PROBLEM 1: DID YOU DO THE READING? (35 points)

The following problem was Problem 1, Quiz 1, 2000. The parts were each worth 5 points.

(a) The Doppler effect for both sound and light waves is named for Johann Christian Doppler, a professor of mathematics at the Realschule in Prague. He predicted the effect for both types of waves in 1842. What are the two digits xx?

(b) When the sky is very clear (i.e., almost transparent to radiation), one can see a band of light across the night sky that has been known since ancient times as the Milky Way. Explain in a sentence or two how this band of light is related to the shape of the galaxy in which we live, which is also called the Milky Way.

(c) The statement that the distant galaxies are on average receding from us with a speed proportional to their distance was first published by Edwin Hubble in 1929, and has become known as Hubble's law. Was Hubble's original paper based on the study of 2, 18, 180, or 1,800 galaxies?

(d) The following diagram, labeled Homogeneity and the Hubble Law, was used by Weinberg to explain how Hubble's law is consistent with the homogeneity of the universe:

The arrows and labels from the "velocities seen by A" and "velocities seen by B" rows have been deleted from this copy of the figure, and it is your job to sketch the figure in your exam book with these arrows and labels included. (Actually, in Weinberg's diagram these arrows were not labeled, but the labels are required here so that the grader does not have to judge the precise length of hand-drawn arrows.)

(e) The horizon is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe, so what matters for purposes of this problem is the current size of the universe. Explain why the horizon would be greater or less than 1% of the current linear size of the universe.

(f) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

(g) At a temperature of 3000K, the nuclei and electrons that filled the universe combined to form neutral atoms, which interact very weakly with the photons of the background radiation. After this process, known as "recombination," the background radiation expanded freely. Since recombination, how have each of the following quantities varied as the size of the universe has changed? (Your answers should resemble statements such as "proportional to the size of the universe," or "inversely proportional to the square of the size of the universe.")

(i) the average distance between photons

(ii) the typical wavelength of the radiation

(iii) the number density of photons in the radiation

(iv) the energy density of the radiation

(v) the temperature of the radiation

The answer to this part of the problem was Problem 1, Quiz 1, 2000. The parts were each worth 5 points.

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY (25 points)

The following problem was Problem 2, Quiz 1, 2000.

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that the universe is infinite in size, and that new matter is continuously created as the universe expands, so that the matter density remains constant. In this model, there is no beginning or end to the universe, and no center.

(a) The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle. Which of the following statements accurately reflects the essence of the steady-state theory? (Choose the best answer.)

(i) The universe has a definite size and shape.

(ii) The universe is finite but unbounded.

(iii) The universe is infinite in size and continuous in extent.

(b) The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $a(t)$ which is consistent with this hypothesis.
Suppose that the mass density of the universe is $\rho_0$, which of course does not change with time. In terms of the general form for $a(t)$ that you found in part (a), calculate the rate at which new matter must be created for $\rho_0$ to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. 

If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $a(t)$. 

PROBLEM 3: DID YOU DO THE READING? (25 points) 

The following problem was Problem 1 on Quiz 1, 2007, where each of the 5 questions was worth 5 points:

(a) In the 1940’s, three astrophysicists proposed a “steady state” theory of cosmology, in which the universe has always looked about the same as it now. State the last name of at least one of these authors. (Bonus points: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.
(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.
(iii) published a catalog, _Nebulae and Star Clusters_, listing 103 objects that astronomers should avoid when looking for comets.
(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.
(v) discovered that the orbital periods of the planets are proportional to the $3/2$ power of the semi-major axis of their elliptical orbits.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were not part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory (ii) MIT (iii) Princeton University (iv) pigeons (v) ground hogs (vi) Hubble’s constant (vii) liquid helium (viii) 7.35 cm

Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer. 

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus’ lifetime.
(ii) approximately two and three decades after Copernicus’ death, respectively.
(iii) about one hundred years after Copernicus’ death.
(iv) approximately two and three centuries after Copernicus’ death, respectively.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this is true? (Approximately two and three centuries after Copernicus’ death.)

If you found $a(t)$, your answer is acceptable if it produced a density proportional to the scale factor $a(t)$.

**PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE** (20 points)

Consider a flat (i.e., $k = 0$, or a Euclidean) universe with scale factor given by

$$a(t) = a_0 e^{\chi t},$$

where $\chi > 0$. 

The following problem was Problem 1 on Quiz 1, 2007, where each of the 5 questions was worth 5 points:

The text continues with additional problems, but the first part of the question is not visible. The student may need to refer to the original document or image for the full question. 

**PROBLEM 3: DID YOU DO THE READING?** (25 points)

The student is asked if they did the reading for the quiz. The question is not visible in the image provided.
where $a_0$ and $\chi$ are constants.

(a) (5 points) Find the Hubble constant $H$ at an arbitrary time $t$.

(b) (5 points) Let $(x,y,z,t)$ be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive $x$-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) (5 points) Suppose that we are living on a galaxy along the positive $x$-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift $z$. Express the time $t$ at which we receive the pulse in terms of $z$, $\chi$, and any relevant physical constants.

(d) (5 points) At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of $z$, $\chi$, and any relevant physical constants.

PROBLEM 5: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift?

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, 10^4 light-years, 10^7 light years, or 10^10 light-years?

[Note for 2011: this question was based on the reading from Joseph Silk's The Big Bang, and therefore would not be a fair question for this year.]

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson-Walker scale factor that behaves as $a(t) = bt^{1/3}$. Here $b$ denotes a constant.

(a) If a light pulse is emitted at time $t_e$ and observed at time $t_o$, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that $t_e$ and $t_o$ are given, find the observed redshift $z$.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of $c$, $t_o$, and $z$.

(d) At an arbitrary time $t$ in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of $c$, $t$, and $z$. Where $n_0$ and $\chi$ are constants.
Problem 5. Suppose the distant object is a galaxy, moving with the Hubble flow from the galaxy. What value would we observe the light coming from this jet? Express your answer in terms of all or some of the variables $c, \gamma, z, H, t$.

Problem 6. Find the “look-back” time as a function of $z$.

Problem 7. Another flat universe with an unusual energy flow.

Problem 8. Did you do the READING? (20 points)
Suppose a light pulse leaves galaxy A at time $t$. At what time does this second pulse arrive at galaxy A?

Consider a flat universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$H(t) = \frac{5}{3} t$$

where $H$ is a constant.

(a) Find the Hubble constant $H$ at an arbitrary time $t$.

(b) Find the Hubble constant $H$ at an arbitrary time $t$.

(c) Find the Hubble constant $H$ at an arbitrary time $t$.

(d) Find the Hubble constant $H$ at an arbitrary time $t$.

(e) Find the Hubble constant $H$ at an arbitrary time $t$.

(f) Find the Hubble constant $H$ at an arbitrary time $t$.

(g) Find the Hubble constant $H$ at an arbitrary time $t$.

(h) Find the Hubble constant $H$ at an arbitrary time $t$.

(i) Find the Hubble constant $H$ at an arbitrary time $t$.

(j) Find the Hubble constant $H$ at an arbitrary time $t$.

(k) Find the Hubble constant $H$ at an arbitrary time $t$.

(l) Find the Hubble constant $H$ at an arbitrary time $t$.

When a light pulse leaves galaxy A at time $t$, what is its speed of light $c$ (units of meters per second)?

When the light pulse is received by galaxy B, what is the speed of light $c$ (units of meters per second)?

Consider a flat universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$H(t) = \frac{5}{3} t$$

where $H$ is a constant.

(a) Find the Hubble constant $H$ at an arbitrary time $t$.

(b) Find the Hubble constant $H$ at an arbitrary time $t$.

(c) Find the Hubble constant $H$ at an arbitrary time $t$.

(d) Find the Hubble constant $H$ at an arbitrary time $t$.

(e) Find the Hubble constant $H$ at an arbitrary time $t$.

(f) Find the Hubble constant $H$ at an arbitrary time $t$.

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(e) Find the Hubble constant $H$ at an arbitrary time $t$.

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(c) Find the Hubble constant $H$ at an arbitrary time $t$.

(d) Find the Hubble constant $H$ at an arbitrary time $t$.

(e) Find the Hubble constant $H$ at an arbitrary time $t$.

(f) Find the Hubble constant $H$ at an arbitrary time $t$.

(g) Find the Hubble constant $H$ at an arbitrary time $t$.

(h) Find the Hubble constant $H$ at an arbitrary time $t$.

(i) Find the Hubble constant $H$ at an arbitrary time $t$.

(j) Find the Hubble constant $H$ at an arbitrary time $t$.

(k) Find the Hubble constant $H$ at an arbitrary time $t$.

(l) Find the Hubble constant $H$ at an arbitrary time $t$. 
than 1% of the mass density of the universe.

\[ \frac{\dot{a}}{a} = \frac{8}{3} \rho \]

where \( a \) is the scale factor, \( \dot{a} \) is the time derivative of \( a \), \( \rho \) is the mass density and \( \rho/\rho_{\text{crit}} \) is the critical mass density (i.e., that mass density of the universe when the universe cooled to about zero).

\[ \Omega = \frac{\rho}{\rho_{\text{crit}}} \]

is the value of the equation of state for a matter-dominated universe.

\[ q = 1 - \frac{\dot{a}}{a} \]

is defined by the equation of state for a cosmological fluid. The parameter is defined by the equation of state for a cosmological fluid.

\[ \Omega = \frac{\rho}{\rho_{\text{crit}}} \]

is necessarily small. (Express your answer in terms of \( \rho/\rho_{\text{crit}} \).)

\[ \frac{(\dot{a})^2}{a^2} b = \rho \]

is the mass density of the gas of photons falls off as 1/\( t^4 \). For a flat (i.e., \( k = 0 \)) matter-dominated universe we learned that the scale factor of the universe then is given by

\[ a(t) = a(0) \left( \frac{t}{t_0} \right)^{1/2} \]

is the time derivative of \( a \), (a) Define the cosmological principle. (b) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (d) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (e) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (f) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (g) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (h) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (i) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (j) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (k) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (l) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (m) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (n) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (o) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (p) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (q) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (r) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (s) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (t) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (u) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (v) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (w) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (x) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (y) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe. (z) Find the relationship between \( \Omega \) and \( q \) for a matter-dominated universe.
Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. How to determine the frequency of the waves received by the observer, given the frequency of the waves emitted by the source.

\[ f_r = f_s \left( \frac{v_f}{c} \right) \]

where:
- \( f_r \) is the received frequency
- \( f_s \) is the emitted frequency
- \( v_f \) is the observed frequency
- \( c \) is the speed of light

To solve this problem, we need to consider the relative motion between the source and the observer. The Doppler shift formula accounts for the change in frequency observed by an observer due to the motion of the source or the observer, relative to the source.

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PROBLEM 16: DID YOU DO THE READING?

(a) (4 points) What was the first external galaxy that was shown to be at a distance significantly greater than the most distant known objects in our galaxy? How was the distance estimated?

(b) (5 points) What is recombination? Did galaxies begin to form before or after recombination? Why?

(c) (4 points) In Chapter IV of his book, Weinberg develops a "recipe for a hot universe," in which the matter of the universe is described as a gas in thermal equilibrium. Which of the following is on this list of conserved quantities? Circle as many as apply.

(i) baryon number
(ii) energy per particle
(iii) proton number
(iv) electric charge
(v) pressure

(d) (4 points) The wavelength corresponding to the mean energy of a CMB (cosmic microwave background) photon today is approximately equal to which of the following quantities? (You may wish to look up the values of various physical constants at the end of the quiz.)

(i) 2 fm (2 \times 10^{-15} m)
(ii) 2 microns (2 \times 10^{-6} m)
(iii) 2 mm (2 \times 10^{-3} m)
(iv) 2 m.

(e) (4 points) What is the equivalence principle?

(f) (4 points) Why is it difficult for Earth-based experiments to look at the small wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?
PROBLEM 18: TRANSVERSE DOPPLER SHIFTS

The following problem was taken fromProblem 4, Quiz 1, 2005, where it counted 20 points.

(a) (8 points) Suppose that a spaceship Xanthu is at rest at location (x = 0, y = a, z = 0) in a Cartesian coordinate system. (We assume that the space is Euclidean, and that the distance scales in the problem are small enough so that the expansion of the universe can be neglected.)

The spaceship Emmerac is moving at speed $v_0$ along the x-axis in the positive direction, as shown in the diagram, where $v_0$ is comparable to the speed of light. As the Emmerac crosses the origin, it receives a radio signal that had been sent some time earlier from the Xanthu. Is the radiation received redshifted or blueshifted? What is the redshift $z$ (where negative values of $z$ can be used to describe blueshifts)?

(b) (7 points) Now suppose that the Emmerac is at the origin, while the Xanthu is moving in the negative x-direction, at y = a and z = 0, as shown in the diagram. That is, the trajectory of the Xanthu can be taken as $(x = -v_0 t, y = a, z = 0)$.

At $t = 0$ the Xanthu crosses the y-axis, and at that instant it emits a radio signal along the y-axis, directed at the origin. The radiation is received some time later by the Emmerac. In this case, is the radiation received redshifted or blueshifted? What is the redshift $z$ (where again negative values of $z$ can be used to describe blueshifts)?

(c) (5 points) Is the sequence of events described in (b) physically distinct from the sequence described in (a), or is it really the same sequence of events described in a reference frame that is moving relative to the reference frame used in part (a)? Explain your reasoning in a sentence or two.

(Hint: note that there are three objects in the problem: Xanthu, Emmerac, and the photons of the radio signal.)

PROBLEM 19: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND

This problem was Problem 3 on Quiz 1, 2007.

Consider a high-speed merry-go-round which is similar to the one discussed in Problem 3 of Problem Set 1, but which has two levels. That is, there are four evenly-spaced cars which travel around a central hub at speed $v$ at a distance $R$ from the center, and also another four cars that are attached to extensions of the four radial arms, each moving at a speed $2v$ at a distance $2R$ from the center. In this problem we will consider only light waves, not sound waves, and we will assume that the merry-go-round is rotating at a constant angular velocity and that there are no radial forces acting on the cars.

We learned in Problem Set 1 that there is no redshift when light from one car at a distance $R$ is received by an observer on another car at a distance $R$. Consider a high-speed merry-go-round which is similar to the one discussed in Problem Set 1.

(a) (5 points) Suppose that cars 5–8 are all emitting light waves in all directions. If an observer in car 1 receives light waves from each of these cars, what redshift $z$ does she observe for each of the four signals?

(b) (10 points) Suppose that a spaceship is receding to the right at a relativistic speed $u$ along a line through the hub, as shown in the diagram. Suppose that an observer in car 6 receives a radio signal from the spaceship, at the time when the car is in the position shown in the diagram. What redshift $z$ is observed?
Suppose that galaxy 1 emits electromagnetic radiation. As the photon travels from the source to us, what is its coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of 

Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a coordinate system with the origin at the point of emission, and us, and at what time does it occur?

The photon travels from the source to us, what is the coordinate (ii) of the photon at the time when it is observed by galaxy 2, (iv) at the time of emission, (v) at the time when it is observed by galaxy 2, (vi) at the time of emission, (vii) at the time of observation?

There is a constant. Let's denote the current time.

\[ \frac{1}{3} \frac{d}{dt} = \frac{1}{3} \] is called the Hubble length.

As the photon travels from the source to us, what is its coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of 

The following problem appeared on Quiz 1 of 2011.

PROBLEM 21: DID YOU DO THE READING?

Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a coordinate system with us at the origin, and the source of the photon along the positive x-axis. What is the coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of emission, (v) at the time of observation?

There is a constant. Let's denote the current time.

\[ \frac{1}{3} \frac{d}{dt} = \frac{1}{3} \] is called the Hubble length.

As the photon travels from the source to us, what is its coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of 

The following problem appeared on Quiz 1 of 2011.

Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a coordinate system with us at the origin, and the source of the photon along the positive x-axis. What is the coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of emission, (v) at the time of observation?

There is a constant. Let's denote the current time.

\[ \frac{1}{3} \frac{d}{dt} = \frac{1}{3} \] is called the Hubble length.

As the photon travels from the source to us, what is its coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of emission, (v) at the time of observation?

There is a constant. Let's denote the current time.

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There is a constant. Let's denote the current time.

\[ \frac{1}{3} \frac{d}{dt} = \frac{1}{3} \] is called the Hubble length.

As the photon travels from the source to us, what is its coordinate (ii) at the time of emission, (ii) at the time of reception, (iii) at the time when it is observed by galaxy 2, (iv) at the time of emission, (v) at the time of observation?
Problem 1: Did you do the reading?

(a) Doppler predicted the Doppler effect in 1842.

(b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look out from the plane of the disk than we do when we look out at right angles to the plane of the disk.

(c) Hubble's original paper on the expansion of the universe was based on a study of only 18 galaxies. His work led to the realization that the universe is expanding.

(d) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by more than 1%. To see why, note that the horizon distance is equal to the scale factor times the comoving horizon distance. The scale factor grows by 1% during this time interval, but the comoving horizon distance also grows, since light from the distant galaxies has had more time to reach us.

(e) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(f) (i) the average distance between photons: proportional to the size of the universe. Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion of the universe.

(ii) the typical wavelength of the radiation: proportional to the size of the universe. See Lecture Notes 3.

(iii) the number density of photons in the radiation: inversely proportional to the cube of the size of the universe. From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a cube is proportional to the cube of the length of a side, the average volume occupied by a photon grows as the cube of the universe. The number density of photons is inversely proportional to the size of the universe, so the only effect is that the number density of photons is stretched with the expansion of the universe.

(iv) the energy density of the radiation: inversely proportional to the fourth power of the size of the universe. The energy of each photon is proportional to its frequency, and hence inversely proportional to its wavelength. So from (ii) the energy of each photon is inversely proportional to the wavelength. The energy of each photon is inversely proportional to the square of the wavelength. The energy of each photon is inversely proportional to the square of the wavelength. Therefore, the energy density of the radiation is inversely proportional to the fourth power of the size of the universe.
(v) the temperature of the radiation:
inversely proportional to the size of the universe.

(The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY

(25 points)

(a) (10 points)

According to Eq. (3.7),

\[ H(t) = \frac{1}{a(t)} \frac{da}{dt} \]

So in this case

\[ \frac{1}{a(t)} \frac{da}{dt} = H_0, \]

which can be rewritten as

\[ \frac{da}{a} = H_0 dt. \]

Integrating,

\[ \ln a = H_0 t + c, \]

where \( c \) is a constant of integration.

Exponentiating,

\[ a = b e^{H_0 t}, \]

where \( b = e^c \) is an arbitrary constant.

(b) (15 points)

Consider a cube of side \( \ell \) drawn on the comoving coordinate system diagram. The physical length of each side is then \( a(t) \ell \), so the physical volume is

\[ V(t) = a^3(t) \ell^3. \]

Since the mass density is fixed at \( \rho = \rho_0 \), the total mass inside the cube at any time \( t \) is

\[ M(t) = a^3(t) \ell^3 \rho_0. \]

In the absence of matter creation, the total mass within a comoving volume would not change, so the increase in mass described by the above equation must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

\[ \frac{dM}{dt} = \frac{\rho_0 (a(t)) A}{V} = \frac{\rho_0 (a(t)) A}{\ell^3}. \]

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us.

You have \( H_0 = 70 \text{ km sec}^{-1} \text{ Mpc}^{-1} \), and take \( \rho_0 \) to be the critical density.

So in this case

\[ \frac{dV}{dt} = \frac{V}{\rho_0} \frac{\rho_0 (a(t)) A}{\ell^3} = \frac{dV}{dt} \frac{\rho_0 (a(t)) A}{\ell^3}. \]

The rate of matter production required for the steady-state universe theory can then be calculated.

To put this number into more meaningful terms, note that the mass of a hydrogen atom is

\[ 1.67 \times 10^{-27} \text{ kg}, \]

and that 1 year = \( 3.15 \times 10^7 \) sec. The rate of matter production required for the steady-state universe theory can then be calculated.

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and that 1 year = \( 3.15 \times 10^7 \) sec. The rate of matter production required for the steady-state universe theory can then be calculated.
PROBLEM 3: DID YOU DO THE READING? (25 points)

The following questions are each worth 5 points:

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it now. State the last name of at least one of these authors. (Bonus points: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)


(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:
(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.
(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.
(iii) published a catalog, Nebulae and Star Clusters, listing 103 objects that astronomers should avoid when looking for comets.
(iv) discovered that the orbital periods of the planets are proportional to the $3/2$ power of the semi-major axis of their elliptical orbits.
(v) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.

Discussion: (i) is false in part because de Sitter was not involved in the measurement of the size of the Milky Way, but the most obvious error is in the size of the Milky Way. Its actual diameter is much smaller. (v) is of course one of Kepler's laws of planetary motion.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were not part of the story behind this spectacular discovery:
(i) Bell Telephone Laboratory (ii) MIT (iii) Princeton University (iv) pigeons (v) ground hogs (vi) Hubble's constant (vii) liquid helium (viii) 7.35 cm

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.)

Discussion: The discovery of the cosmic background radiation was described in some detail by Weinberg in Chapter 3. The observation was done at Bell Telephone Laboratories, in Holmdel, New Jersey. The detector was cooled with liquid helium to minimize electrical noise, and the measurements were made at a wavelength of 7.35 cm. During the course of the experiment, the astronomers had to eject a pair of pigeons who were roosting in the antenna. Penzias and Wilson were not initially aware that the radiation they discovered might have come from the big bang, but Bernard Burke of MIT put them in touch with a group at Princeton University (Robert Dicke, James Peebles, P.G. Roll, and David Wilkinson) who were actively working on this hypothesis.

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were important because they confirmed the Copernican theory of the Earth's orbit around the Sun. The discovery of the aberration of starlight was made about one hundred years after Copernicus died. The Foucault pendulum was invented about one hundred years after Copernicus died.

Ryden discusses this on p. 5. The aberration of starlight was discovered in 1725, while the Foucault pendulum was invented in 1851.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous. How large must the averaging scale be before this is true?

(i) 1 AU (1 AU = 1.496 × 10^11 m).
(ii) 100 kpc (1 kpc = 1000 pc, 1 pc = 3.086 × 10^16 m = 3.262 light-year).
(iii) 1 Mpc (1 Mpc = 10^6 pc).
(iv) 10 Mpc.
(v) 100 Mpc.
(vi) 1000 Mpc.

This issue is discussed in Ryden's book on p. 11.

The following 2 questions are each worth 5 points:

PROBLEM 3: DID YOU DO THE READING?
PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by
\[ H = \frac{\dot{a}}{a} \]
So
\[ H = \chi a_0 e^{\chi t} \]
\[ = \chi. \]

(b) According to Eq. (3.8), the coordinate velocity of light is given by
\[ \frac{dx}{dt} = c a(t) = c a_0 e^{\chi t} \]
Integrating,
\[ x(t) = c a_0 \int_0^t e^{\chi t'} dt' = c a_0 \left[ \frac{-1}{\chi} e^{\chi t'} \right]_0^t = c a_0 \left[ 1 - e^{-\chi t} \right]. \]

(c) From Eq. (3.11), or from the front of the quiz, one has
\[ 1 + z = \frac{a(t_r)}{a(t_e)} \]
Here \( t_e = 0 \), so
\[ 1 + z = \frac{a_0 e^{\chi t_r}}{a_0} = \Rightarrow e^{\chi t_r} = 1 + z = \Rightarrow t_r = \frac{1}{\chi} \ln(1 + z). \]

(d) The coordinate distance is
\[ x(t_r), \text{ where } x(t) \text{ is the function found in part (b), and } t_r \text{ is the time found in part (c)}. \]
So
\[ e^{\chi t_r} = 1 + z, \]
\[ x(t_r) = c \chi a_0 \left[ 1 - e^{-\chi t_r} \right] = c \chi a_0 \left[ 1 - e^z \right] = c \frac{Z}{\chi a_0} (1 + z). \]

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so
\[ \frac{(z + 1) a_0}{\chi a_0^{\frac{1}{2}}} \]
The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception,
\[ \frac{(z + 1) a_0}{\chi a_0^{\frac{1}{2}}} \]
\[ = \frac{(z + 1)}{1} \frac{a_0}{\chi} = \frac{(z + 1)}{\chi} \]

PROBLEM 5: "DID YOU DO THE READING?"

(a) The distinguishing quantity is \( \Omega \equiv \frac{\rho}{\rho_c} \). The universe is open if \( \Omega < 1 \), flat if \( \Omega = 1 \), or closed if \( \Omega > 1 \).

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date\(^*\) was made by the COBE satellite, which measured the temperature as \( 2.728 \pm 0.004 \) Kelvin. The error here is quoted with a 95\% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5\%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years.

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

(g) The absolute luminosity (i.e., the total light output) of a Cepheid variable star appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to Cepheids, by measuring the period and the apparent luminosity. From the period one can estimate the absolute luminosity, and then use the 1\( /r^2 \) law for the intensity of a point source to determine the distance.

(h) \( 10^{7} \) light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(j) Princeton University.


PROBLEM 6: DID YOU DO THE READING?

\[ \frac{\chi}{z} = \frac{(z + 1)}{\chi a^2} = \left( \frac{(z + 1) a_0}{\chi a_0^{\frac{1}{2}}} \right) \]
According to Eq. (g.8), the coordinate velocity of light is given by
\[ \frac{\chi}{z} \]
\[ \chi = \frac{\rho_{\text{cr}}}{\rho_{\text{c}}} = H \]
\[ \frac{\chi}{z} = \frac{\rho_{\text{cr}}}{\rho_{\text{c}}} = H \]

(a) According to Eq. (g.9), the Hubble parameter is related to the scale factor by
\[ \frac{\chi}{z} = \frac{\rho_{\text{cr}}}{\rho_{\text{c}}} = H \]

(b) The coordinate distance is \( x(t) \), where \( x(t) \) is the function found in part (d).
(b) The coordinates \( t \) and \( \ell \) are cosmic time coordinates. The "look-back time" is when the integral \( \int_0^t \) is taken. We can write\( I - \left( \frac{\ell}{0_1} \right) = z \)

and\( \left( \frac{\ell}{0_1} \right) = \frac{(\ell)_p}{(0_1)_p} = z + 1 \)

The cosmological redshift is given by the usual form,

\[
\gamma = \frac{1}{1 - \frac{\ell}{0_1}} = \frac{1}{1 - z}
\]

Thus, the coordinate distance at time \( t \) is equal to \( (\ell)_p \) times the coordinate velocity, \( \frac{d}{dt} \), so \( \int_0^t \frac{d}{dt} \) times the coordinate velocity, \( \frac{d}{dt} \), gives the coordinate distance. The coordinate time \( t \) is defined in the exam is when the integral \( \int_0^t \) is evaluated. The "look-back time" is when the integral \( \int_0^t \) is taken. We can write

\[
\left( \frac{\ell}{0_1} \right) = \frac{(\ell)_p}{(0_1)_p} = z + 1
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This is the coordinate distance at time \( t \) is equal to \( (\ell)_p \) times the coordinate velocity, \( \frac{d}{dt} \), so \( \int_0^t \frac{d}{dt} \) times the coordinate velocity, \( \frac{d}{dt} \), gives the coordinate distance. The coordinate time \( t \) is defined in the exam is when the integral \( \int_0^t \) is evaluated. The "look-back time" is when the integral \( \int_0^t \) is taken. We can write

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\[
\left( \frac{\ell}{0_1} \right) = \frac{(\ell)_p}{(0_1)_p} = z + 1
\]
We can use the result of part (a) to eliminate $t_0$ in favor of $z$. From (a),

$$t_0 = \frac{1 + z}{\gamma} - 1.$$

Therefore,

$$t_0 - t = t_0 \left[ 1 - \frac{1 + z}{\gamma} - 1 \right].$$

**c)** (10 points) The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = a(t_0) \int_{t_0}^{t_{\gamma}} a(t) \, dt.$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{c t_0}{\gamma} \left[ 1 - \frac{1 + z}{\gamma} - 1 \right].$$

Factoring $t_0$ out of the parentheses gives

$$\ell_p(t_0) = c t_0 \gamma \left[ 1 - \frac{1 + z}{\gamma} - 1 \right].$$

This can be rewritten in terms of $z$ and $H_0$ using the result of part (a) as well.

Finally, then,

$$\ell_p(t_0) = c H_0 \gamma \left[ 1 - \frac{1 + z}{\gamma} - 1 \right].$$

**d)** (10 points) A nearly identical problem was worked through in Problem 8 of Problem Set 1. The energy of the observed photons will be redshifted by a factor of $(1 + z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of emission, reducing the flux by another factor of $(1 + z)$. Consequently, the observed power will be redshifted by two factors of $(1 + z)$ to

$$P/(1 + z)^2.$$

Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but

$$\left[ \frac{\zeta}{1 - \zeta} (z + 1) - 1 \right] \frac{\zeta - 1}{\zeta} \, dH = \frac{dJ}{\zeta},$$

where we used $\zeta = \frac{\zeta}{1 - \zeta}$, and $dH = d\gamma/(\gamma)^2$. Using the result of part (e) to write $\gamma$ in terms

$$\gamma(\gamma_{\text{or}})^{2} = (\gamma_{\text{or}})^{2} \frac{d\gamma}{d\zeta}$$

the radiation energy flux $J$, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4 \pi \ell_p(t_0)^2} \left[ \frac{\zeta}{1 - \zeta} (z + 1) - 1 \right] \frac{\zeta - 1}{\zeta} \, dH = \frac{(\gamma_{\text{or}})^{2} d\gamma}{d\zeta}.$$

Finally, then,

$$\frac{\gamma_{\text{or}}}{\zeta} = \frac{(\gamma_{\text{or}})^{2}}{(\gamma_{\text{or}})^{2}} = \frac{dH}{d\zeta}.$$

This can be rewritten in terms of $z$ and $H_0$ using the result of part (a) as well.

$$\left[ \frac{\zeta}{1 - \zeta} (z + 1) - 1 \right] \frac{\zeta - 1}{\zeta} \, dH = \frac{(\gamma_{\text{or}})^{2} d\gamma}{d\zeta}.$$

Calculating this integral gives

$$\frac{dH}{d\zeta} \int_{\gamma_{\text{or}}}^{\gamma} (\gamma_{\text{or}})^{2} = (\gamma_{\text{or}})^{2} d\gamma.$$

Finally, from the present value of the physical distance to the object, $\gamma$ is

$$\left[ \frac{\zeta}{1 - \zeta} (z + 1) - 1 \right] \frac{\zeta - 1}{\zeta} \, dH = \frac{(\gamma_{\text{or}})^{2}}{d\zeta}.$$

Therefore, $\gamma$ is

$$\left[ \frac{\zeta}{1 - \zeta} (z + 1) - 1 \right] \frac{\zeta - 1}{\zeta} \, dH = \frac{(\gamma_{\text{or}})^{2}}{d\zeta}.$$
Located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1}{1 - \frac{v}{c}} \Delta t_{\text{source}}}.$$ 

Note that I have used the formula from the front of the exam, but I have changed the size of $v$, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1 + z) \Delta t_{\text{relay station}}.$$ 

Thus,

$$1 + z \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} = \left(1 + \frac{z}{1 + \frac{z}{1 - \frac{v}{c}}}\right) \Delta t_{\text{source}}.$$ 

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler formula to convert the redshift $z$ of the galaxy to a velocity of recession, then subtracted from this the speed $v$ of the jet, and then again used the special relativity Doppler formula to find the Doppler shift corresponding to this composite velocity.

PROBLEM 8: DID YOU DO THE READING?

(a) The lines were dark, caused by absorption of the radiation in the cooler, outer layers of the sun.

(b) Individual stars in the Andromeda Nebula were resolved by Hubble in 1923. [The other names and dates are not without significance. In 1609 Galileo built his first telescope; during 1609-10 he resolved the individual stars of the Milky Way, and also discovered that the surface of the moon is irregular, that Jupiter has moons of its own, that Saturn has handles (later recognized as rings), that the sun has spots, and that Venus has phases. In 1755 Immanuel Kant published his Universal Natural History and Theory of the Heavens, in which he suggested that at least some of the nebulae are galaxies like our own. In 1912 Henrietta Leavitt discovered the relationship between the period and luminosity of Cepheid variable stars. In the 1950s Walter Baade and Allan Sandage recalibrated the extra-galactic distance scale, reducing the accepted value of the Hubble constant by about a factor of 10.

(c) (i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3 °K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until the invention of cosmic background radiation detectors in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave background is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating $z$ to $v$ does not apply.]
\[ \left( \frac{V_f}{c} - \frac{\theta_f}{\theta} \right) \frac{\theta F}{\theta} = \left( \frac{V_f}{c} - \frac{\theta_f}{\theta} \right) \frac{\theta C}{\theta} \]

Solving for \( b \):

\[ b = \frac{c}{\theta} \]

5) According to Hubble's law, the speed is equal to Hubble's constant times the

\[ (V_f \cdot d) = H \]

In this case one has

\[ \frac{\rho}{q} \]

In general, the (physical) horizon distance is given by

\[ \frac{\rho}{q} = \frac{c}{\theta} \frac{\theta c}{\theta} \]

6) Aristarchus. \[ \text{The heliocentric picture was never accepted by other Greeks.} \]

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PROBLEM 9: A FLAT UNIVERSE WITH

\[ \frac{\rho}{q} \times (\Delta) \]

In this case

\[ \frac{\rho}{q} \]

The physical distance is the coordinate distance, the speed is the

\[ \frac{\rho}{q} \]

In general, the horizon distance is given by

\[ \frac{\rho}{q} = H \]

7) Any patch of the night sky could look as bright as the surface of the sun.

\[ \text{The intensity of the microwave radiation is determined by the intensity of the microwave radiation.} \]

\[ \text{This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by} \]

\[ \text{clouds,} \]

\[ \text{which are measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by} \]

\[ \text{clouds.} \]
g) The redshift for radiation observed at time \( t \) is given by

\[
1 + z = \frac{a(t)}{a(t_0)}
\]

where \( t_0 \) is the time that the radiation was emitted. Solving for \( t_0 \),

\[
t_0 = t \left( 1 + \frac{1}{z} \right)^{\frac{5}{3}}
\]

As found in part (d), the physical distance that the light travels between \( t \) and \( t_0 \), measured at time \( t \), is given by

\[
\ell_p(t) = \frac{a(t)}{a(t_0)} \int_{t_0}^{t} c a(t') dt' = \frac{5}{2} ct \left[ 1 - \left( \frac{t}{t_0} \right)^{2/5} \right]
\]

Substituting the expression for \( t_0 \), one has

\[
\left[ \frac{\ell_p(t)}{\ell_p(t_0)} \right] = \left( \frac{a(t)}{a(t_0)} \right)^{\frac{4}{5}}
\]

and, as measured at time \( t_0 \), is given by

\[
\frac{\ell_p(t_0)}{\ell_p(t)} = \frac{1}{\left( \frac{t}{t_0} \right)^{2/5}}
\]

As found in part (d), the physical distance that the light travels between \( t_0 \) and \( t \) is given by

\[
\frac{\ell_p(t)}{\ell_p(t_0)} = z + 1
\]

h) Again we will view the problem in comoving coordinates. Put galaxy B at the origin, and galaxy A at a coordinate distance \( \ell_c \) along the \( x \)-axis. Draw a sphere of radius \( \ell_c \), centered galaxy A. Also draw a detector on galaxy B, with the origin, and galaxy A at a coordinate distance \( c t \). Draw a photon distance to multiple redshift \( z \), which is exactly equal to the horizon distance. If \( t \) is a general time that the bid hits detector

\[
\rho_c t = (d_j)^{\frac{z}{z} = \infty}
\]
The problem is worded so that \( t_A \) and not \( z \) is the given variable that determines how far galaxy A is from galaxy B. In practice, however, it is usually more useful to express the answer in terms of the redshift \( z \) of the received radiation. One can do this by using the above expression for \( 1 + z \) to eliminate \( t_A \) in favor of \( z \), finding

\[
J = \frac{P}{25 \pi c^2 t_B^2} \left[ \left(1 + \frac{1}{1 - \frac{t_A^2}{5A^2} - \frac{t_B^2}{5B^2}} \right)^2 - 1 \right]^{2/3}.
\]

\( t_A \) be the time at which the light pulse arrives back at galaxy A. The pulse must therefore travel a coordinate distance \( \ell_c \), so

\[
\int_{t_A}^{t_B} c a(t') \, dt' = \ell_c.
\]

Using the answer from (c) and integrating the left-hand side,

\[
5c^2 b \left( t_{A/5} - t_{B/5} \right) = 5c^2 b \left( t_{B/5} - t_{A/5} \right).
\]

Solving for \( t_A' \),

\[
t_A' = \left( \frac{2t_{B/5} - t_{A/5}}{2} \right).
\]
radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.

**Problem 11: Another Flat Universe with**

\[ a(t) \propto t^{3/5} \]

a) According to Eq. (3.7) of the Lecture Notes, \[ H(t) = \frac{1}{a(t)} \frac{da}{dt} \]. For the special case of \[ a(t) = bt^{3/5} \], this gives \[ H(t) = \frac{1}{bt^{3/5}} \frac{3}{5} \frac{bt^{-2/5}}{5} = \frac{3}{5} t \].

b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by \[ \frac{dx}{dt} = c \frac{a(t)}{a} \]. Since galaxies A and B have physical separation \[ \ell_0 \] at time \( t_1 \), their coordinate separation is \[ \frac{\ell_0}{bt^{3/5}} \]. The radio signal must cover this coordinate distance in the time interval from \( t_1 \) to \( t_2 \), which implies that 

\[
\int_{t_2}^{t_1} c \frac{a}{a} dt = \frac{\ell_0}{bt^{3/5}} \]

Using the expression for \( a(t) \) and integrating, we get

\[
\frac{c^2 b}{5} \left( \frac{t_2^{2/5}}{2} - \frac{t_1^{2/5}}{2} \right) = \frac{\ell_0}{bt^{3/5}}
\]

which can be solved for \( t_2 \) to give

\[
t_2 = \left( \frac{t_1^{2/5} + \ell_0^{2/5}}{c^2 b^{2/5}} \right)^{5/2} t_1
\]

c) The method is the same as in part (b). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from \( t_2 \) to \( t_3 \). So, 

\[
\int_{t_3}^{t_2} c \frac{a}{a} dt = \frac{\ell_0}{bt^{3/5}}
\]

Solving for \( t_3 \) gives 

\[
t_3 = \left( \frac{t_2^{2/5} + \ell_0^{2/5}}{c^2 b^{2/5}} \right)^{5/2} t_1
\]

d) Cosmic time is defined by the reading of suitably synchronized clocks which are each at rest with respect to the matter of the universe at the same location. (For this problem we will not need to think about the method of synchronization.) Thus, the cosmic time interval between the receipt of the message and the
Putting this back into the Taylor series gives

\[ \left. \frac{\partial}{\partial t} \delta \right|_{t=0} = 0 = \left. \frac{\partial}{\partial t} \theta \right|_{t=0} \]

which when specialized to \( t = \frac{1}{2} \) becomes

\[ \left. \frac{\partial}{\partial t} \theta \right|_{t=0} = \frac{1}{2} \frac{\partial \theta}{\partial t} \]

Evaluating the necessary derivative gives

\[ \delta \theta + \left. \frac{\partial}{\partial t} \theta \right|_{t=0} = 0 = \left. \frac{\partial}{\partial t} \theta \right|_{t=0} \]

Perform those steps except the brute force approach the answer to part (d) can be replaced by \( \frac{1}{2} \)

where to first order in the \( \ell \) in the numerator could equally well have been

\[ \left. \frac{\partial}{\partial \ell} \theta \right|_{\ell=0} = 0 = \left. \frac{\partial}{\partial \ell} \theta \right|_{\ell=0} \]

which can be solved for \( \frac{\partial}{\partial \ell} \theta \) to give

\[ \left. \frac{\partial}{\partial \ell} \theta \right|_{\ell=0} = \left. \frac{\partial}{\partial \ell} \left( \frac{1}{2} \right) \right|_{\ell=0} \]

Integration gives

\[ \int \frac{1}{\ell^2} \left( \frac{1}{2} \right) \]
PROBLEM 12: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that
\[ \ddot{a} = -\frac{4\pi}{3} G \rho a \]
and
\[ (\dot{a} a)^2 = 8\pi G \rho - \frac{k c^2}{a^2}, \]
where a dot denotes a derivative with respect to time \( t \).

The critical mass density \( \rho_c \) is defined to be the mass density that corresponds to a flat (\( k = 0 \)) universe, so from the equation above it follows that
\[ (\dot{a} a)^2 = 8\pi G \rho_c. \]

Substituting into the definition of \( q \), we find
\[ q = -\frac{\ddot{a}}{a (\dot{a} a)} = \left( \frac{4\pi}{3} G \rho \right) \left( \frac{3}{8\pi G \rho_c} \right) = \frac{1}{2} \frac{\rho}{\rho_c}. \]

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that \( k = 0 \), so
\[ (\dot{a} a)^2 = 8\pi G \rho. \]

Since \( \rho(t) \propto \frac{1}{a^4(t)} \), it follows that
\[ \frac{da}{dt} = \text{const} a. \]
Rewriting this as \( da/a = \text{const} dt \), the indefinite integral becomes
\[ \frac{1}{2} a^2 = (\text{const}) t + c', \]
where \( c' \) is a constant of integration. Different choices for \( c' \) correspond to different choices for the definition of \( t = 0 \). We will follow the standard convention of choosing \( c' = 0 \), which sets \( t = 0 \) to be the time when \( a = 0 \). Thus the above equation implies that \( \rho c = 0 \), which sets the universe to be closed when \( a = 0 \). The above equation implies that the definition of \( t = 0 \) we will follow is consistent with our convention.

The Hubble expansion of the model universe means that \( k = 0 \), so
\[ \frac{dC}{ds} = \left( \frac{v}{r} \right). \]

PROBLEM 14: DID YOU DO THE READING? (25 points)

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it? (Ryden, Chapter 2, Pages 6-8)
Ans: Olber's paradox is that the night sky appears to be dark, instead of being uniformly bright. The primary resolution is that the universe has a finite age, and so the light from stars beyond the horizon distance hasn't reached us yet. However, even in the steady-state model of the universe, the paradox is not resolved, and so the light from stars beyond the horizon distance is still reaching us. This resolution is that the universe has a finite age. These paradoxes are then the light sky appears to be dark, instead of being visible.

(b) What is the value of the Newtonian gravitational constant \( G \) in Planck units? The Planck length is of the order of \( 10^{-15} \) m, \( 10^{-16} \) m, or \( 10^{-17} \) m? (Ryden, Chapter 1, Page 3)
Ans: \( G = 1 \) in Planck units, by definition. The Planck length is of the order of \( 10^{-35} \) m. (Note that this answer could be obtained by a process of elimination as long as you remember that the Planck length is much smaller than 10^{-15} m, which is the typical size of a nucleus).

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (Weinberg, Chapter 2, Pages 21-23; Ryden, Chapter 2, Page 11)
Ans: The Cosmological Principle states that there is nothing special about our location in the universe. Thus the universe is homogenous and isotropic. Yes, the Hubble expansion is consistent with it (since there is no center of expansion).
Note that it guarantees that \(|v| \leq c\) and \(\sqrt{\frac{v}{c^2}} \leq 1\). The final formula is the relativistic expression for the addition of velocities.

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = v / c
\]

(18.4)

Combining Eqs. (18.2) and (18.3),

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.5)

Alpha-7, but the wavelength found by the observer will be the same as if Alpha-7 acted as a relay station, receiving the photons and retransmitting them at the received wavelength. So, applying Eq. (18.1) again, the wavelength seen by the observer can be written as

\[
\lambda_{\text{observed}} = \lambda_{\text{emitted}} / \sqrt{1 - \beta^2}
\]

(18.6)

where \(\beta = v / c\), and \(\lambda = \lambda_{\text{emitted}}\).

Problem 1.2: Special Relativity Doppler Shift (20 points)

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler shift formula, which was given on the front of the exam:

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.7)

(Weinberg, Chapter 1, Page 5)

(b) Although we used the presence of Alpha-7 in determining the redshift of the photons that are detected by the observer, we can still apply the special-relativity Doppler shift formula, which was given on the front of the exam.

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.8)

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

(c) If the redshift is not exactly zero, we can calculate it using the standard special-relativity Doppler shift formula.

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.9)

We refer to Chapter 3, Ryden, Chapter 2, Page 22.

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

(d) In other words, the redshift is not exactly zero.

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.10)

(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

(f) Combining Eqs. (18.2) and (18.3),

\[
\frac{\gamma}{1} = \frac{\gamma}{1}
\]

(18.11)

8.286 Q128 QUIZ I REVIEW PROBLEM SOLUTIONS, FALL 2011
The wavelength corresponding to the mean energy of a CMB (cosmic microwave) photon today is approximately equal to which of
the 
small 
hyperapplication wavelengths, and it is identified to a good approximation with

\[ \frac{\pi}{\nu} = \lambda \]

or

\[ \frac{\pi}{\nu} = \lambda \]

Wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?

Why is it difficult for Earth-based experiments to look at the small 
sized galaxy in our own galaxy? How was the first external galaxy in our galaxy?

What was the first external galaxy that was shown to be at a distance beyond the 


PROBLEM 16: DID YOU DO THE READING?

PROBLEM 16: DID YOU DO THE READING?
The Earth's atmosphere is increasingly opaque for wavelengths shorter than 0.3 cm. Therefore, radiation at these wavelengths will be absorbed and re-scattered by the Earth's atmosphere; observations of the cosmic microwave background at small wavelengths must be performed above the Earth's atmosphere.

**Problem 17:** Tracing a Light Pulse through a Radiation-Dominated Universe

(a) The physical horizon distance is given in general by

\[ \ell_p,\text{horizon} = a(t) \frac{\int_0^{t_f} c a(t) \, dt}{c} \]

so in this case

\[ \ell_p,\text{horizon} = bt^{1/2} \int_0^{t_f} c bt^{1/2} \, dt = 2ct^{1/2} \]

(b) If the source is at the horizon distance, it means that a photon leaving the source at \( t = 0 \) would just be reaching the origin at \( t_f \). So,

\[ t = 0 \]

(c) The coordinate distance between the source and the origin is the coordinate horizon distance, given by

\[ \ell_c,\text{horizon} = \int_0^{t_f} c bt^{1/2} \, dt = 2ct^{1/2} \]

(d) The photon starts at coordinate distance \( 2c\sqrt{t_f/b} \), and by time \( t \) it will have traveled a coordinate distance

\[ \frac{q}{x'^{1/2}c} = \int_0^{t_f} \frac{c}{x'^{1/2}c} \, dx' = \text{integration} \]

(e) The time interval between wave crests is the same as measured by the source, the Xanthu:

\[ \Delta t_s = \Delta t \]

Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wave crests will travel the same distance, so the time interval of reception is also the time it would take for a photon to travel the horizon distance, given by

\[ \Delta t_r = \Delta t_s \]

Thus, there is a blueshift, and hence a physical distance

\[ \left(\frac{\lambda - \lambda_0}{\lambda_0}\right) \frac{q}{c} = \gamma \]

from the origin, and hence a physical distance

\[ \left(\frac{\lambda - \lambda_0}{\lambda_0}\right) \frac{q}{c} = \gamma \]

Problem 18: Transverse Doppler Shifts

(a) Describing the events in the coordinate system shown, the Xanthu is at rest, so its clocks run at the same speed as the coordinate system time variable, \( t \). The emission of the wave crests of the radio signal are therefore separated by a time interval equal to the time interval as measured by the source, the Xanthu:

\[ \Delta t = \Delta t_s \]

Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wave crests will travel the same distance, so the time interval of reception is also the time it would take for a photon to travel the horizon distance, given by

\[ \Delta t_r = \Delta t_s \]

Thus, there is a blueshift.

The redshift parameter \( z \) is defined by

\[ \frac{\Delta t_r}{\Delta t_s} = 1 + z \]

Thus, there is a blueshift, and hence a physical distance

\[ \left(\frac{\lambda - \lambda_0}{\lambda_0}\right) \frac{q}{c} = \gamma \]

Problem 19: Tracking a Light Pulse through a Radiation-Dominated Universe
so $\gamma = 1 + \frac{1}{z}$, or $z = \frac{1}{\gamma} - 1$.

Recall that $\gamma > 1$, so $z$ is negative.

(b) Describing this situation in the coordinate system shown, this time the source on the Xanthu is moving, so the clocks at the source are running slowly. The time between wavecrests, measured in coordinate time $t$, is therefore larger by a factor of $\gamma$ than $\Delta t_s$, the time as measured by the clock on the source:

\[ \Delta t = \gamma \Delta t_s. \]

Since the radio signal is emitted when the Xanthu is at its minimum separation from the Emmerac, the rate of change of the separation is zero, so each wavecrest travels the same distance (again assuming that $c \Delta t \ll a$). Since the Emmerac is at rest, its clocks run at the same speed as the coordinate time $t$, and hence the time interval between crests, as measured by the receiver, is

\[ \Delta t_r = \Delta t = \gamma \Delta t_s. \]

Thus the time interval as measured by the receiver is longer than that measured by the source, and hence it is a redshift.

The redshift parameter $z$ is given by

\[ 1 + z = \frac{\gamma}{\gamma - 1} \]

so

\[ z = \frac{1}{\gamma} - 1. \]

(c) The events described in (a) can be made to look a lot like the events described in (b) by transforming to a frame of reference that is moving to the right at speed $v_0$—i.e., by transforming to the rest frame of the Emmerac. In this frame the Emmerac is of course at rest, and the Xanthu is traveling on the $x$-axis, as in part (b). However, just as the transformation causes the $x$-component of the velocity to change from zero to a negative value, so the transformation will cause the $y$-component of the velocity to change from zero to a negative value, so the $x$-component of the velocity of the radio signal will be transformed from zero to a negative value, so the $x$-component of the velocity of the radio signal will be transformed from zero to a negative value.

\[ z = \frac{1 - \gamma}{\gamma} \]

so

\[ z = \frac{1}{\gamma} - 1. \]
measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by
\[ \gamma^2 \Delta t_S = \Delta t_{Lab} \]
and
\[ \gamma^1 \Delta t_O = \Delta t_{Lab} \].
To make sure that the \( \gamma \)-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that
\[ \Delta t_O = \gamma^2 \gamma^1 \Delta t_S. \]

The redshift \( z \) is defined by
\[ \Delta t_O \equiv (1 + z) \Delta t_S, \]
so
\[ z = \gamma^2 \gamma^1 - 1 = \sqrt{1 - \frac{v^2}{c^2}} - 1. \]

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.

The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and a change in path lengths between successive pulses, but we do not need to do it. It is the standard situation of a source and observer moving directly away from each other, as discussed at the end of Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed \( u \), it becomes
\[ (1 + z) |_{\text{relay}} = \sqrt{1 + \frac{u}{c}} \]
if we again use the symbol \( \Delta t_S \) for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is
\[ \Delta t_R = \sqrt{1 + \frac{u}{c}} \Delta t_S. \]

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in \( \Delta t_S \), so we can ignore it. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station.

Combining the formulas above,
\[ \gamma^2 \gamma^1 \Delta t_S = \Delta t_{Lab} \]
and
\[ \gamma^1 \gamma^2 \Delta t_O = \Delta t_{Lab} \].

Therefore, putting together the equations above, it becomes immediately that
\[ \gamma^2 \gamma^1 \Delta t_S = \Delta t_{Lab} = \gamma^1 \gamma^2 \Delta t_O \]
and
\[ \gamma^1 \gamma^2 \Delta t_O = \Delta t_{Lab} = \gamma^2 \gamma^1 \Delta t_S. \]

To make sure that the \( t \)-values on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that
\[ \Delta t_O = \gamma^2 \gamma^1 \Delta t_S. \]

The redshift \( z \) is defined by
\[ \Delta t_O \equiv (1 + z) \Delta t_S, \]
so
\[ z = \gamma^2 \gamma^1 - 1 = \sqrt{1 - \frac{v^2}{c^2}} - 1. \]
This is what should be expected, since the speed of separation of the light of high, which should always give the standard value. The speed at time of emission is really just a local measurement of the speed of light, which should be expected, since the speed of separation of the light signal at the time of emission is really just a local measurement of the speed of light.

\[ v = \frac{\delta p}{\delta s^d} \]

\[
\int \frac{q}{\delta s^d} \left( \frac{1}{c^2} - 1 \right) \delta \xi = \\
\left[ \frac{1}{c^2} \frac{q}{\delta s^d} \right] = (i)^2 y(i)^2 = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \left( \frac{1}{c^2} - 1 \right) \delta \xi = \\
\left[ \frac{1}{c^2} \frac{q}{\delta s^d} \right] = (i)^2 y(i)^2 = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \left( \frac{1}{c^2} - 1 \right) \delta \xi = \\
\left[ \frac{1}{c^2} \frac{q}{\delta s^d} \right] = (i)^2 y(i)^2 = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \left( \frac{1}{c^2} - 1 \right) \delta \xi = \\
\left[ \frac{1}{c^2} \frac{q}{\delta s^d} \right] = (i)^2 y(i)^2 = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \left( \frac{1}{c^2} - 1 \right) \delta \xi = \\
\left[ \frac{1}{c^2} \frac{q}{\delta s^d} \right] = (i)^2 y(i)^2 = (i)^2 y
\]

\[
\int \frac{q}{\delta s^d} \frac{1}{c^2} \frac{q}{\delta s^d} = (i)^2 y
\]
\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} = (\partial g_1)_p = \partial g_z + I \]  

Then

\[ x = \frac{\partial g_1}{\partial t} = \frac{(\partial g_1)}{(1)_p} = \partial g_z + I \]

which can be solved to find

\[ x = \frac{\partial g_1}{\partial t} = \frac{(\partial g_1)}{(1)_p} = \partial g_z + I \]

Thus

\[ x = \frac{(\partial g_1)}{(1)_p} = \partial g_z + I \]

The redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The coordinate difference shown by Eq. (20.3) and (20.9) is just the scale factor times the coordinate separation, so that if light at galaxy $A$ at time $t$ is emitted from galaxy $B$, the coordinate distance from galaxy $C$ to $B$ at time $t$ is also $t_1$. The coordinate distance from galaxy $C$ to $B$ at time $t$ is therefore the physical distance from galaxy $C$ to galaxy $B$. Thus, the redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

and that it equals $3$ when $x = 1$. Therefore, the redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The time of reception is given by

\[ \frac{t}{x - 1} = \frac{\partial g_1}{\partial t} \]

for

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The speed of approach is

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

and that it equals $3$ when $x = 1$. Therefore, the redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The time of reception is given by

\[ \frac{t}{x - 1} = \frac{\partial g_1}{\partial t} \]

for

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The speed of approach is

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

and that it equals $3$ when $x = 1$. Therefore, the redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The time of reception is given by

\[ \frac{t}{x - 1} = \frac{\partial g_1}{\partial t} \]

for

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The speed of approach is

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

and that it equals $3$ when $x = 1$. Therefore, the redshift is given by

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The time of reception is given by

\[ \frac{t}{x - 1} = \frac{\partial g_1}{\partial t} \]

for

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

which can be solved to give

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]

The speed of approach is

\[ \frac{x}{x^2 - 1} = \frac{\partial g_1}{\partial t} \]
and $z_{BC} = 1 - \sqrt{2}$

\[(20.26)\]

Full credit will be given for the answer in the form above, but it can be simplified by rationalizing the fraction:

\[
\frac{1}{\sqrt{1 - \sqrt{2}}} = \frac{1 + \sqrt{2}}{3}
\]

\[(20.27)\]

Numerically, $z_{BC} = 1.0657$.

Following the solution to Problem 6 of Problem Set 2, we draw a diagram in comoving coordinates, putting the source at the center of a sphere:

![Diagram of Galaxy A emitting radiation in a sphere, with a detector at a distance $d$ from the source.]

The energy from Galaxy A will radiate uniformly over the sphere. If the detector has physical area $A_D$, then in the comoving coordinate picture it has coordinate area $A_D/a^2(t_2^2)$, since the detection occurs at time $t_2$.

The useful coordinate area of the sphere is $4\pi\ell c, B A$, so the fraction of photons that hit the detector is

\[
\text{fraction} = \frac{A_D/a^2(t_2^2)}{4\pi\ell c, B A}
\]

\[(20.28)\]

As in Problem 6, the power hitting the detector is reduced by two factors of $(1 + z)$: one factor because the energy of each photon is proportional to the frequency, and hence is reduced by the redshift, and one more factor because the rate of arrival of photons is also reduced by the redshift factor $(1 + z)$. Thus,

\[
\text{Power hitting detector} = \frac{P}{4\pi\ell c, B A} \frac{A_D/a^2(t_2^2)}{a(t_1^2/a(t_2^2))} = \frac{P}{4\pi\ell c, B A} \frac{a(t_1^2)}{a(t_2^4)} = \frac{P}{4\pi\ell c, B A} \frac{a(t_1^2)}{a(t_2^4)}
\]

\[(20.29)\]

The energy flux is given by

\[
J = \text{Power hitting detector} / A
\]

\[(20.30)\]

so

\[
J = \frac{P}{4\pi\ell c, B A} \frac{A_D/a^2(t_2^2)}{a(t_1^2/a(t_2^2))} = \frac{P}{4\pi\ell c, B A} \frac{a(t_1^2)}{a(t_2^4)}
\]

\[(20.31)\]

From here it is just algebra, using Eqs. (20.9) and (20.11), and $a(t) = b t^2/3$:

\[
J = \frac{P}{4\pi\ell c, B A} \left(\frac{3c^2b}{t_1^2}\right)^2 b^2 t_4^4 / b^4 t_8^8 = \frac{256}{59049} \pi P c^2 t_2^2
\]

\[(20.32)\]
Consider two observers, and two arbitrary points. For the case of Euclidean geometry, isotropy around two or more points does imply homogeneity. Weinberg shows this in chapter 2, page 24. Returning now to 1929: Hubble estimated the distance to 18 galaxies from their Doppler shifts.

...
Problem 22: The Trajectory of a Photon Originating at the Horizon

(a) True. The key idea is that the coordinate speed of light is given by $c/a$.

(b) A counter-example we mentioned in class is a two-dimensional universe consisting of the surface of a sphere. Think of the two-dimensional universe as a two-dimensional non-Euclidean universe.

(c) The photon starts at $t = 0$. Since the coordinate distance between us and the photon is measured in directions of constant $x$-coordinate, the photon's position at time $t$ is given by $(t/c)(x)$.

(d) The photon's trajectory is determined by solving the equations of motion in a non-Euclidean universe. If we relax the hypothesis of Euclidean geometry, the two circles will not intersect. On your next problem set you will have a chance to invent a better proof.

Therefore, the photon must have been at $x = 0$, and hence we are at the origin, at $t = 0$. The photon's trajectory is given by $(t/c)(x)$.

Thus, the corresponding physical distance is the horizon distance:

$$\int_0^t \frac{dp}{\ell_p} \int_0^\infty \frac{d\ell_p}{\ell_p} = \left( \int \frac{d\ell_p}{\ell_p} \right) \frac{d\gamma}{d\ell_p}$$

where $\gamma$ is the Einstein factor for a two-dimensional universe.