10 points)* According to Eq. (3.7),

$$H(t) = \frac{1}{a(t)}\frac{da}{dt} .$$

So in this case

$$\frac{1}{a(t)}\frac{da}{dt} = H_0 ,$$

which can be rewritten as

$$\frac{da}{a} = H_0\, dt .$$

Integrating,

$$\ln a = H_0\, t + c ,$$

where c is a constant of integration. Exponentiating,

$$\boxed{a = b e^{H_0\, t} ,}$$

where $b = e^c$ is an arbitrary constant.

b) *(15 points)* Consider a cube of side ℓ_c drawn on the comoving coordinate system diagram. The physical length of each side is then $a(t)\,\ell_c$, so the physical volume is

$$V(t) = a^3(t)\,\ell_c^3 .$$

Since the mass density is fixed at $\rho = \rho_0$, the total mass inside this cube at any given time is given by

$$M(t) = a^3(t)\,\ell_c^3\,\rho_0 .$$

In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation

must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

$$\begin{aligned}\text{Rate} &= \frac{1}{V(t)}\frac{dM}{dt}\\ &= \frac{1}{a^3(t)\,\ell_c^3}\,3a^2(t)\frac{da}{dt}\,\ell_c^3\,\rho_0\\ &= \frac{3}{a}\frac{da}{dt}\rho_0\\ &= \boxed{3H_0\,\rho_0\ .}\end{aligned}$$

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose we take $H_0 = 70$ km-sec^{-1}-Mpc^{-1}, and take ρ_0 to be the critical density, $\rho_c = 3H_0^2/8\pi G$. Then

$$\begin{aligned}\text{Rate} &= \frac{9H_0^3}{8\pi G}\\ &= \frac{9\times(70\ \text{km-s}^{-1}\text{-Mpc}^{-1})^3}{8\pi\times 6.67260\times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}}\\ &= \frac{9\times(70\ \cancel{\text{km}}\text{-s}^{-1}\text{-}\cancel{\text{Mpc}}^{-1})^3}{8\pi\times 6.67260\times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}}\\ &\qquad\times\left(\frac{1\ \cancel{\text{Mpc}}}{3.086\times 10^{22}\ \cancel{\text{m}}}\right)^3\times\left(\frac{10^3\ \cancel{\text{m}}}{\cancel{\text{km}}}\right)^3\\ &= 6.26\times 10^{-44}\ \text{kg-m}^{-3}\text{-s}^{-1} .\end{aligned}$$

To put this number into more meaningful terms, note that the mass of a hydrogen atom is 1.67×10^{-27} kg, and that 1 year $= 3.156\times 10^7$ s. The rate of matter production required for the steady-state universe theory can then be expressed as roughly one hydrogen atom per cubic meter per billion years! Needless to say, such a rate of matter production is totally undetectable, so the steady-state theory cannot be ruled out by the failure to detect matter production.

PROBLEM 3: DID YOU DO THE READING? *(25 points)*

The following 5 questions are each worth 5 points:

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (*Bonus points*: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

Ans: (Weinberg, page 8, or Ryden, page 16): Hermann Bondi, Thomas Gold, and Fred Hoyle.

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.

(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.

(iii) published a catalog, *Nebulae and Star Clusters*, listing 103 objects that astronomers should avoid when looking for comets.

$\boxed{\text{(iv)}}$ published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.

(v) discovered that the orbital periods of the planets are proportional to the 3/2 power of the semi-major axis of their elliptical orbits.

Discussion: (i) is false in part because de Sitter was not involved in the measurement of the size of the Milky Way, but the most obvious error is in the size of the Milky Way. Its actual diameter is reported by Weinberg (p. 16) to be about 100,000 light-years, although now it is believed to be about twice that large. (ii) is an accurate description of an observation by Edwin Hubble in 1923 (Weinberg, pp. 19-20). (iii) describes the work of Charles Messier in 1781 (Weinberg, p. 17). (v) is of course one of Kepler's laws of planetary motion.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were **not** part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory	(ii) MIT	(iii) Princeton University
(iv) pigeons	$\boxed{\text{(v)}}$ ground hogs	$\boxed{\text{(vi)}}$ Hubble's constant
(vii) liquid helium	(viii) 7.35 cm	

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.)

Discussion: The discovery of the cosmic background radiation was described in some detail by Weinberg in Chapter 3. The observation was done at Bell Telephone Laboratories, in Holmdel, New Jersey. The detector was cooled with liquid helium to minimize electrical noise, and the measurements were made at a wavelength of 7.35 cm. During the course of the experiment the astronomers had to eject a pair of pigeons who were roosting in the antenna. Penzias and Wilson were not initially aware that the radiation they discovered might have come from the big bang, but Bernard Burke of MIT put them in touch with a group at Princeton University (Robert Dicke, James Peebles, P.G. Roll, and David Wilkinson) who were actively working on this hypothesis.

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus' lifetime.

(ii) approximately two and three decades after Copernicus' death, respectively.

(iii) about one hundred years after Copernicus' death.

$\boxed{\text{(iv)}}$ approximately two and three centuries after Copernicus' death, respectively.

Ryden discusses this on p. 5. The aberration of starlight was discovered in 1728, while the Foucault pendulum was invented in 1851.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

(ii) 100 kpc (1 kpc = 1000 pc, 1 pc $= 3.086 \times 10^{16}$ m = 3.262 light-year).

(iii) 1 Mpc (1 Mpc $= 10^6$ pc).

(iv) 10 Mpc.

$\boxed{\text{(v)}}$ 100 Mpc.

(vi) 1000 Mpc.

This issue is discussed in Ryden's book on p. 11.

PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{a}/a \ .$$

So

$$H = \frac{\chi a_0 e^{\chi t}}{a_0 e^{\chi t}} = \boxed{\chi \ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{a(t)} = \frac{c}{a_0} e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{a_0} \int_0^t e^{-\chi t'}\, dt' \\ &= \frac{c}{a_0} \left[-\frac{1}{\chi} e^{-\chi t'} \right]_0^t \\ &= \boxed{\frac{c}{\chi a_0} \left[1 - e^{-\chi t} \right] \ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1 + z = \frac{a(t_r)}{a(t_e)} \ .$$

Here $t_e = 0$, so

$$\begin{aligned} 1 + z = \frac{a_0 e^{\chi t_r}}{a_0} & \\ \Longrightarrow \quad & e^{\chi t_r} = 1 + z \\ \Longrightarrow \quad & \boxed{t_r = \frac{1}{\chi} \ln(1 + z) \ .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1 + z \ ,$$

and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi a_0} \left[1 - e^{-\chi t_r} \right] \\ &= \frac{c}{\chi a_0} \left[1 - \frac{1}{1+z} \right] \\ &= \frac{cZ}{\chi a_0 (1 + z) \ .} \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = a(t_r) x(t_r) = \frac{cze^{\chi t_r}}{\chi(1+z)} = \boxed{\frac{cz}{\chi} \ .}$$

PROBLEM 5: "DID YOU DO THE READING?"

(a) The distinguishing quantity is $\Omega \equiv \rho/\rho_c$. The universe is open if $\Omega < 1$, flat if $\Omega = 1$, or closed if $\Omega > 1$.

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date* was made by the COBE satellite, which measured the temperature as 2.728 ± 0.004 Kelvin. The error here is quoted with a 95% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years.

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

(g) The absolute luminosity (*i.e.*, the total light output) of a Cepheid variable star appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to the Cepheid, by measuring the period and the apparent luminosity. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the $1/r^2$ law for the intensity of a point source to determine the distance r.

(h) 10^7 light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(j) Princeton University.

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* Astrophysical Journal, vol. **473**, p. 576 (1996): *The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data Sets*, D.J. Fixsen, E.S. Cheng, J.M. Gales, J.C. Mather, R.A. Shafer, and E.L. Wright.

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The key to this problem is to work in comoving coordinates.

[Some students have asked me why one cannot use "physical" coordinates, for which the coordinates really measure the physical distances. In principle one can use any coordinate system on likes, but the comoving coordinates are the simplest. In any other system it is difficult to write down the trajectory of either a particle or a light-beam. In comoving coordinates it is easy to write the trajectory of either a light beam, or a particle which is moving with the expansion of the universe (and hence standing still in the comoving coordinates). Note, by the way, that when one says that a particle is standing still in comoving coordinates, one has not really said very much about it's trajectory. One has said that it is moving with the matter which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor, $a(t)$.]

(a) The physical separation at t_o is given by the scale factor times the coordinate distance. The coordinate distance is found by integrating the coordinate velocity, so

$$\ell_p(t_o) = a(t_o)\int_{t_e}^{t_o}\frac{c\,dt'}{a(t')} = bt_o^{1/3}\int_{t_e}^{t_o}\frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct_o^{1/3}\left[t_o^{2/3}-t_e^{2/3}\right]$$

$$\boxed{=\frac{3}{2}ct_o\left[1-(t_e/t_o)^{2/3}\right]\ .}$$

(b) From the front of the exam,

$$1+z = \frac{a(t_o)}{a(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/3}$$

$$\Longrightarrow \quad \boxed{z = \left(\frac{t_o}{t_e}\right)^{1/3} - 1\ .}$$

(c) By combining the answers to (a) and (b), one has

$$\boxed{\ell_p(t_o) = \frac{3}{2}ct_o\left[1-\frac{1}{(1+z)^2}\right]\ .}$$

(d) The physical distance of the light pulse at time t is equal to $a(t)$ times the coordinate distance. The coordinate distance at time t is equal to the starting coordinate distance, $\ell_c(t_e)$, minus the coordinate distance that the light pulse travels between time t_e and time t. Thus,

$$\begin{aligned}\ell_p(t) &= a(t)\left[\ell_c(t_e) - \int_{t_e}^{t}\frac{c\,dt'}{a(t')}\right]\\ &= a(t)\left[\int_{t_e}^{t_o}\frac{c\,dt'}{a(t')} - \int_{t_e}^{t}\frac{c\,dt'}{a(t')}\right]\\ &= a(t)\int_{t}^{t_o}\frac{c\,dt'}{a(t')}\\ &= bt^{1/3}\int_{t}^{t_o}\frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct^{1/3}\left[t_o^{2/3}-t^{2/3}\right]\\ &= \boxed{\frac{3}{2}ct\left[\left(\frac{t_o}{t}\right)^{2/3}-1\right]\ .}\end{aligned}$$

PROBLEM 7: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

a) *(5 points)* The cosmological redshift is given by the usual form,

$$1+z = \frac{a(t_0)}{a(t_e)}\ .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma}\ .$$

So,

$$\boxed{z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1\ .}$$

b) *(5 points)* The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0\left(1-\frac{t_e}{t_0}\right)\ .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1+z)^{-1/\gamma} \quad .$$

Therefore,

$$\boxed{t_0 - t_e = t_0\left[1 - (1+z)^{-1/\gamma}\right] \quad .}$$

c) *(10 points)* The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = a(t_0)\int_{t_e}^{t_0} \frac{c}{a(t)}dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{ct_0^\gamma}{1-\gamma}\left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Factoring $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0) = \frac{ct_0}{1-\gamma}\left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

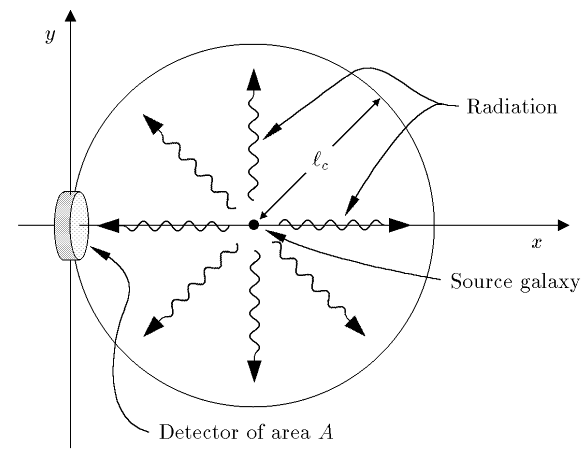
$$H_0 = \frac{\dot a(t_0)}{a(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1 - (1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) *(10 points)* A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/a(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/a(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/a(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \quad .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = a(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \quad .}$$

e) *(10 points)* Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \times \Delta t_{\text{source}} \ .$$

Note that I have used the formula from the front of the exam, but I have changed the size of v, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1+z)\Delta t_{\text{relay station}} \ .$$

Thus,

$$\begin{aligned} 1 + z_J \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} &= \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{relay station}}} \frac{\Delta t_{\text{relay station}}}{\Delta t_{\text{source}}} \\ &= (1+z)\big|_{\text{cosmological}} \ \times \ (1+z)\big|_{\text{special relativity}} \\ &= (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \ . \end{aligned}$$

Thus,

$$\boxed{z_J = (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} - 1 \ .}$$

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler shift formula to convert the redshift z of the galaxy to a velocity of recession, then subtracted from this the speed v of the jet, and then again used the special relativity Doppler shift formula to find the Doppler shift corresponding to this composite velocity.

However, as discussed at the end of Lecture Notes 3, the cosmological Doppler shift is given by

$$1 + z \equiv \frac{\Delta t_o}{\Delta t_e} = \frac{a(t_o)}{a(t_e)} \ , \tag{3.11}$$

and is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating z to v does not apply.

PROBLEM 8: DID YOU DO THE READING?

a) The lines were dark, caused by absorption of the radiation in the cooler, outer layers of the sun.

b) Individual stars in the Andromeda Nebula were resolved by Hubble in 1923.

[The other names and dates are not without significance. In 1609 Galileo built his first telescope; during 1609-10 he resolved the individual stars of the Milky Way, and also discovered that the surface of the moon is irregular, that Jupiter has moons of its own, that Saturn has handles (later recognized as rings), that the sun has spots, and that Venus has phases. In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens*, in which he suggested that at least some of the nebulae are galaxies like our own. In 1912 Henrietta Leavitt discovered the relationship between the period and luminosity of Cepheid variable stars. In the 1950s Walter Baade and Allan Sandage recalibrated the extra-galactic distance scale, reducing the accepted value of the Hubble constant by about a factor of 10.]

c)

(i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3° K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until high altitude balloons and rockets were introduced into cosmic background radiation research in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state, a state in which the C and N atoms rotate about each other. The population of this low-lying state is therefore determined by the intensity of the microwave radiation. This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by transitions between the low-lying state and higher energy excited states.]

(v) False. [No chemical reactions are seen.]

d) Aristarchus. [The heliocentric picture was never accepted by other Greek philosophers, however, and was not revived until the publication of *De Revolutionibus Orbium Coelestium* (*On the Revolutions of the Celestial Spheres*) by Copernicus in 1543.]

e) (ii) Any patch of the night sky would look as bright as the surface of the sun. [Explanation: The crux of the argument is that the brightness of an object, measured for example by the power per area (i.e., flux) hitting the retina of your eye, does not change as the object is moved further away. The power falls off with the square of the distance, but so does the area of the image on your retina — so the power per area is independent of distance. Under the assumptions stated, your line of sight will eventually hit a star no matter what direction you are looking. The energy flux on your retina will therefore be the same as in the image of the sun, so the entire sky will appear as bright as the surface of the sun.]

PROBLEM 9: A FLAT UNIVERSE WITH $a(t) \propto t^{3/5}$

a) In general, the Hubble constant is given by $H = \dot{a}/a$, where the overdot denotes a derivative with respect to cosmic time t. In this case

$$H = \frac{1}{bt^{3/5}}\,\frac{3}{5}\,bt^{-2/5} = \boxed{\frac{3}{5t}\,.}$$

b) In general, the (physical) horizon distance is given by

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt'\,.$$

In this case one has

$$\ell_{p,\text{horizon}}(t) = bt^{3/5}\int_0^t \frac{c}{bt'^{3/5}}dt' = ct^{3/5}\,\frac{5}{2}\left[t^{2/5} - 0^{2/5}\right] = \boxed{\frac{5}{2}\,ct\,.}$$

c) The coordinate speed of light is $c/a(t)$, so the coordinate distance that light travels between t_A and t_B is given by

$$\ell_c = \int_{t_A}^{t_B} \frac{c}{a(t')}\,dt' = \int_{t_A}^{t_B} \frac{c}{bt'^{3/5}}\,dt' = \boxed{\frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right)\,.}$$

d) The physical separation is just the scale factor times the coordinate separation, so

$$\ell_p(t_A) = a(t_A)\,\ell_c = \boxed{\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\,.}$$

$$\ell_p(t_B) = a(t_B)\,\ell_c = \boxed{\frac{5}{2}ct_B\left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]\,.}$$

e) Let t_{eq} be the time at which the light pulse is equidistant from the two galaxies. At this time it will have traveled a coordinate distance $\ell_c/2$, where ℓ_c is the answer to part (c). Since the coordinate speed is $c/a(t)$, the time t_{eq} can be found from:

$$\int_{t_A}^{t_{\text{eq}}} \frac{c}{a(t')}\,dt' = \frac{1}{2}\ell_c$$

$$\frac{5c}{2b}\left(t_{\text{eq}}^{2/5} - t_A^{2/5}\right) = \frac{5c}{4b}\left(t_B^{2/5} - t_A^{2/5}\right)$$

Solving for t_{eq},

$$\boxed{t_{\text{eq}} = \left[\frac{t_A^{2/5} + t_B^{2/5}}{2}\right]^{5/2}\,.}$$

f) According to Hubble's law, the speed is equal to Hubble's constant times the physical distance. By combining the answers to parts (a) and (d), one has

$$v = H(t_A)\,\ell_p(t_A)$$

$$= \frac{3}{5t_A}\,\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] = \boxed{\frac{3}{2}\,c\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\,.}$$

g) The redshift for radiation observed at time t can be written as

$$1 + z = \frac{a(t)}{a(t_e)} ,$$

where t_e is the time that the radiation was emitted. Solving for t_e,

$$t_e = \frac{t}{(1+z)^{5/3}} .$$

As found in part (d), the physical distance that the light travels between t_e and t, as measured at time t, is given by

$$\ell_p(t) = a(t) \int_{t_e}^{t} \frac{c}{a(t')}\, dt' = \frac{5}{2} ct \left[1 - \left(\frac{t_e}{t} \right)^{2/5} \right] .$$

Substituting the expression for t_e, one has

$$\boxed{\ell_p(t) = \frac{5}{2}\, ct \left[1 - \frac{1}{(1+z)^{2/3}} \right] .}$$

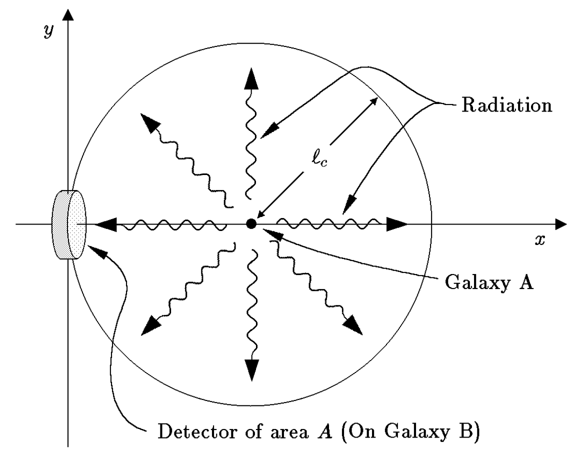
As $z \to \infty$, this expression approaches

$$\boxed{\lim_{z\to\infty} \ell_p(t) = \frac{5}{2}\, ct ,}$$

which is exactly equal to the horizon distance. It is a general rule that the horizon distance corresponds to infinite redshift z.

h) Again we will view the problem in comoving coordinates. Put galaxy B at the origin, and galaxy A at a coordinate distance ℓ_c along the x-axis. Draw a sphere of radius ℓ_c, centered galaxy A. Also draw a detector on galaxy B, with

physical area A (measured at the present time).



The energy from the quasar will radiate uniformly on the sphere. The detector has a physical area A, so in the comoving coordinate picture its area in square notches would be $A/a(t_B)^2$. The detector therefore occupies a fraction of the sphere given by

$$\frac{[A/a(t_B)^2]}{4\pi\ell_c^2} = \frac{A}{4\pi\ell_p(t_B)^2} ,$$

so this fraction of the emitted photons will strike the detector.

Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time t_A and observed at time t_B, then the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor $1 + z = a(t_B)/a(t_A)$. The same argument will apply to the rate of arrival of photons, so the rate of photon arrival at the sphere will be slower than the rate of emission by the factor $1+z$, reducing the energy flux by this factor. In addition, each photon is redshifted in frequency by $1 + z$. Since the energy of each photon is proportional to its frequency, the energy flux is reduced by an additional factor of $1+z$. Thus, the rate at which energy reaches the detector is

$$\text{Power hitting detector} = \frac{A}{4\pi\ell_p(t_B)^2} \frac{P}{(1+z)^2} .$$

The red shift z of the light pulse received at galaxy B is given by

$$1 + z = \frac{a(t_B)}{a(t_A)} = \left(\frac{t_B}{t_A} \right)^{3/5} .$$

Using once more the expression for $\ell_P(t_B)$ from part (d), one has

$$\boxed{J = \frac{\text{Power hitting detector}}{A} = \frac{P(t_A/t_B)^{6/5}}{25\pi\, c^2\, t_B^2 \left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]^2} \ .}$$

The problem is worded so that t_A, and not z, is the given variable that determines how far galaxy A is from galaxy B. In practice, however, it is usually more useful to express the answer in terms of the redshift z of the received radiation. One can do this by using the above expression for $1+z$ to eliminate t_A in favor of z, finding

$$J = \frac{P}{25\pi\, c^2\, t_B^2 (1+z)^{2/3}\, \left[(1+z)^{2/3} - 1\right]^2} \ .$$

i) Let t'_A be the time at which the light pulse arrives back at galaxy A. The pulse must therefore travel a coordinate distance ℓ_c (the answer to part (c)) between time t_B and t'_A, so

$$\int_{t_B}^{t'_A} \frac{c}{a(t')}\, dt' = \ell_c \ .$$

Using the answer from (c) and integrating the left-hand side,

$$\frac{5c}{2b}\left(t_A'^{2/5} - t_B^{2/5}\right) = \frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right) \ .$$

Solving for t'_A,

$$\boxed{t'_A = \left(2t_B^{2/5} - t_A^{2/5}\right)^{5/2} \ .}$$

PROBLEM 10: DID YOU DO THE READING?

a) Einstein believed that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attractive force of normal gravity. [The repulsive effect of a cosmological constant grows linearly with distance, so if the coefficient is small it is important only when the separations are very large. Such a term can be important cosmologically while still being too small to be detected by observations of the solar system or even the galaxy. Recent measurements of distant supernovas ($z \approx 1$), which you

may have read about in the newspapers, make it look like maybe there is a cosmological constant after all! Since the cosmological constant is the hot issue in cosmology this season, we will want to look at it more carefully. The best time will be after Lecture Notes 7.]

b) At the time of its discovery, de Sitter's model was thought to be static [although it was known that the model predicted a redshift which, at least for nearby galaxies, was proportional to the distance]. From a modern perspective the model is thought to be expanding.

[It seems strange that physicists in 1917 could not correctly determine if the theory described a universe that was static or expanding, but the mathematical formalism of general relativity can be rather confusing. The basic problem is that when space is not Euclidean there is no simple way to assign coordinates to it. The mathematics of general relativity is designed to be valid for any coordinate system, but the underlying physics can sometimes be obscured by a peculiar choice of coordinates. A change of coordinates can not only distort the apparent geometry of space, but it can also mix up space and time. The de Sitter model was first written down in coordinates that made it look static, so everyone believed it was. Later Arthur Eddington and Hermann Weyl (independently) calculated the trajectories of test particles, discovering that they flew apart.]

c) $n_1 = 3$, and $n_2 = 4$.

d) Above 3,000 K the universe was so hot that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the universe was cool enough so that the nuclei and electrons combined to form neutral atoms.

[This process is usually called "recombination," although the prefix "re-" is totally inaccurate, since in the big bang theory these constituents had never been previously combined. As far as I know the word was first used in this context by P.J.E. Peebles, so I once asked him why the prefix was used. He replied that this word is standard terminology in plasma physics, and was carried over into cosmology.]

[Regardless of its name, recombination was crucial for the clumping of matter into galaxies and stars, because the pressure of the photons in the early universe was enormous. When the matter was ionized, the free electrons interacted strongly with the photons, so the pressure of these photons prevented the matter from clumping. After recombination, however, the matter became very transparent to radiation, and the pressure of the radiation became ineffective.]

[Incidentally, at roughly the same time as recombination (with big uncertainties), the mass density of the universe changed from being dominated by radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker than the effect of recombination— because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.]

PROBLEM 11: ANOTHER FLAT UNIVERSE WITH $a(t) \propto t^{3/5}$

a) According to Eq. (3.7) of the Lecture Notes,

$$H(t) = \frac{1}{a(t)}\frac{da}{dt} .$$

For the special case of $a(t) = bt^{3/5}$, this gives

$$H(t) = \frac{1}{bt^{3/5}}\frac{3}{5}bt^{-2/5} = \boxed{\frac{3}{5t}} .$$

b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by

$$\frac{dx}{dt} = \frac{c}{a(t)} .$$

Since galaxies A and B have physical separation ℓ_0 at time t_1, their coordinate separation is given by

$$\ell_c = \frac{\ell_0}{bt_1^{3/5}} .$$

The radio signal must cover this coordinate distance in the time interval from t_1 to t_2, which implies that

$$\int_{t_1}^{t_2} \frac{c}{a(t)}\,dt = \frac{\ell_0}{bt_1^{3/5}} .$$

Using the expression for $a(t)$ and integrating,

$$\frac{5c}{2b}\left(t_2^{2/5} - t_1^{2/5}\right) = \frac{\ell_0}{bt_1^{3/5}} ,$$

which can be solved for t_2 to give

$$\boxed{t_2 = \left(1 + \frac{2\ell_0}{5ct_1}\right)^{5/2} t_1 .}$$

c) The method is the same as in part (b). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from t_2 to t_3. So,

$$\int_{t_2}^{t_3} \frac{c}{a(t)}\,dt = \frac{\ell_0}{bt_1^{3/5}} ,$$

which leads to

$$\frac{5c}{2b}\left(t_3^{2/5} - t_2^{2/5}\right) = \frac{\ell_0}{bt_1^{3/5}} .$$

Solving for t_3 gives

$$\boxed{t_3 = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 .}$$

The above answer is perfectly acceptable, but one could also replace t_2 by using the answer to part (b), which gives

$$\boxed{t_3 = \left(1 + \frac{4\ell_0}{5ct_1}\right)^{5/2} t_1 .}$$

[Alternatively, one could have begun the problem by considering the full round trip of the radio signal, which travels a coordinate distance $2\ell_c$ during the time interval from t_1 to t_3. The problem then becomes identical to part (b), except that the coordinate distance ℓ_c is replaced by $2\ell_c$, and t_2 is replaced by t_3. One is led immediately to the answer in the form of the previous equation.]

d) Cosmic time is defined by the reading of suitably synchronized clocks which are each at rest with respect to the matter of the universe at the same location. (For this problem we will not need to think about the method of synchronization.) Thus, the cosmic time interval between the receipt of the message and the response is the same as what is measured on the galaxy B clocks, which is Δt. The response is therefore sent at cosmic time $t_2+\Delta t$. The coordinate distance between the galaxies is still $\ell_0/a(t_1)$, so

$$\int_{t_2+\Delta t}^{t_4} \frac{c}{a(t)}\,dt = \frac{\ell_0}{b t_1^{3/5}} .$$

Integration gives

$$\frac{5c}{2b}\left[t_4^{2/5} - (t_2+\Delta t)^{2/5}\right] = \frac{\ell_0}{b t_1^{3/5}} ,$$

which can be solved for t_4 to give

$$\boxed{t_4 = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 .}$$

e) From the formula at the front of the exam,

$$1+z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} = \frac{a(t_4)}{a(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} .$$

So,

$$\boxed{z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} = \frac{a(t_4)}{a(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} - 1 .}$$

f) If Δt is small compared to the time that it takes $a(t)$ to change significantly, then the interval between a signal sent at t_3 and a signal sent at $t_3+\Delta t$ will be received with a redshift identical to that observed between two successive crests of a wave. Thus, the separation between the receipt of the acknowledgement and the receipt of the response will be a factor $(1+z)$ times longer than the time interval between the sending of the two signals, and therefore

$$\begin{aligned} t_4 - t_3 &= (1+z)\Delta t + \mathcal{O}(\Delta t^2) \\ &= \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) . \end{aligned}$$

Since the answer contains an explicit factor of Δt, the other factors can be evaluated to zeroth order in Δt:

$$\boxed{t_4 - t_3 = \left(\frac{t_4}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) ,}$$

where to first order in Δt the t_4 in the numerator could equally well have been replaced by t_3.

For those who prefer the brute force approach, the answer to part (d) can be Taylor expanded in powers of Δt. To first order one has

$$t_4 = t_3 + \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} \Delta t + \mathcal{O}(\Delta t^2) .$$

Evaluating the necessary derivative gives

$$\frac{\partial t_4}{\partial \Delta t} = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2+\Delta t}{t_1}\right)^{-3/5} ,$$

which when specialized to $\Delta t = 0$ becomes

$$\left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2}{t_1}\right)^{-3/5} .$$

Using the first boxed answer to part (c), this can be simplified to

$$\begin{aligned} \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} &= \left(\frac{t_3}{t_1}\right)^{3/5} \left(\frac{t_2}{t_1}\right)^{-3/5} \\ &= \left(\frac{t_3}{t_2}\right)^{3/5} . \end{aligned}$$

Putting this back into the Taylor series gives

$$t_4 - t_3 = \left(\frac{t_3}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) ,$$

in agreement with the previous answer.

PROBLEM 12: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that

$$\ddot{a} = -\frac{4\pi}{3} G\rho a$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} ,$$

where a dot denotes a derivative with respect to time t. The critical mass density ρ_c is defined to be the mass density that corresponds to a flat ($k = 0$) universe, so from the equation above it follows that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c .$$

Substituting into the definition of q, we find

$$\begin{aligned} q &= -\ddot{a}(t)\frac{a(t)}{\dot{a}^2(t)} = -\frac{\ddot{a}}{a}\left(\frac{a}{\dot{a}}\right)^2 \\ &= \left(\frac{4\pi}{3} G\rho\right)\left(\frac{3}{8\pi G\rho_c}\right) = \frac{1}{2}\frac{\rho}{\rho_c} = \boxed{\frac{1}{2}\,\Omega\ .} \end{aligned}$$

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that $k = 0$, so

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho .$$

Since

$$\rho(t) \propto \frac{1}{a^4(t)} ,$$

it follows that

$$\frac{da}{dt} = \frac{\text{const}}{a} .$$

Rewriting this as

$$a\,da = \text{const}\,dt ,$$

the indefinite integral becomes

$$\frac{1}{2}a^2 = (\text{const})t + c' ,$$

where c' is a constant of integration. Different choices for c' correspond to different choices for the definition of $t = 0$. We will follow the standard convention of choosing $c' = 0$, which sets $t = 0$ to be the time when $a = 0$. Thus the above equation implies that $a^2 \propto t$, and therefore

$$\boxed{a(t) \propto t^{1/2}}$$

for a photon-dominated flat universe.

PROBLEM 14: DID YOU DO THE READING? *(25 points)*

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(Ryden, Chapter 2, Pages 6-8)

Ans: Olber's paradox is that the night sky appears to be dark, instead of being uniformly bright. The primary resolution is that the universe has a finite age, and so the light from stars beyond the horizon distance has not reached us yet. (However, even in the steady-state model of the universe, the paradox is resolved because the light from distant stars will be red-shifted beyond the visible spectrum).

(b) What is the value of the Newtonian gravitational constant G in Planck units? The Planck length is of the order of 10^{-35} m, 10^{-15} m, 10^{15} m, or 10^{35} m?

(Ryden, Chapter 1, Page 3)

Ans: $G = 1$ in Planck units, by definition.

The Planck length is of the order of 10^{-35} m. (Note that this answer could be obtained by a process of elimination as long as you remember that the Planck length is much smaller than 10^{-15} m, which is the typical size of a nucleus).

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it?

(Weinberg, Chapter 2, Pages 21-23; Ryden, Chapter 2, Page 11)

Ans: The Cosmological Principle states that there is nothing special about our location in the universe, i.e. the universe is homogeneous and isotropic.

Yes, the Hubble expansion is consistent with it (since there is no center of expansion).

(d) In the "Standard Model" of the universe, when the universe cooled to about 3×10^a K, it became transparent to photons, and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^b K. What are the integers a and b?

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

$a = 3$, $b = 0$.

(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

(Weinberg, Chapter 1, Page 5)

Ans: Electrons, positrons, neutrinos, and photons.

PROBLEM 15: SPECIAL RELATIVITY DOPPLER SHIFT *(20 points)*

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler shift formula, which was given on the front of the exam:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \,, \tag{18.1}$$

where $\beta = v/c$. Remembering that the wavelength is stretched by a factor $1 + z$, we find immediately that the wavelength of the radio wave received at Alpha-7 is given by

$$\lambda_{\text{Alpha}-7} = \sqrt{\frac{1+v_s/c}{1-v_s/c}}\,\lambda_{\text{emitted}} \,. \tag{18.2}$$

The photons that are received by the observer are in fact never received by Alpha-7, but the wavelength found by the observer will be the same as if Alpha-7 acted as a relay station, receiving the photons and retransmitting them at the received wavelength. So, applying Eq. (18.1) again, the wavelength seen by the observer can be written as

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\,\lambda_{\text{Alpha}-7} \,. \tag{18.3}$$

Combining Eqs. (18.2) and (18.3),

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}}\,\lambda_{\text{emitted}} \,, \tag{18.4}$$

so finally

$$\boxed{z = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \,.} \tag{18.5}$$

(b) Although we used the presence of Alpha-7 in determining the redshift z of Eq. (18.5), the redshift is not actually affected by the space station. So the special-relativity Doppler shift formula, Eq. (18.1), must directly describe the redshift resulting from the relative motion of the source and the observer. Thus

$$\sqrt{\frac{1+v_{\text{tot}}/c}{1-v_{\text{tot}}/c}} - 1 = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \,. \tag{18.6}$$

The equation above determines v_{tot} in terms of v_o and v_s, so the rest is just algebra. To simplify the notation, let $\beta_{\text{tot}} \equiv v_{\text{tot}}/c$, $\beta_o \equiv v_o/c$, and $\beta_s \equiv v_s/c$. Then

$$1 + \beta_{\text{tot}} = \frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s}(1-\beta_{\text{tot}})$$

$$\beta_{\text{tot}}\left[1 + \frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s}\right] = \frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s} - 1$$

$$\beta_{\text{tot}}\left[\frac{(1-\beta_o-\beta_s+\beta_o\beta_s)+(1+\beta_o+\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}\right] = \frac{(1+\beta_o+\beta_s+\beta_o\beta_s)-(1-\beta_o-\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}$$

$$\beta_{\text{tot}}[2(1+\beta_o\beta_s)] = 2(\beta_o+\beta_s)$$

$$\beta_{\text{tot}} = \frac{\beta_o+\beta_s}{1+\beta_o\beta_s}$$

$$\boxed{v_{\text{tot}} = \frac{v_o+v_s}{1+\dfrac{v_o v_s}{c^2}} \,.} \tag{18.7}$$

The final formula is the relativistic expression for the addition of velocities. Note that it guarantees that $|v_{\text{tot}}| \le c$ as long as $|v_o| \le c$ and $|v_s| \le c$.

PROBLEM 16: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* What was the first external galaxy that was shown to be at a distance significantly greater than the most distant known objects in our galaxy? How was the distance estimated?

Ans: (Weinberg, page 20) The first galaxy shown to be at a distance beyond the size of our galaxy was Andromeda, also known by its Messier number, M31. It is the nearest spiral galaxy to our galaxy. The distance was determined (by Hubble) using Cepheid variable stars, for which the absolute luminosity is proportional to the period. A measurement of a particular Cepheid's period determines the star's absolute luminosity, which, compared to the measured luminosity, determines the distance to the star. (Hubble's initial measurement of the distance to Andromeda used a badly-calibrated version of this period-luminosity relationship and consequently underestimated the distance by more than a factor of two; nonetheless, the initial measurement still showed that the Andromeda Nebula was an order of magnitude more distant than the most distant known objects in our own galaxy.)

(b) *(5 points)* What is recombination? Did galaxies begin to form before or after recombination? Why?

Ans: (Weinberg, pages 64 and 73) Recombination refers to the formation of neutral atoms out of charged nuclei and electrons. Galaxies began to form after recombination. Prior to recombination, the strong electromagnetic interactions between photons and matter produced a high pressure which effectively counteracted the gravitational attraction between particles. Once the universe became transparent to radiation, the matter no longer interacted significantly with the photons and consequently began to undergo gravitational collapse into large clumps.

(c) *(4 points)* In Chapter IV of his book, Weinberg develops a "recipe for a hot universe," in which the matter of the universe is described as a gas in thermal equilbrium at a very high temperature, in the vicinity of 10^9 K (several thousand million degrees Kelvin). Such a thermal equilibrium gas is completely described by specifying its temperature and the density of the conserved quantities. Which of the following is on this list of conserved quantities? Circle as many as apply.

(i) baryon number (ii) energy per particle (iii) proton number

(iv) electric charge (v) pressure

Ans: (Weinberg, page 91) The correct answers are (i) and (iv). A third conserved quantity, lepton number, was not included in the multiple-choice options.

(d) *(4 points)* The wavelength corresponding to the mean energy of a CMB (cosmic microwave background) photon today is approximately equal to which of the following quantities? (You may wish to look up the values of various physical constants at the end of the quiz.)

(i) 2 fm (2×10^{-15} m)

(ii) 2 microns (2×10^{-6} m)

(iii) 2 mm (2×10^{-3} m)

(iv) 2 m.

Ans: (Ryden, page 23) The correct answer is (iii).

If you did not remember this number, you could estimate the answer by remembering that the characteristic temperature of the cosmic microwave background is approximately 3 Kelvin. The typical photon energy is then on the order of kT, from which we can find the frequency as $E = h\nu$. The wavelength of the photon is then $\lambda = \nu/c$. This approximation gives $\lambda = 5.3$ mm, which is not equal to the correct answer, but it is much closer to the correct answer than to any of the other choices.

(e) *(4 points)* What is the equivalence principle?

Ans: (Ryden, page 27) In its simplest form, the equivalence principle says that the gravitational mass of an object is identical to its inertial mass. This equality implies the equivalent statement that it is impossible to distinguish (without additional information) between an observer in a reference frame accelerating with acceleration $\vec{a}$ and an observer in an inertial reference frame subject to a gravitational force $-m_{obs}\vec{a}$.

(Actually, what the equivalence principle really says is that the ratio of the gravitational to inertial masses m_g/m_i is universal, that is, independent of the material properties of the object in question. The ratio does not necessarily need to be 1. However, once we know that the two types of masses are proportional, we can simply define the gravitational coupling G to make them equal. To see this, consider a theory of gravity where $m_g/m_i = q$. Then the gravitational force law is

$$m_i a = -\frac{GMm_g}{r^2} \ ,$$

or

$$a = -\frac{GqM}{r^2} \ .$$

At this point, if we define $G' = Gq$, we have a gravitational theory with gravitational coupling G' and inertial mass equal to gravitational mass.)

(f) *(4 points)* Why is it difficult for Earth-based experiments to look at the small wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?

Ans: (Weinberg, page 67) The Earth's atmosphere is increasingly opaque for wavelength shorter than .3 cm. Therefore, radiation at these wavelengths will be absorbed and rescattered by the Earth's atmosphere; observations of the cosmic microwave background at small wavelengths must be performed above the Earth's atmosphere.

PROBLEM 17: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE

(a) The physical horizon distance is given in general by

$$\ell_{p,\mathrm{horizon}} = a(t)\int_0^{t_f} \frac{c}{a(t)}\,dt\ ,$$

so in this case

$$\ell_{p,\mathrm{horizon}} = bt^{1/2}\int_0^{t_f} \frac{c}{bt^{1/2}}\,dt = \boxed{2ct_f\ .}$$

(b) If the source is at the horizon distance, it means that a photon leaving the source at $t = 0$ would just be reaching the origin at t_f. So, $\boxed{t_e = 0\ .}$

(c) The coordinate distance between the source and the origin is the coordinate horizon distance, given by

$$\ell_{c,\mathrm{horizon}} = \int_0^{t_f} \frac{c}{bt^{1/2}}\,dt = \boxed{\frac{2ct_f^{1/2}}{b}\ .}$$

(d) The photon starts at coordinate distance $2c\sqrt{t_f}/b$, and by time t it will have traveled a coordinate distance

$$\int_0^t \frac{c}{bt'^{1/2}}\,dt' = \frac{2c\sqrt{t}}{b}$$

toward the origin. Thus the photon will be at coordinate distance

$$\ell_c = \frac{2c}{b}\left(\sqrt{t_f} - \sqrt{t}\right)$$

from the origin, and hence a physical distance

$$\ell_p(t) = a(t)\ell_c = \boxed{2c\left(\sqrt{t\,t_f} - t\right)\ .}$$

(e) To find the maximum of $\ell_p(t)$, we differentiate it and set the derivative to zero:

$$\frac{d\ell_p}{dt} = \left(\sqrt{\frac{t_f}{t}} - 2\right)c\ ,$$

so the maximum occurs when

$$\sqrt{\frac{t_f}{t_{\max}}} = 2\ ,$$

or

$$\boxed{t_{\max} = \frac{1}{4}t_f\ .}$$

PROBLEM 18: TRANSVERSE DOPPLER SHIFTS

(a) Describing the events in the coordinate system shown, the Xanthu is at rest, so its clocks run at the same speed as the coordinate system time variable, t. The emission of the wavecrests of the radio signal are therefore separated by a time interval equal to the time interval as measured by the source, the Xanthu:

$$\Delta t = \Delta t_s\ .$$

Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wavecrests will travel the same distance, as long as $c\Delta t \ll a$. Since the wavecrests travel the same distance, the time separation of their arrival at the Emmerac is Δt, the same as the time separation of their emission. The clocks on the Emmerac, however, and running slowly by a factor of

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\ .$$

The time interval between wave crests as measured by the receiver, on the Emmerac, is therefore smaller by a factor of γ,

$$\Delta t_r = \frac{\Delta t_s}{\gamma}\ .$$

Thus, there is a $\boxed{\text{blueshift.}}$ The redshift parameter z is defined by

$$\frac{\Delta t_r}{\Delta t_s} = 1 + z\ ,$$

so

$$\frac{1}{\gamma} = 1 + z\ ,$$

or

$$\boxed{z = \frac{1-\gamma}{\gamma}\ .}$$

Recall that $\gamma > 1$, so z is negative.

(b) Describing this situation in the coordinate system shown, this time the source on the Xanthu is moving, so the clocks at the source are running slowly. The time between wavecrests, measured in coordinate time t, is therefore larger by a factor of γ than Δt_s, the time as measured by the clock on the source:

$$\Delta t = \gamma\, \Delta t_s\ .$$

Since the radio signal is emitted when the Xanthu is at its minimum separation from the Emmerac, the rate of change of the separation is zero, so each wavecrest travels the same distance (again assuming that $c\Delta t \ll a$). Since the Emmerac is at rest, its clocks run at the same speed as the coordinate time t, and hence the time interval between crests, as measured by the receiver, is

$$\Delta t_r = \Delta t = \gamma\, \Delta t_s\ .$$

Thus the time interval as measured by the receiver is longer than that measured by the source, and hence it is a $\boxed{\text{redshift.}}$ The redshift parameter z is given by

$$1 + z = \frac{\Delta t_r}{\Delta t_s} = \gamma\ ,$$

so

$$\boxed{z = \gamma - 1\ .}$$

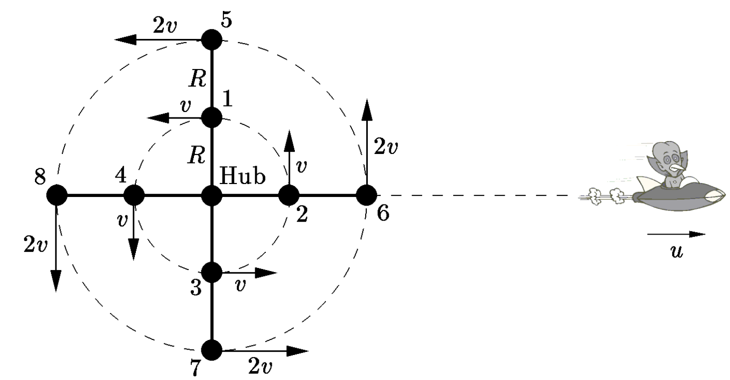
(c) The events described in (a) can be made to look a lot like the events described in (b) by transforming to a frame of reference that is moving to the right at speed v_0 — i.e., by transforming to the rest frame of the Emmerac. In this frame the Emmerac is of course at rest, and the Xanthu is traveling on the trajectory

$$(x{=}-v_0 t, y{=}a, z{=}0)\ ,$$

as in part (b). However, just as the transformation causes the x-component of the velocity of the Xanthu to change from zero to a negative value, so the x-component of the velocity of the radio signal will be transformed from zero to a negative value. Thus in this frame the radio signal will not be traveling along the y-axis, so the events will not match those described in (b). The situations described in (a) and (b) are therefore physically distinct (which they must be if the redshifts are different, as we calculated above).

PROBLEM 19: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND

(15 points)



(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed v, and the outer cars are moving at speed $2v$. In the laboratory frame, the time interval between the wave crests emitted by the source $\Delta t_S^{\rm Lab}$ will be exactly equal to the time interval $\Delta t_O^{\rm Lab}$ between two crests reaching the observer:

$$\Delta t_O^{\rm Lab} = \Delta t_S^{\rm Lab}\ .$$

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slowly by the factor

$$\gamma_1 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

for the inner cars, and by the factor

$$\gamma_2 = \frac{1}{\sqrt{1 - 4v^2/c^2}}$$

for the outer cars. Thus, if we let Δt_S denote the time between crests as measured by a clock on the source, and Δt_O as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma_2 \Delta t_S = \Delta t_S^{\rm Lab} \quad \text{and} \quad \gamma_1 \Delta t_O = \Delta t_O^{\rm Lab} \; .$$

To make sure that the γ-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that

$$\Delta t_O = \frac{\gamma_2}{\gamma_1} \Delta t_S \; .$$

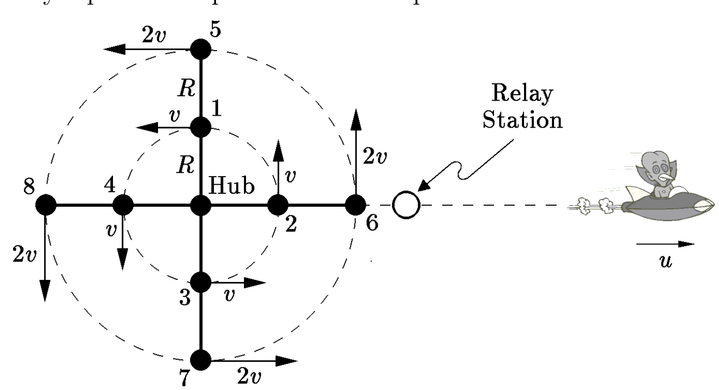
The redshift z is defined by

$$\Delta t_O \equiv (1+z)\, \Delta t_S \; ,$$

so

$$\boxed{z = \frac{\gamma_2}{\gamma_1} - 1 = \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 - \frac{4v^2}{c^2}}} - 1 \; .}$$

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.



The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and

a change in path lengths between successive pulses, but we do not need to do it. It is the standard situation of a source and observer moving directly away from each other, as discussed at the end of Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed u, it becomes

$$(1+z)|_{\rm relay} = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} \; .$$

If we again use the symbol Δt_S for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is

$$\Delta t_R = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\, \Delta t_S \; .$$

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in Δt, so it can be ignored. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station:

$$\Delta t_O^{\rm Lab} = \Delta t_R \; .$$

As in part (a), the time dilation implies that

$$\gamma_2 \Delta t_O = \Delta t_O^{\rm Lab} \; .$$

Combining the formulas above,

$$\Delta_O = \frac{1}{\gamma_2} \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\, \Delta t_S \; .$$

Again $\Delta t_O \equiv (1+z)\, \Delta t_S$, so

$$\boxed{z = \frac{1}{\gamma_2} \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} - 1 = \sqrt{\frac{\left(1 - \frac{4v^2}{c^2}\right)\left(1+\frac{u}{c}\right)}{1-\frac{u}{c}}} - 1 \; .}$$

PROBLEM 20: SIGNAL PROPAGATION IN A FLAT MATTER-DOMINATED UNIVERSE *(55 points)*

(a)-(i) If we let $\ell_c(t)$ denote the coordinate distance of the light signal from A, then we can make use of Eq. (3.8) from the lecture notes for the coordinate velocity of light:

$$\frac{d\ell_c}{dt} = \frac{c}{a(t)} . \tag{20.1}$$

Integrating the velocity,

$$\begin{aligned}\ell_c(t) &= \int_{t_1}^{t} \frac{c\,dt'}{a(t')} = \frac{c}{b}\int_{t_1}^{t} \frac{dt'}{t'^{2/3}} \\ &= \frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right] .\end{aligned} \tag{20.2}$$

The physical distance is then

$$\begin{aligned}\ell_{p,sA}(t) &= a(t)\ell_c(t) = bt^{2/3}\frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right] \\ &= 3c\left(t - t^{2/3}t_1^{1/3}\right) \\ &= 3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .\end{aligned} \tag{20.3}$$

We now need to differentiate, which is done most easily with the middle line of the above equation:

$$\boxed{\frac{d\ell_{p,sA}}{dt} = c\left[3 - 2\left(\frac{t_1}{t}\right)^{1/3}\right] .} \tag{20.4}$$

(ii) At $t = t_1$, the time of emission, the above formula gives

$$\boxed{\frac{d\ell_{p,sA}}{dt} = c .} \tag{20.5}$$

This is what should be expected, since the speed of separation of the light signal at the time of emission is really just a local measurement of the speed of light, which should always give the standard value c.

(iii) At arbitrarily late times, the second term in brackets in Eq. (20.4) becomes negligible, so

$$\boxed{\frac{d\ell_{p,sA}}{dt} \to 3c .} \tag{20.6}$$

Although this answer is larger than c, it does not violate relativity. Once the signal is far from its origin it is carried by the expansion of the universe, and relativity places no speed limit on the expansion of the universe.

(b) This part of the problem involves $H(t_1)$, so we can start by evaluating it:

$$H(t) = \frac{\dot a(t)}{a(t)} = \frac{\frac{d}{dt}(bt^{2/3})}{bt^{2/3}} = \frac{2}{3t} . \tag{20.7}$$

Thus, the physical distance from A to B at time t_1 is

$$\ell_{p,BA} = \frac{3}{2}ct_1 . \tag{20.8}$$

The coordinate distance is the physical distance divided by the scale factor, so

$$\ell_{c,BA} = \frac{cH^{-1}(t_1)}{a(t_1)} = \frac{\frac{3}{2}ct_1}{bt_1^{2/3}} = \frac{3c}{2b}t_1^{1/3} . \tag{20.9}$$

Since light travels at a coordinate speed $c/a(t)$, the light signal will reach galaxy B at time t_2 if

$$\begin{aligned}\ell_{c,BA} &= \int_{t_1}^{t_2} \frac{c}{bt'^{2/3}}\,dt' \\ &= \frac{3c}{b}\left[t_2^{1/3} - t_1^{1/3}\right] .\end{aligned} \tag{20.10}$$

Setting the expressions (20.9) and (20.10) for $\ell_{c,BA}$ equal to each other, one finds

$$\frac{1}{2}t_1^{1/3} = t_2^{1/3} - t_1^{1/3} \quad\Longrightarrow\quad t_2^{1/3} = \frac{3}{2}t_1^{1/3} \quad\Longrightarrow\quad \boxed{t_2 = \frac{27}{8}t_1 .} \tag{20.11}$$

(c)-(i) Physical distances are additive, so if one adds the distance from A and the light signal to the distance from the light signal to B, one gets the distance from A to B:

$$\ell_{p,sA} + \ell_{p,sB} = \ell_{p,BA} . \tag{20.12}$$

But $\ell_{p,BA}(t)$ is just the scale factor times the coordinate separation, $a(t)\ell_{c,BA}$. Using the previous relations (20.3) and (20.9) for $\ell_{p,sA}(t)$ and $\ell_{c,BA}$, we find

$$3ct\left[1-\left(\frac{t_1}{t}\right)^{1/3}\right]+\ell_{p,sB}(t)=\frac{3}{2}ct_1^{1/3}t^{2/3}\ , \tag{20.13}$$

so

$$\ell_{p,sB}(t)=\frac{9}{2}ct_1^{1/3}t^{2/3}-3ct=3ct\left[\frac{3}{2}\left(\frac{t_1}{t}\right)^{1/3}-1\right]\ . \tag{20.14}$$

As a check, one can verify that this expression vanishes for $t=t_2=(27/8)\,t_1$, and that it equals $(3/2)ct_1$ at $t=t_1$. But we are asked to find the speed of approach, the negative of the derivative of Eq. (20.14):

$$\begin{aligned}\text{Speed of approach} &= -\frac{\mathrm{d}\ell_{p,sB}}{\mathrm{d}t}\\ &= -3ct_1^{1/3}t^{-1/3}+3c\\ &= \boxed{3c\left[1-\left(\frac{t_1}{t}\right)^{1/3}\right]}\ .\end{aligned} \tag{20.15}$$

(ii) At the time of emission, $t=t_1$, Eq. (20.15) gives

$$\boxed{\text{Speed of approach}=0\ .} \tag{20.16}$$

This makes sense, since at $t=t_1$ galaxy B is one Hubble length from galaxy A, which means that its recession velocity is exactly c. The recession velocity of the light signal leaving A is also c, so the rate of change of the distance from the light signal to B is initially zero.

(iii) At the time of reception, $t=t_2=(27/8)\,t_1$, Eq. (20.15) gives

$$\boxed{\text{Speed of approach}=c\ ,} \tag{20.17}$$

which is exactly what is expected. As in part (a)-(ii), this is a local measurement of the speed of light.

(d) To find the redshift, we first find the time t_{BA} at which a light pulse must be emitted from galaxy B so that it arrives at galaxy A at time t_1. Using the coordinate distance given by Eq. (20.9), the time of emission must satisfy

$$\frac{3c}{2b}\,t_1^{1/3}=\int_{t_{BA}}^{t_1}\frac{c}{bt'^{2/3}}\,\mathrm{d}t'=\frac{3c}{b}\left(t_1^{1/3}-t_{BA}^{1/3}\right)\ , \tag{20.18}$$

which can be solved to give

$$t_{BA}=\frac{1}{8}\,t_1\ . \tag{20.19}$$

The redshift is given by

$$1+z_{BA}=\frac{a(t_1)}{a(t_{BA})}=\left(\frac{t_1}{t_{BA}}\right)^{2/3}=4\ . \tag{20.20}$$

Thus,

$$\boxed{z_{BA}=3\ .} \tag{20.21}$$

(e) Applying Euclidean geometry to the triangle C-A-B shows that the physical distance from C to B, at time t_1, is $\sqrt{2}cH^{-1}$. The coordinate distance is also larger than the A-B separation by a factor of $\sqrt{2}$. Thus,

$$\ell_{c,BC}=\frac{3\sqrt{2}c}{2b}\,t_1^{1/3}\ . \tag{20.22}$$

If we let t_{BC} be the time at which a light pulse must be emitted from galaxy B so that it arrives at galaxy C at time t_1, we find

$$\frac{3\sqrt{2}c}{2b}\,t_1^{1/3}=\int_{t_{BC}}^{t_1}\frac{c}{bt'^{2/3}}\,\mathrm{d}t'=\frac{3c}{b}\left(t_1^{1/3}-t_{BC}^{1/3}\right)\ , \tag{20.23}$$

which can be solved to find

$$t_{BC}=\left(1-\frac{\sqrt{2}}{2}\right)^3 t_1\ . \tag{20.24}$$

Then

$$1+z_{BC}=\frac{a(t_1)}{a(t_{BC})}=\left(\frac{t_1}{t_{BC}}\right)^{2/3}=\frac{1}{\left(1-\frac{\sqrt{2}}{2}\right)^2}\ , \tag{20.25}$$

and

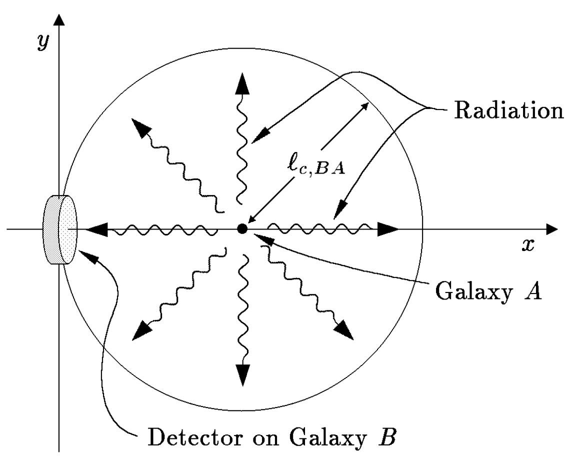
$$\boxed{z_{BC} = \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} - 1\ .} \qquad (20.26)$$

Full credit will be given for the answer in the form above, but it can be simplified by rationalizing the fraction:

$$\begin{aligned} z_{BC} &= \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} \frac{\left(1 + \frac{\sqrt{2}}{2}\right)^2}{\left(1 + \frac{\sqrt{2}}{2}\right)^2} - 1 \\ &= \frac{1 + \sqrt{2} + \frac{1}{2}}{\frac{1}{4}} - 1 \\ &= \boxed{5 + 4\sqrt{2}\ .} \end{aligned} \qquad (20.27)$$

Numerically, $z_{BC} = 10.657$.

(f) Following the solution to Problem 6 of Problem Set 2, we draw a diagram in comoving coordinates, putting the source at the center of a sphere:



The energy from galaxy A will radiate uniformly over the sphere. If the detector has physical area A_D, then in the comoving coordinate picture it has coordinate area $A_D/a^2(t_2)$, since the detection occurs at time t_2 The full coordinate area

of the sphere is $4\pi\ell_{c,BA}^2$, so the fraction of photons that hit the detector is

$$\text{fraction} = \frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\ . \qquad (20.28)$$

As in Problem 6, the power hitting the detector is reduced by two factors of $(1+z)$: one factor because the energy of each photon is proportional to the frequency, and hence is reduced by the redshift, and one more factor because the rate of arrival of photons is also reduced by the redshift factor $(1+z)$. Thus,

$$\begin{aligned} \text{Power hitting detector} &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\frac{1}{(1+z)^2} \\ &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\left[\frac{a(t_1)}{a(t_2)}\right]^2 \\ &= P\frac{A}{4\pi\ell_{c,BA}^2}\frac{a^2(t_1)}{a^4(t_2)}\ . \end{aligned} \qquad (20.29)$$

The energy flux is given by

$$J = \frac{\text{Power hitting detector}}{A}\ , \qquad (20.30)$$

so

$$J = \frac{P}{4\pi\ell_{c,BA}^2}\frac{a^2(t_1)}{a^4(t_2)}\ . \qquad (20.31)$$

From here it is just algebra, using Eqs. (20.9) and (20.11), and $a(t) = bt^{2/3}$:

$$\begin{aligned} J &= \frac{P}{4\pi\left[\frac{3c}{2b}t_1^{1/3}\right]^2}\frac{b^2 t_1^{4/3}}{b^4 t_2^{8/3}} \\ &= \frac{P}{4\pi\left[\frac{3c}{2b}t_1^{1/3}\right]^2}\frac{b^2 t_1^{4/3}}{\left(\frac{27}{8}\right)^{8/3} b^4 t_1^{8/3}} \\ &= \frac{P}{4\pi\left[\frac{3c}{2}t_1^{1/3}\right]^2}\frac{t_1^{4/3}}{\left(\frac{3}{2}\right)^8 t_1^{8/3}} \\ &= \boxed{\frac{2^8}{3^{10}\pi}\frac{P}{c^2 t_1^2}} \\ &= \boxed{\frac{256}{59,049\pi}\frac{P}{c^2 t_1^2}\ .} \end{aligned} \qquad (20.32)$$

It is debatable which of the last two expressions is the simplest, so I have boxed both of them. One could also write

$$J = 1.380 \times 10^{-3} \frac{P}{c^2 t_1^2} \ . \tag{20.33}$$

PROBLEM 21: DID YOU DO THE READING? *(25 points)*†

(a) *(10 points)* To determine the distance of the galaxies he was observing Hubble used so called *standard candles*. Standard candles are astronomical objects whose intrinsic luminosity is known and whose distance is inferred by measuring their apparent luminosity. First, he used as standard candles variable stars, whose intrinsic luminosity can be related to the period of variation. Quoting Weinberg's *The First Three Minutes*, chapter 2, pages 19-20:

In 1923 Edwin Hubble was for the first time able to resolve the Andromeda Nebula into separate stars. He found that its spiral arms included a few bright variable stars, with the same sort of periodic variation of luminosity as was already familiar for a class of stars in our galaxy known as Cepheid variables. The reason this was so important was that in the preceding decade the work of Henrietta Swan Leavitt and Harlow Shapley of the Harvard College Observatory had provided a tight relation between the observed periods of variation of the Cepheids and their absolute luminosities. (Absolute luminosity is the total radiant power emitted by an astronomical object in all directions. Apparent luminosity is the radiant power received by us in each square centimeter of our telescope mirror. It is the apparent rather than the absolute luminosity that determines the subjective degree of brightness of astronomical objects. Of course, the apparent luminosity depends not only on the absolute luminosity, but also on the distance; thus, knowing both the absolute and the apparent luminosities of an astronomical body, we can infer its distance.) Hubble, observing the apparent luminosity of the Cepheids in the Andromeda Nebula, and estimating their absolute luminosity from their periods, could immediately calculate their distance, and hence the distance of the Andromeda Nebula, using the simple rule that apparent luminosity is proportional to the absolute luminosity and inversely proportional to the square of the distance.

He also used particularly bright stars as standard candles, as we deduce from page 25:

Returning now to 1929: Hubble estimated the distance to 18 galaxies from the apparent luminosity of their brighest stars, and compared these distances with the galaxies' respective velocities, determined spectroscopically from their Doppler shifts.

Note: since from reading just the first part of Weinberg's discussion one could be induced to think that Hubble used just Cepheids as standard candles, students who mentioned only Cepheids got 9 points out of 10. In fact, however, Hubble was able to identify Cepheid variables in only a few galaxies. The Cepheids were crucial, because they served as a calibration for the larger distances, but they were not in themselves sufficient.

(b) *(5 points)* Quoting Weinberg's *The First Three Minutes*, chapter 2, page 21:

We would expect intuitively that at any given time the universe ought to look the same to observers in all typical galaxies, and in whatever directions they look. (Here, and below, I will use the label "typical" to indicate galaxies that do not have any large peculiar motion of their own, but are simply carried along with the general cosmic flow of galaxies.) This hypothesis is so natural (at least since Copernicus) that it has been called **the** *Cosmological Principle by the English astrophysicist Edward Arthur Milne.*

So the Cosmological principle basically states that the universe appears as homogeneous and isotropic (on scales of distance large enough) to any typical observer, where typical is referred to observers with small local motion compared to the expansion flow. Ryden gives a more general definition of Cosmological Principle, which is valid as well. Quoting Ryden's *Introduction to Cosmology*, chapter 2, page 11 or 14 (depending on which version):

However, modern cosmologists have adopted the **cosmological principle,** *which states: There is nothing special about our location in the universe. The cosmological principle holds true only on large scales (of 100 Mpc or more).*

(c) *(10 points)* Quoting again Ryden's *Introduction to Cosmology*, chapter 2, page 9 or 11:

Saying that the universe is **isotropic** *means that there are no preferred directions in the universe; it looks the same no matter which way you point your telescope. Saying that the universe is* **homogeneous** *means that there are no preferred locations in the universe; it looks the same no matter where you set up your telescope.*

(i) $\boxed{\text{False.}}$ If the universe is isotropic around one point it does not need to be homogeneous. A counter-example is a distribution of matter with spherical symmetry, that is, with a density which is only a function of the radius but does not depend on the direction: $\rho(r, \theta, \phi) \equiv \rho(r)$. In this case for an observer at the center of the distribution the universe looks isotropic but it is not homogeneous.

(ii) $\boxed{\text{True.}}$ For the case of Euclidean geometry isotropy around two or more distinct points does imply homogeneity. Weinberg shows this in chapter 2, page 24. Consider two observers, and two arbitrary points A and B

which we would like to prove equivalent. Consider a circle through point A, centered on observer 1, and another circle through point B, centered on observer 2. If C is a point on the intersection of the two circles, then isotropy about the two observers implies that $A = C$ and $B = C$, and hence $A = B$. (This argument was good enough for Weinberg and hence good enough to deserve full credit, but it is actually incomplete: one can find points A and B for which the two circles will not intersect. On your next problem set you will have a chance to invent a better proof.)

(d) *(2 points extra credit)* $\boxed{\text{False.}}$ If we relax the hypothesis of Euclidean geometry, then isotropy around two points does not necessarily imply homogeneity. A counter-example we mentioned in class is a two-dimensional universe consisting of the surface of a sphere. Think of the sphere in three Euclidean dimensions, but the model "universe" consists only of its two-dimensional surface. Imagine latitude and longitude lines to give coordinates to the surface, and imagine a matter distribution that depends only on latitude. This would not be homogeneous, but it would look isotropic to observers at both the north and south poles. While this example describes a two-dimensional universe, which therefore cannot be our universe, we will learn shortly how to construct a three-dimensional non-Euclidean universe with these same properties.

† Solution written by Daniele Bertolini.

PROBLEM 22: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)*

(a) They key idea is that the coordinate speed of light is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} ,$$

so the coordinate distance (in notches) that light can travel between $t = 0$ and now $(t = t_0)$ is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} .$$

The corresponding physical distance is the horizon distance:

$$\ell_{p,\mathrm{horizon}}(t_0) = a(t_0) \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} .$$

Evaluating,

$$\ell_{p,\mathrm{horizon}}(t_0) = bt_0^{2/3} \int_0^{t_0} \frac{c\,\mathrm{d}t}{bt^{2/3}} = t_0^{2/3}\left[3ct_0^{1/3}\right] = \boxed{3ct_0 .}$$

(b) As stated in part (a), the coordinate distance that light can travel between $t = 0$ and $t = t_0$ is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{3ct_0^{1/3}}{b} .$$

Thus, if we are at the origin, at $t = 0$ the photon must have been at

$$\boxed{x_0 = \frac{3ct_0^{1/3}}{b} .}$$

(c) The photon starts at $x = x_0$ at $t = 0$, and then travels in the negative x-direction at speed $c/a(t)$. Thus, it's position at time t is given by

$$x(t) = x_0 - \int_0^t \frac{c\,\mathrm{d}t'}{a(t')} = \frac{3ct_0^{1/3}}{b} - \frac{3ct^{1/3}}{b} = \boxed{\frac{3c}{b}\left(t_0^{1/3} - t^{1/3}\right) .}$$

(d) Since the coordinate distance between us and the photon is $x(t)$, measured in notches, the physical distance (in, for example, meters) is just $a(t)$ times $x(t)$. Thus.

$$\ell_p(t) = a(t)x(t) = \boxed{3ct^{2/3}\left(t_0^{1/3} - t^{1/3}\right) .}$$

(e) To find the maximum of $\ell_p(t)$, we set the derivative equal to zero:

$$\frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[3c\left(t^{2/3}t_0^{1/3} - t\right)\right] = 3c\left[\frac{2}{3}\left(\frac{t_0}{t}\right)^{1/3} - 1\right] = 0 ,$$

so

$$\left(\frac{t_0}{t_{\max}}\right)^{1/3} = \frac{3}{2} \quad\Longrightarrow\quad \boxed{t_{\max} = \left(\frac{2}{3}\right)^3 t_0 = \frac{8}{27}\,t_0 .}$$

The maximum distance is then

$$\ell_{p,\max} = \ell_p(t_{\max}) = 3c\left(\frac{2}{3}\right)^2 t_0^{2/3}\left[t_0^{1/3} - \left(\frac{2}{3}\right)t_0^{1/3}\right] = 3c\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)t_0$$

$$= \boxed{\frac{4}{9}ct_0 .}$$