MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 7, 2013
Prof. Alan Guth

# QUIZ 2

**Reformatted to Remove Blank Pages***

**A FORMULA SHEET IS AT THE END OF THE EXAM.**

You may rip off and keep the formula sheet.

Please answer all questions in this stapled booklet.

________________________________
Your Name

* A few corrections announced at the quiz have been incorporated.



## PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity $v$ as a function of radius $R$. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii? Explain your answer.

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state? No need to explain your answer.

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

## PROBLEM 2: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} .$$

Assuming that the model universe is flat, calculate

(a) *(5 points)* The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) *(5 points)* The value of the Hubble parameter $H(t)$, as a function of $t$.

(c) *(5 points)* The physical horizon distance, $\ell_{p,\mathrm{horizon}}(t)$.

(d) *(5 points)* The mass density $\rho(t)$.



## PROBLEM 3: ROTATING FRAMES OF REFERENCE *(35 points)*

*The following problem was Problem 17 of Review Problems for Quiz 2.*

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] , \tag{P3.1}$$

which corresponds to a coordinate system rotating about the $z$-axis, where $\phi$ is the azimuthal angle around the $z$-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

*EXTRA INFORMATION*

*To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (P3.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, and $\bar{z}$,*

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] ,$$

*and then introducing new coordinates $t$, $r$, $\phi$, and $z$ that are related by*

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z ,$$

*so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, and $\mathrm{d}\bar{z} = \mathrm{d}z$.*

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, $g_{11}$ (which can also be called $g_{rr}$) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \tag{P3.2}$$

— *Problem 3 continues* —

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{P3.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index $\mu$ is equal to 1, corresponding to the coordinate $r$.

(c) *(7 points)* Explicitly write the equation that results when the free index $\mu$ is equal to 2, corresponding to the coordinate $\phi$.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants $c$ and $\omega$. Be sure to note that your answer should depend on the derivatives of $t$, $\phi$, and $z$ with respect to $t$, not $\tau$. *(Hint: first find an expression for* $\mathrm{d}\tau/\mathrm{d}t$*, in terms of the quantities indicated, and then ask yourself how this result can be used to find* $\mathrm{d}t/\mathrm{d}\tau$*.)*

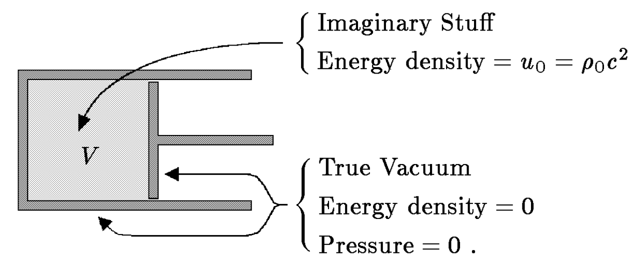


**PROBLEM 4: PRESSURE AND ENERGY DENSITY OF IMAGINARY STUFF** *(20 points)*

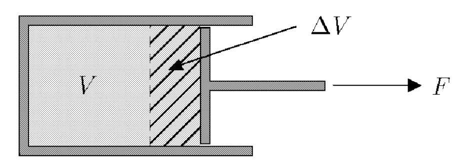
*This problem is very similar to Problem 19 of the Quiz 2 Review Problems, but be careful: it is not the same problem.*

In Lecture Notes 6, with further calculations in Problem 4 of Problem Set 6, a thought experiment involving a piston was used to show that $p = \frac{1}{3}\rho c^2$ for radiation. In this problem you will apply the same technique to calculate the pressure of ***imaginary stuff***, which has the property that the energy density falls off in proportion to $1/V^{3/2}$ as the volume $V$ is increased.

If the initial energy density of the imaginary stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as




The piston is then pulled outward, so that its initial volume $V$ is increased to $V + \Delta V$. You may consider $\Delta V$ to be infinitesimal, so $\Delta V^2$ can be neglected.




(a) *(10 points)* Using the fact that the energy density of imaginary stuff falls off as $1/V^{3/2}$, find the amount $\Delta U$ by which the energy inside the piston changes when the volume is enlarged by $\Delta V$. Define $\Delta U$ to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the imaginary stuff is called $p$, how much work $\Delta W$ is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure $p$ of the imaginary stuff in terms of its energy density $u$. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

| Problem | Maximum | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 20 | |
| 3 | 35 | |
| 4 | 20 | |
| **TOTAL** | 100 | |

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# QUIZ 2 FORMULA SHEET

## SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\text{coord}} = \frac{c}{a(t)} \ .$$

## DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

## COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

## SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: $\gamma$

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$ .

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

## COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right) a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\,\rho_r(t_i) \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

## EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $\quad a(t) \propto t^{2/3}$

$$\Omega = 1 \ .$$

Closed ($k > 0$): $\quad ct = \alpha(\theta - \sin\theta) \ , \quad \dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$): $\quad ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \dfrac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$



## ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + \tilde a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde a(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + \tilde a^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde a(t) = a(t)/\sqrt{-k}$. Note that $\tilde a$ can be called $a$ if there is no need to relate it to the $a(t)$ that appears in the first equation above.

## HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt'$$

$$= 3ct \ \ \text{(flat, matter-dominated).}$$

## SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2$$
$$+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \ ,$$

## GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\frac{dx^\ell}{ds}$$

$$\text{or:} \quad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$