MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

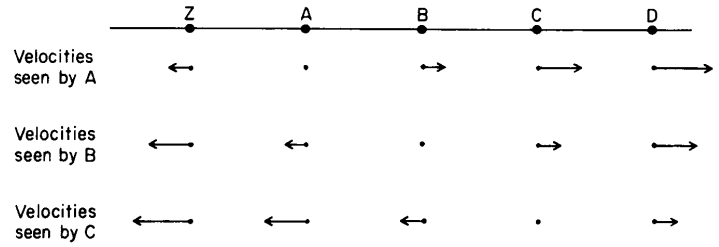
Physics 8.286: The Early Universe — October 27, 2013
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 3, 2013

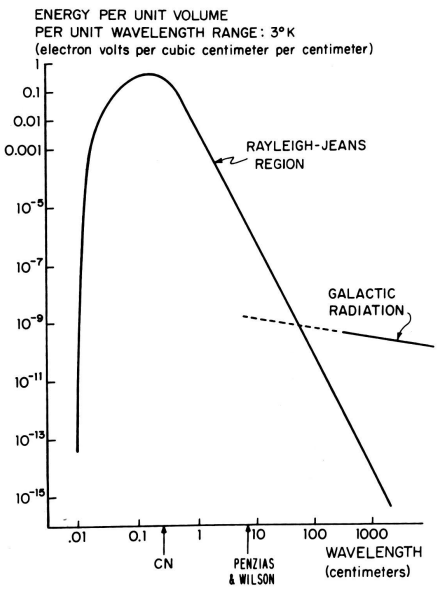
PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(5 points)*



The diagram was used to explain how Hubble's Law is consistent with homogeneity. Naively Hubble's law sounds like it makes us the center of the universe, since all distant galaxies are receding from us. The diagram shows, however, that if observers on Galaxy A see a Hubble expansion pattern, as shown on the top line, then observers on Galaxies B and C would also see a Hubble expansion pattern, centered on themselves.

(b) *(5 points)* The intensity at short wavelengths is suppressed by the effect of quantum theory, which implies that the electomagnetic field is composed of quanta, or photons, each with an energy of Planck's constant h times the frequency $\nu = c/\lambda$, where λ is the wavelength. When the wavelength is so small that the photon energy is large compared to kT, the typical thermal energy, then it is rare for these high-energy photons to exist.

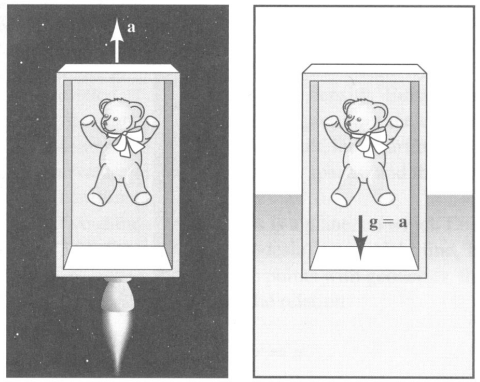


(c) *(5 points)* By observing multiple closely spaced absorption lines of the cyanogen radicals, astronomers can infer the relative population of the ground state and an excited state.

This allows astronomers to measure the effective temperature of interstellar gas clouds that are in equilibrium with the CMB.

(d) *(5 points)* Arno Penzias and Robert Wilson discovered the cosmic microwave background *(3 points)*, while employed at Bell Telephone Laboratories *(2 points)*.

(e) *(5 points)* Ryden used the teddy-bear diagram at the right to illustrate the equivalence principle, a key assumption of general relativity. The principle implies that they teddy bear has no way of distinguishing whether she is experiencing the gravity of the Earth, or whether she is in an acclerating rocket in otherwise empty space.



PROBLEM 2: AN EXPONENTIALLY EXPANDING FLAT UNIVERSE *(25 points)*

(a) *(5 points)* As given in the formula sheet, the redshift factor $(1+z)$ is just the ratio of the scale factors at the two times:

$$1+z = \frac{a(t_{\rm observed})}{a(t_{\rm emitted})} = \frac{a(t_2)}{a(t_1)} = e^{\chi(t_2-t_1)} \ , \tag{2.1}$$

so

$$\boxed{z = e^{\chi(t_2-t_1)} - 1 \ .} \tag{2.2}$$

(b) *(10 points)* The time t_2 of course depends on how far away the other galaxy is. One indicator of this distance is z, so t_2 can be found by solving the answer to part (a) for t_2. The question, however, explicitly says that z cannot appear in the answer. The other indicator of the distance is ℓ_p, so this variable must be the key to the answer. The coordinate distance between the galaxies is

$$\ell_c = \int_{t_1}^{t_2} \frac{c}{a_0 e^{\chi t}}\, {\rm d}t = \frac{c}{\chi a_0}\left[e^{-\chi t_1} - e^{-\chi t_2}\right] \ , \tag{2.3}$$

so the physical distance at time t_1 is

$$\ell_p = a(t_1)\ell_c = \frac{c}{\chi}\left[1 - e^{-\chi(t_2 - t_1)}\right] \ . \tag{2.4}$$

Now we need only solve for t_2:

$$\begin{aligned} e^{-\chi(t_2-t_1)} &= 1 - \frac{\chi\ell_p}{c} \\ \Longrightarrow \quad \chi(t_2 - t_1) &= -\ln\left(1 - \frac{\chi\ell_p}{c}\right) \\ \Longrightarrow \quad & \boxed{t_2 = t_1 - \frac{1}{\chi}\ln\left(1 - \frac{\chi\ell_p}{c}\right) \ .} \end{aligned} \tag{2.5}$$

(c) *(5 points)* From the answer to the previous part we can see that as ℓ_p approaches c/χ, $t_2 \to \infty$.

Yes. For $\ell_p \geq \dfrac{c}{\chi}$, the light pulse will never reach the other galaxy.

(d) *(5 points)* The coordinate distance $\ell_{c,\mathrm{eq}}$ that the pulse will travel between time t_1 and time t_{eq} is given by

$$\ell_{c,\mathrm{eq}} = \int_{t_1}^{t_{\mathrm{eq}}} \frac{c}{a_0 e^{\chi t}}\,\mathrm{d}t = \frac{c}{\chi a_0}\left[e^{-\chi t_1} - e^{-\chi t_{\mathrm{eq}}}\right] \ . \tag{2.6}$$

This should be half of the coordinate distance $\ell_c = \ell_p/a(t_1)$ to the other galaxy, so

$$\frac{c}{\chi a_0}\left[e^{-\chi t_1} - e^{-\chi t_{\mathrm{eq}}}\right] = \frac{\ell_p}{2a_0 e^{\chi t_1}} \ . \tag{2.7}$$

Solving for t_{eq}, we find

$$\boxed{t_{\mathrm{eq}} = t_1 - \frac{1}{\chi}\ln\left(1 - \frac{\chi\ell_p}{2c}\right) \ .} \tag{2.8}$$

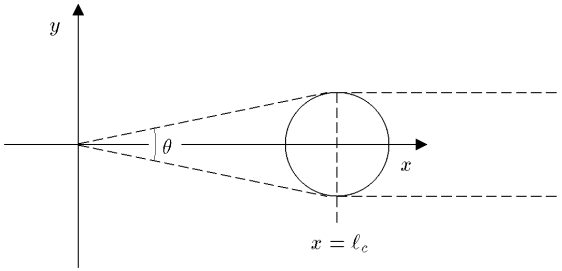
(e) *(5 points)* All time intervals are stretched by the same factor, so the duration of the pulse is stretched by the same factor as the period of the light waves. Thus

$$\boxed{\Delta t_{\mathrm{obs}} = (1+z)\Delta t \ .} \tag{2.9}$$

PROBLEM 3: A CYLINDRICAL UNIVERSE *(25 points)*

No solution is included, since this problem was on Problem Set 3.

PROBLEM 4: ANGULAR SIZE AND RADIATION FLUX RECEIVED FROM A DISTANT GALAXY *(25 points)*



(a) *(10 points)* Assuming that the diagram is labeled in comoving coordinates (i.e., in "notches"), and assuming that $\theta \ll 1$, then θ is given geometrically by

$$\theta = \frac{2R_{1,c}}{\ell_c} \ ,$$

where $R_{1,c}$ is the coordinate radius of the galaxy at the time of emission. We are assuming a flat (i.e., Euclidean) universe, an assumption that was unstated on Quiz 1, since at that time we had not yet discussed any other kind of geometry. (Note that galaxies do not expand with the Hubble expansion, but instead evolve in complicated ways. Thus the coordinate radius of the galaxy depends on time, and $R_{1,c}$ is generally not equal to $R_{0,c}$, the coordinate radius of the galaxy today.) By the usual relationship between coordinate distances and physical distances,

$$\begin{aligned} \ell_p &\equiv \ell_p(t_0) = a(t_0)\ell_c \\ R_1 &\equiv R_{1,p}(t_1) = a(t_1)R_{1,c} \ . \end{aligned}$$

Thus,

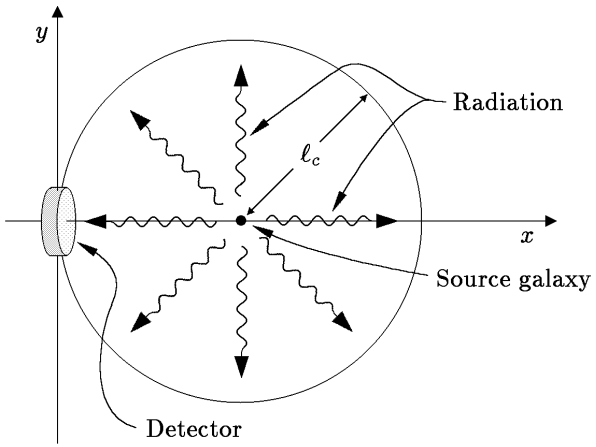
$$\theta = 2\frac{R_1}{a(t_1)}\frac{a(t_0)}{\ell_p} = \frac{2R_1}{\ell_p}\frac{a(t_0)}{a(t_1)} = \boxed{\frac{2R_1}{\ell_p}(1+z) \ ,}$$

where we used the fact that

$$\frac{a(t_0)}{a(t_1)} = 1 + z \ ,$$

which is the standard relationship for the cosmological redshift.

(b) *(15 points)*



As shown in the diagram, it is useful to think about a sphere with the source galaxy at the center, and our detector on the surface of the sphere. The power P that was emitted by the galaxy at the time of emission is now arriving at this sphere. The total power P' arriving at the sphere today is given by

$$P' = \frac{P}{(1+z)^2} \ .$$

Here one power of $(1+z)$ is caused by the redshifting of photons, as each photon has an energy proportional to its frequency, and hence inversely proportional to $(1+z)$. The second power of $(1+z)$ is caused by the arrival rate of photons, which is also a type of frequency, so the value measured by the observer is $1/(1+z)$ times the value measured by the source. The fraction of this power received by the detector is just equal to the fraction of the area subtended by the detector. If we measure both areas in current physical units, then the power P_D hitting the detector is

$$P_D = \frac{P}{(1+z)^2}\frac{A_D}{4\pi\ell_p^2} \ ,$$

where A_D is the area of the detector. The radiation energy flux J is then

$$J = \frac{P_D}{A_D} = \boxed{\frac{P}{4\pi\ell_p^2(1+z)^2} \ .}$$

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QUIZ 1 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1-v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta\ell_0/c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3}G\rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

$$\text{Flat } (k=0): \qquad a(t) \propto t^{2/3} \ ,$$

$$\Omega = 1 \ .$$