Time Dilation Factor:
\[
\frac{z'}{a} = \frac{\sqrt{1 - \frac{v}{c}}}{1 - \frac{v}{c}} \equiv z
\]

Special Relativity:
\[
\frac{\text{Observed}}{\text{Unlensed}} = \frac{z + 1}{\frac{1}{1 - \frac{v}{c^2}}} = z
\]

Cosmological Redshift:
\[
1 + z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}
\]

Doppler Shift (For motion along a line):
\[
\frac{\nu}{\nu_0} = \sqrt{1 - \frac{v}{c} - \frac{v}{c^2}}
\]

The starred problems are the ones that I recommend that you review most carefully: Problems 4, 5, 6, 11, 13, 15, 17, 19. There are only three reading questions, Problems 1, 2, and 3. There are other useful equations:

**Purpose:** The starred problems are not to be handed in, but are designed to help you study. They come from problems that have appeared on previous quizzes. You will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, and 2011. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you may still need to consult those quizzes.
where \( \theta < \frac{\pi}{2} \)

\[
\left( \frac{d\tau}{c} \right) \left( \frac{d\sigma}{c} \right) = \left( \frac{d\varphi}{c} \right) = \left( \frac{d\psi}{c} \right) = 0
\]

**Covariant Equation:**

\[
\varepsilon^{\lambda\mu
\nu}(\theta, \varphi, \vartheta + \varphi_d) \cdot \frac{\varepsilon^\lambda}{\varepsilon^\nu} = \varepsilon^{\lambda\mu
\nu}(\theta, \varphi, \vartheta)
\]

**Schwarzschild Metric:**

\[
\text{here, if } \mu = (\text{null metric}) \quad \rho \left( \frac{d\rho}{c} \right) = \int_0^\rho \left( \frac{d\rho}{c} \right) = \rho^{\text{(null metric)}}
\]

**Horizon Distance:**

Above in red to illustrate the fact that an event in the past is where \( \gamma \equiv \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} \)

\[
\left( \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} + \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} \right) \left( \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} - \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} \right) = \varepsilon^{\lambda\mu\nu} \varepsilon^{\lambda\mu\nu} = \varepsilon^\rho
\]

Alternatively, for \( \theta < \frac{\pi}{2} \), we can define \( \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} = \rho \cdot \frac{d\rho}{c} = (\text{null metric}) \)

\[
\left( \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} + \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} \right) \left( \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} - \frac{\varepsilon^{\lambda\mu\nu}}{\varepsilon^{\lambda\mu\nu}} \right) = \varepsilon^{\lambda\mu\nu} \varepsilon^{\lambda\mu\nu} = \varepsilon^\rho
\]

**Robertson-Walker Metric:**

\[
\varepsilon^{\lambda\mu\nu}(\theta, \varphi, \vartheta + \varphi_d) \cdot \frac{\varepsilon^\lambda}{\varepsilon^\nu} = \varepsilon^{\lambda\mu\nu}(\theta, \varphi, \vartheta)
\]

**Evolution of a Matter-dominated Universe:**

\[
\varepsilon^{\lambda\mu\nu}(\theta, \varphi, \vartheta + \varphi_d) \cdot \frac{\varepsilon^\lambda}{\varepsilon^\nu} = \varepsilon^{\lambda\mu\nu}(\theta, \varphi, \vartheta)
\]

\[
\left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right)
\]

\[
\left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right)
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\left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right)
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\[
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\]

\[
\left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) \equiv \left( \frac{d\rho}{c} \right) = \left( \frac{d\rho}{c} \right)
\]
PROBLEM 1: DID YOU DO THE READING?

Parts (a) - (c) of this problem come from Quiz 4, 2000, and parts (d) and (e) come from Quiz 6, 2000, and parts (f) and (g) come from Quiz 3, 2002.

(a) (5 points) By what factor does the lepton number per comoving volume of the universe change between temperatures of \( kT = 10 \text{ MeV} \) and \( kT = 0 \text{ MeV} \)? You should assume the existence of the normal three species of neutrinos for your answer.

(b) (5 points) Measurements of the primordial deuterium abundance in the sun’s photosphere become increasingly difficult as the photosphere becomes hotter. For example, less deuterium is detected than would have escaped from the Earth's surface.

(c) (5 points) Give three examples of hadrons.

(d) (6 points) In Chapter 6 of The First Three Minutes, Steven Weinberg posed the question, “Why was there no experimental search for the cosmic background radiation years before 1965?” In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these particles. (If you name both correctly, you will get 3 extra points. However, if you name just one right and one wrong, you will get 4 points.)

Answer:

Problems & Solutions:

1. Did You Do the Reading (2000)?
2. Did You Do the Reading (2007)?
3. Did You Do the Reading (2011)?
In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3 ◦K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose these three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1 ◦K, and such a background would be too weak to detect.
(ii) There was a breakdown in communication between theorists and experimentalists.
(iii) It was not technologically possible to detect a signal as weak as a 3 ◦K microwave background until about 1965.
(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
(v) It was extraordinarily difficult for physicists to take seriously any theory of the early universe.
(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

PROBLEM 2: DID YOU DO THE READING? (24 points)

The following problem was Problem 1 of Quiz 2 in 2007.

(a) (6 points)
In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the... for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed.
(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

(b) (6 points)
In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons?

- Electric Charge:
  (i) ∼10^9
  (ii) ∼1000
  (iii) ∼1
  (iv) ∼10^{-6}
  (v) either zero or negligible

- Baryon Number:
  (i) ∼10^{-20}
  (ii) ∼10^{-9}
  (iii) ∼10^{-6}
  (iv) ∼1
  (v) anywhere from 10^{-5} to 1

- Lepton Number:
  (i) ∼10^9
  (ii) ∼1000
  (iii) ∼1
  (iv) ∼10^{-6}
  (v) could be as high as ∼1, but is assumed to be very small.
During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\begin{align*}
\text{antineutrino} + \text{proton} & \rightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} & \rightarrow \text{positron} + \text{proton}
\end{align*}
\]

(E) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\begin{align*}
\text{antineutrino} + \text{proton} & \rightarrow \text{positron} + \text{neutron} \\
\text{neutrino} + \text{neutron} & \rightarrow \text{electron} + \text{proton}
\end{align*}
\]

(F) Neutrons and protons can be created and destroyed by reactions such as:

\[
\begin{align*}
\text{proton} + \text{neutrino} & \rightarrow \text{positron} + \text{antineutrino} \\
\text{neutron} + \text{antineutrino} & \rightarrow \text{electron} + \text{positron}
\end{align*}
\]

During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\begin{align*}
\text{antineutrino} + \text{proton} & \rightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} & \rightarrow \text{positron} + \text{proton}
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\text{neutrino} + \text{neutron} & \rightarrow \text{electron} + \text{proton}
\end{align*}
\]

(F) Neutrons and protons can be created and destroyed by reactions such as:

\[
\begin{align*}
\text{proton} + \text{neutrino} & \rightarrow \text{positron} + \text{antineutrino} \\
\text{neutron} + \text{antineutrino} & \rightarrow \text{electron} + \text{positron}
\end{align*}
\]

The masses of the neutron and proton are not exactly equal, but:

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.
1. $\phi p = \frac{\psi^4 - 1}{\psi^4}$

Then

\[ \phi = \frac{\psi}{\sqrt{4 - \psi^4}} \]

The mass density stored in radiation is proportional to the temperature to the fourth power: i.e., $\rho \propto T^4$.

(ii) The mass density stored in radiation is proportional to the temperature to the fourth power: $\rho \propto T^4$.

(i) For a radiation-dominated universe, the scale-factor:

\[ t \propto a(t) \]

(b) Explain briefly what is the cosmological constant.

(c) Find the time $t$ at which the scale factor $a(t)$ would collapse to 0.

\[ \frac{\psi^2}{\sqrt{4 - \psi^4}} = \frac{2}{\psi^4} \]

Use the Friedmann equation for a flat universe to express the Hubble expansion rate in terms of the temperature. $\alpha$ is the curvature parameter. Choose one.

1. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate $r \equiv \sqrt{r^2 + \psi^2}$. Then

\[ \frac{\partial}{\partial \psi} (r^2 \theta \phi) \]

(ii) The Robertson-Walker formula:

\[ ds^2 = -d\tau^2 + k \left( \frac{r^2}{c^2} - 1 \right) \frac{d\psi^2}{\psi^4} + \frac{r^2}{c^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

(i) For a radiation-dominated universe, the scale-factor:

\[ t \propto \sqrt{\psi} \]

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(i) For a radiation-dominated universe, the scale-factor:

\[ t \propto \sqrt{\psi} \]
8.286 QUIZ 2 REVIEW PROBLEMS, FALL 2011

**Problem 7: Lengths and Areas in a Two-Dimensional Metric**

The following problem was Problem 3, Quiz 2, 1994:

Suppose a two-dimensional space, described in polar coordinates, has a metric given by

\[ ds^2 = (1 + ar)dr^2 + r^2(1 + br)d\theta^2, \]

where \( a \) and \( b \) are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the \( \theta = 0 \) line to \( r = r_0 \), then moving at fixed \( r \) to the origin at fixed \( \theta \), and then moving back to the origin at fixed \( \theta \). The path is shown below:

Path is shown above: moving at fixed \( r \) to \( r = r_0 \), and then moving back to the origin at fixed \( \theta \). Then

\[ a) (10 \text{ points}) \text{ Find the total length of this path.} \]

\[ b) (15 \text{ points}) \text{ Find the area enclosed by this path.} \]

**Bonus Problem 7.1:**

Suppose that a photon leaves the origin of the coordinate system \( \psi = 0 \) at \( t = 0 \). How long will it take for the photon to return to the origin, considering the speed of light is the maximum speed in the universe? Express your answer as a function of the total time of travel, \( T \).

\[ T = \text{function of } T \]

**Problem 8: Geometry in a Closed Universe**

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with \( k = 1 \). The questions below all pertain to some fixed time \( t \), so the scale factor can be written simply as \( a \), dropping its explicit \( t \)-dependence.

A small rod has one end at the point \( (r = h, \theta = 0, \phi = 0) \) and the other end at the point \( (r = h, \theta = \Delta \theta, \phi = 0) \). Assume that \( \Delta \theta \ll 1 \).

\[ \text{Find an expression for the physical distance} \quad \text{travelled in time} \quad 0 < t < T \]}

**Problem 9: Metrics (25 points)**

The following problem was Problem 9, Quiz 2, 1994:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with \( k = 1 \). Suppose that a photon leaves the origin at \( t = 0 \). How long will it take for the photon to return to the origin, considering the speed of light is the maximum speed in the universe? Express your answer as a function of the total time of travel, \( T \).

\[ T = \text{function of } T \]

**Problem 10: Lengths and Areas in a Two-Dimensional Metric**

Suppose a two-dimensional space, described in polar coordinates, has a metric given by

\[ ds^2 = (1 + ar)dr^2 + r^2(1 + br)d\theta^2, \]

where \( a \) and \( b \) are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the \( \theta = 0 \) line to \( r = r_0 \), then moving at fixed \( r \) to \( \theta = \pi/2 \), and then moving back to the origin at fixed \( \theta \). The path is shown below:

Path is shown above: moving at fixed \( r \) to \( r = r_0 \), and then moving back to the origin at fixed \( \theta \). Then

\[ a) (10 \text{ points}) \text{ Find the total length of this path.} \]

\[ b) (15 \text{ points}) \text{ Find the area enclosed by this path.} \]
8.286 QUIZ 2 REVIEW PROBLEMS, FALL 2011 p. 15

(a) Find the physical distance \( \ell_p \) from the origin \((r=0)\) to the first end \((h,0,0)\) of the rod. You may find one of the following integrals useful:

\[
\int dr \sqrt{1 - r^2} = \sin^{-1} r
\]

\[
\int dr \frac{1}{1 - r^2} = \frac{1}{2} \ln \left( \frac{1 + r}{1 - r} \right)
\]

(b) Find the physical length \( s_p \) of the rod. Express your answer in terms of the scale factor \(a\), and the coordinates \(h\) and \(\Delta \theta\).

(c) Note that \(\Delta \theta\) is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance \(\ell_p\), the physical length \(s_p\), and the scale factor \(a\).

PROBLEM 9: THE GENERAL SPHERICALLY SYMMETRIC METRIC (20 points)

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three-dimensional space which is spherically symmetric about a particular point, coordinates can be found whose metric has the form

\[
ds^2 = dr^2 + \rho^2(r)[d\theta^2 + \sin^2 \theta d\phi^2]
\]

for some function \(\rho(r)\). The coordinates \(\theta\) and \(\phi\) have their usual ranges: \(\theta\) varies between 0 and \(\pi\), and \(\phi\) varies from 0 to \(2\pi\), where \(\phi = 0\) and \(\phi = 2\pi\) are identified.

Given this metric, consider the sphere whose outer boundary is defined by \(r = r_0\).

(a) Find the physical radius \(a\) of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate \(\sigma\) is introduced, where \(\sigma = \frac{r}{2}\). Express the metric in terms of this new variable.

PROBLEM 10: VOLUMES IN A ROBERTSON-WALKER UNIVERSE (20 points)

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

\[
ds^2 = a^2(t)\left\{dr^2 + \frac{1}{1 - Kr^2}r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right\}
\]

Calculate the volume \(V(\sigma)\) of the sphere described by

\[
\left\{(\phi, \theta, \phi + \epsilon \theta) : r = r_{\text{max}}\right\}
\]

The metric for a Robertson-Walker universe is given by

\[
ds^2 = a^2(t)\left\{dr^2 + \frac{1}{1 - Kr^2}r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right\}
\]

The physical length \(a\), and the scale factor \(a\), and the coordinates \(\theta\) and \(\phi\) are defined.

(a) Find the physical length \(a\) of the rod. Express your answer in terms of the scale factor \(a\) and the coordinates \(\theta\) and \(\phi\).

(b) Find the physical length \(a\) of the rod. Express your answer in terms of the scale factor \(a\) and the coordinates \(\theta\) and \(\phi\).

(c) Find the physical area of the surface of the sphere.

(d) Find the physical distance \(d\theta^2\) from the origin \((r=0)\) to the first end \((r=0)\) of the rod.

PROBLEM 11: THE SCHWARZSCHILD METRIC (25 points)

The following problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass \(M\) is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated \(A\) and \(B\), are located along the same radial line, with values of the coordinate \(r\) given by \(r_A\) and \(r_B\), respectively, with \(r_A < r_B\). You should assume that both observers lie outside the Schwarzschild horizon.
The geodesic is described by functions \( \theta \) and \( r \) in the form of a definite integral, making sure to specify the limits of integration. For the space described in part (b), consider a geodesic described by the usual geodesic equation, where the param- 

eater is the arc length measured along the curve. Use the general formula on 

defined integral, for the length 

\[ s(\theta) \]

and 

\[ s(r) \]

To show that the line 

\[ \theta = \frac{\pi}{2} \]

is a geodesic curve. Write explicitly both (i.e., for 

\( \theta \)) and 

\( r \)).

Consider first a two-dimensional space with coordinates \( \theta \) and 

\( r \).

The curved is given by 

\[ \theta \partial P \partial t + r \partial P = \frac{\partial P}{\partial x} \]

where \( \partial P \) is the interval \( \Delta \) of this curve. Does 

\[ \theta \partial P \partial t + r \partial P = \frac{\partial P}{\partial x} \]

hold on the Schwarzschild horizon, 

\[ A \equiv \frac{\partial P}{\partial x} \]

lies on the Schwarzschild horizon, so the Schwarzschild metric is valid for all \( t \). Now suppose that 

\[ A \equiv \frac{\partial P}{\partial x} \]

is the interval \( \Delta \) of this curve. Does 

\[ \theta \partial P \partial t + r \partial P = \frac{\partial P}{\partial x} \]

hold on the Schwarzschild horizon, 

\[ A \equiv \frac{\partial P}{\partial x} \]

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\[ A \equiv \frac{\partial P}{\partial x} \]

lies on the Schwarzschild horizon, so the Schwarzschild metric is valid for all \( t \). Now suppose that 

\[ A \equiv \frac{\partial P}{\partial x} \]
Consider a two-dimensional curved space described by

\[ \theta d\phi + \sin \phi d\xi = ds \]

Given this expression of the line element, the geodesic equation can be written as:

\[ \frac{\ddot{\xi}}{c^2} = -2 \frac{\dot{\xi}}{c} \left( \frac{\partial F}{\partial \xi} \right) \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]

We will assume that this metric is given, and that the following conditions hold:

\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

\[ \frac{d^2 \xi}{dt^2} + \frac{\partial F}{\partial \xi} = 0 \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]

We will assume that this metric is given, and that the following conditions hold:

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We will assume that this metric is given, and that the following conditions hold:

\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

\[ \frac{d^2 \xi}{dt^2} + \frac{\partial F}{\partial \xi} = 0 \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]

We will assume that this metric is given, and that the following conditions hold:

\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

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\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

\[ \frac{d^2 \xi}{dt^2} + \frac{\partial F}{\partial \xi} = 0 \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]

We will assume that this metric is given, and that the following conditions hold:

\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

\[ \frac{d^2 \xi}{dt^2} + \frac{\partial F}{\partial \xi} = 0 \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]

We will assume that this metric is given, and that the following conditions hold:

\[ k_n = 0, \quad k_\theta = k_\phi = 0 \]

where \( k_n, k_\theta, k_\phi \) are the curvatures in the normal, \( \theta \), and \( \phi \) directions, respectively. Hence, the geodesic equation can be written as:

\[ \frac{d^2 \xi}{dt^2} + \frac{\partial F}{\partial \xi} = 0 \]

where \( F \) is the potential function. The geodesic equation is therefore:

\[ \frac{d^2 \xi}{dt^2} + \left( \frac{\partial F}{\partial \xi} \right) = 0 \]
where 0 \leq \phi \leq \phi_0 \text{ and } 0 \leq \theta \leq \theta_0 \text{ and } 0 \leq r \leq r_0 \text{ and } -\infty \leq z \leq \infty.

\begin{align*}
\left[ \dd^2 \phi \dd^2 \theta + \dd \dd \right] + \dd \dd = 0
\end{align*}

The problem will concern the geodesics of the metric.

The problem will concern the geodesics of the metric.

The problem will concern the geodesics of the metric.

\text{(17 points)}

\text{EXTRA INFORMATION}

\text{PROBLEM 17: ROTATING FRAMES OF REFERENCE}

The problem will concern the geodesics of the metric.

The problem will concern the geodesics of the metric.

The problem will concern the geodesics of the metric.

\text{PROBLEM 17: ROTATING FRAMES OF REFERENCE}
**Problem 18: The Stability of Schwarzschild Orbits**

(a) Use the metric to find an expression for \( \frac{dt}{d\tau} \). You may allow your answer to contain \( \frac{dt}{d\tau} \) with respect to \( \tau \) and the derivative with respect to \( \tau \) of any coordinate, including \( \tau \), and the derivative with respect to \( r \) of the quantities indicated, and then ask yourself how this result can be used to work out the explicit form of the geodesic equation

\[
\frac{dr}{d\tau} = \frac{\frac{\partial g_{\mu\nu}}{\partial r} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}{\frac{\partial g_{\mu\nu}}{\partial \tau} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}.
\]

(b) Explicitly write the equation that results when the free index \( c \) in the geodesic equation is equal to 1, \( \mu = 1 \), \( r \) will not be a constant in this solution, will \( \beta = \tau \), \( \theta = \theta \), \( \phi = \phi \), (P18.3)

(c) Use the metric to find an expression for \( \frac{d\tau}{dt} \). You should use this result to find an explicit expression for \( r(t) \) for the case \( r = 2 \), corresponding to the coordinate \( \tau \) equal to 2, which is the Schwarzschild radius. That is, for the Schwarzschild geometry show that the smallest circular orbits are possible for \( r = 2 \).

(d) Use the metric to find an expression for \( \frac{d\mu}{d\tau} \). You must then find an expression for \( \frac{d\tau}{dt} \), in terms of \( \frac{d\mu}{d\tau} \) and \( \frac{dt}{d\tau} \), and the derivative \( \frac{d\mu}{d\tau} \) also depends on the constants \( g_{\mu\nu} \) and \( c \) of the nonzero entries in the matrix \( g_{\mu\nu} \). You may allow your answer to contain \( \frac{dt}{d\tau} \) with respect to \( \tau \) and the derivative \( \frac{d\mu}{d\tau} \) with respect to \( \tau \) of any coordinate, including \( \tau \), and the derivative with respect to \( r \) of the quantities indicated, and then ask yourself how this result can be used to work out the explicit form of the geodesic equation

\[
\frac{d\tau}{dt} = \frac{\frac{\partial g_{\mu\nu}}{\partial \tau} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}{\frac{\partial g_{\mu\nu}}{\partial \mu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}.
\]

(e) Explicitly write the equation that results when the free index \( c \) in the geodesic equation is equal to 1, \( \mu = 1 \), \( r \) will not be a constant in this solution, will \( \beta = \tau \), \( \theta = \theta \), \( \phi = \phi \), (P18.4)

(f) If you cannot answer part (a), you can introduce unspecified functions \( f, g \) into your answer. If \( f \) and \( g \) match the form of the metric, then

\[
\begin{align*}
\frac{dt}{d\tau} &= \frac{\partial g_{\mu\nu}}{\partial \tau} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \\
\frac{d\tau}{dt} &= \frac{\partial g_{\mu\nu}}{\partial \mu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}.
\end{align*}
\]
It is useful to consider $r$ and $\phi$ to be the independent variables, while $t$ is a dependent variable. Find an expression for $(\text{d}t/\text{d}\tau)^2$ in terms of $r$, $dr/d\tau$, $d\phi/d\tau$, $h(r)$, and $c$. Use this equation to simplify the expression for $d^2 r/d\tau^2$ obtained in part (a). The goal is to obtain an expression of the form

$$d^2 r/d\tau^2 = f_0(r) + f_1(r)(d\phi/d\tau)^2,$$

where the functions $f_0(r)$ and $f_1(r)$ might depend on $R$ or $c$, and might be positive, negative, or zero. Note that the intermediate steps in the calculation involve a term proportional to $(dr/d\tau)^2$, but the net coefficient for this term vanishes.

To understand the orbit we will also need the equation of motion for $\phi$. Evaluate the geodesic equation (P18.2) for $\mu = \phi$, and write the result in terms of the quantity $L$, defined by

$$L \equiv r^2 (d\phi/d\tau).$$

Finally, we come to the question of stability. Substituting Eq. (P18.4) into Eq. (P18.3), the equation of motion for $r$ can be written as

$$d^2 r/d\tau^2 = f_0(r) + f_1(r)L^2 r^4.$$

Now consider a small perturbation about the circular orbit at $r = r_0$, and write an equation that determines the stability of the orbit. (That is, if some external force gives the orbiting body a small kick in the radial direction, how can you determine whether the perturbation will lead to stable oscillations, or whether it will start to grow?) You should express the stability requirement in terms of the unspecified functions $f_0(r)$ and $f_1(r)$. You are NOT asked to carry out the algebra of inserting the explicit forms that you have found for these functions.

---

**Problem 19: Pressure and Energy Density of Mysterious Stuff**

In Lecture Notes 10, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of mysterious stuff, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume $V$ is increased.

(a) (15 points) Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount $\Delta U$ by which the energy inside the piston changes when the volume is enlarged by $\Delta V$. Define $\Delta U$ to be positive if the energy increases.

(b) (5 points) How much work $\Delta W$ is done by the agent that pulls out the piston?

(c) (5 points) Use your results from (a) and (b) to express the pressure $p$ of the mysterious stuff in terms of its energy density $u$. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)
Problem 1: Did You Do the Reading?

(a) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(b) The correct answer is (i). The others are all real reasons why it’s hard to measure, although Weinberg’s book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(c) The most obvious answers would be proton, neutron, and pi meson. However, there are many other possibilities, including many that were not mentioned by Weinberg. See Weinberg chapter 7, pages 136-8.

(d) The correct answers were the neutrino and the antiproton. The neutrino was first hypothesized by Wolfgang Pauli in 1932 (in order to explain the kinematics of beta decay), and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.

(e) The correct answers were (ii), (v) and (vi). The others were incorrect for the following reasons:

(i) The earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the 3◦K radiation could have been observed “long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s.” To Weinberg, however, the historically interesting question is not when the radiation could have been observed but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: “It was extraordinarily difficult for physicists to take seriously any theory of the early universe.”

PROBLEM 2: Did You Do the Reading?

In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a model of the universe that was consistent with the current observational data. The model assumed that the universe was in thermal equilibrium, and that the particles were in thermal equilibrium with the radiation. The model was able to explain the observed anisotropies in the cosmic microwave background, which are a result of the large-scale structure of the universe.

(a) 1950s.

(b) The correct answer was that the universe was hotter in the past.

(c) The correct answer was that the universe started with nearly equal densities of protons and neutrons.

(d) The correct answer was that the universe started with mainly alpha particles.

(e) The correct answer was that the conversion of neutrons into protons took place through collisions with electrons, positrons, neutrinos, and antineutrinos.

(f) The correct answer was that the ratio of photons to nuclear particles in the early universe is now believed to be about 1:10^3, not 1:10^9 as Alpher and Herman concluded.

(g) The correct answer was that the neutron was thought to be absolutely stable, not that the neutron could decay.

(h) In Weinberg’s book, the neutrino was first hypothesized by Wolfgang Pauli in 1932, and first detected in the 1950s.

(i) In Weinberg’s book, the antiproton was first hypothesized by W. E. Lamb, Jr. in 1947, and first detected in the 1950s.

(j) In Weinberg’s book, the CMB was first hypothesized by E. M. Purcell in 1946, and first detected in the 1960s.

(k) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(l) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(m) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(n) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(o) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(p) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(q) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(r) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(s) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(t) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(u) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(v) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(w) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(x) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(y) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

(z) In Weinberg’s book, the first detection of the CMB by radio astronomers was in 1965.

PROBLEM 3: Did You Do the Reading?

(a) 1950.

(b) 1950.

(c) 1950.

(d) 1950.

(e) 1950.

(f) 1950.

(g) 1950.

(h) 1950.

(i) 1950.

(j) 1950.

(k) 1950.

(l) 1950.

(m) 1950.

(n) 1950.

(o) 1950.

(p) 1950.

(q) 1950.

(r) 1950.

(s) 1950.

(t) 1950.

(u) 1950.

(v) 1950.

(w) 1950.

(x) 1950.

(y) 1950.

(z) 1950.
Electric Charge:
(i) $\sim 10^{-9}$
(ii) $\sim 1000$
(iii) $\sim 1$
(iv) $\sim 10^{-6}$
(v) either zero or negligible

Baryon Number:
(i) $\sim 10^{-20}$
(ii) $\sim 10^{-9}$
(iii) $\sim 10^{-6}$
(iv) $\sim 1$
(v) anywhere from $10^{-5}$ to 1

Lepton Number:
(i) $\sim 10^9$
(ii) $\sim 1000$
(iii) $\sim 1$
(iv) $\sim 10^{-6}$
(v) could be as high as $\sim 1$, but is assumed to be very small

(F) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{neutron} + \text{antineutrino} \leftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \leftrightarrow \text{positron} + \text{proton}$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \leftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \leftrightarrow \text{electron} + \text{positron}$$

(e) 12 points. During the period labeled “thermal equilibrium” the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.
(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.
(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.
(D) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \leftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \leftrightarrow \text{positron} + \text{proton}$$

(F) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \leftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \leftrightarrow \text{electron} + \text{proton}$$

(c) 12 points. The figure below comes from Weinberg's Chapter 5, and is labeled "The Shifting Neutron-Proton Balance.

The figure shows the change in neutron and proton fractions over time. The neutron fraction decreases over time, while the proton fraction increases. The curve indicates that the system is moving towards thermal equilibrium. The figure is labeled "Thermal equilibrium."
The masses of the neutron and proton are not exactly equal, but instead
(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).
(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).
(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).
(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.
(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.
(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

During the period labeled "era of nucleosynthesis," (choose one:)
(A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.
(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.
(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half remain free.
(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.
(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

Students who described chains different from those of Weinberg, but that can still take place (in general with different probabilities) for example:

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place in general with different probabilities. For example:

\[ p + n \rightarrow H_2 + \gamma H_2 \]  
\[ H_2 + n \rightarrow H_3 + \gamma H_3 \]  
\[ p \rightarrow He_4 + \gamma. \]

The deuterium bottleneck is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium (\( H_2 \)) plays a crucial role in the nuclear reactions. Therefore, different chains of reactions can take place in general with different probabilities.

Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.
Numerically, $t \approx 1.039967 \alpha$.

We have

$$\alpha = \frac{\theta}{\sqrt{\frac{1}{2} \left( 1 - \cosh^{-1} \theta \right)}}$$

We can use these results in the first equation to solve for $1 = 0.9567$. Nothing that

$\cosh^{-1} 2 = \theta = 0.9567$

$\cosh^{-1} \left( \frac{\theta}{\sqrt{\frac{1}{2} \left( 1 - \cosh^{-1} \theta \right)}} \right) = \theta$

Evaluating the second of these equations at $\theta = 0.9567$ yields a solution for $\theta$.

$$\theta = \frac{\theta}{\sqrt{\frac{1}{2} \left( 1 - \cosh^{-1} \theta \right)}}$$

Pronounce parametric equations:

The evolution of an open, matter-dominated universe is described by the following parametric equations:

**Problem 4: Evolution of an Open Universe**

Solution written by Daniele Bertolini.

Using the Friedmann equation with $a(t)$, for some constant $\rho$, we get for the Hubble expansion rate:

$$\frac{\dot{a}}{a} = \frac{H}{t}$$

where we need the fact that $H$ = 2.0 x 10$^{-20}$ s$^{-1}$ and $\frac{\dot{a}}{a} = 2.0$.

Temperature is expressed in degree Kelvin (K). In detail, we see:

$$T = T_{\text{nucl}} =$$

If we substitute the given numerical values

$$T = T_{\text{nucl}} =$$

we get:

$$T_{\text{nucl}} = 2.0 \times 10^9 K$$

and $\frac{\dot{a}}{a} = 2.0 \times 10^2 s^{-1}$.

By using the Friedmann equation with $a(t)$, we get:

$$\frac{\dot{a}}{a} = \frac{H}{t} = \frac{H}{t}$$

We can use these results in the first equation to solve for $\theta$.

$$\frac{\dot{a}}{a} = \frac{H}{t} = \frac{H}{t}$$

For some constant $\rho$, we get for the Hubble expansion rate:

$$\frac{\dot{a}}{a} = \frac{H}{t}$$

where $H$ = 2.0 x 10$^{-20}$ s$^{-1}$ and $\frac{\dot{a}}{a} = 2.0$.

Notice that the units correctly combine to give $H$ = 2.0 x 10$^{-20}$ s$^{-1}$ and $\frac{\dot{a}}{a} = 2.0$.

The temperature is expressed in degree Kelvin (K). In detail, we see:

$$T = T_{\text{nucl}} =$$

If we substitute the given numerical values

$$T = T_{\text{nucl}} =$$

we get:

$$T_{\text{nucl}} = 2.0 \times 10^9 K$$

and $\frac{\dot{a}}{a} = 2.0 \times 10^2 s^{-1}$.

By using the Friedmann equation with $a(t)$, we get:

$$\frac{\dot{a}}{a} = \frac{H}{t} = \frac{H}{t}$$

We can use these results in the first equation to solve for $\theta$.

$$\theta = \frac{\theta}{\sqrt{\frac{1}{2} \left( 1 - \cosh^{-1} \theta \right)}}$$

Pronounce parametric equations:

The evolution of an open, matter-dominated universe is described by the following parametric equations:
PROBLEM 5: ANTICIPATING A BIG CRUNCH

The critical density is given by
\[ \rho_c = \frac{3}{8\pi G}, \]
so the mass density is given by
\[ \rho = \Omega_0 \rho_c = 2 \rho_c = \frac{3}{4\pi G}, \]
where \( \Omega_0 \) is the density parameter.

Substituting this relation into
\[ H_0 = 8\pi G \rho - \frac{k}{a^2}, \]
we find
\[ H_0 = 2H_0 - \frac{k}{a^2}, \]
from which it follows that
\[ a \sqrt{k} = c H_0. \]

Now use
\[ \alpha = 4\pi \rho a^3 / 2 c^2. \]
Substituting the values we have from Eqs. (S5.1) and (S5.2) for \( \rho \) and \( a / \sqrt{k} \), we have
\[ \alpha = c H_0. \]

To determine the value of the parameter \( \theta \), we use
\[ \sqrt{k} = \alpha (1 - \cos \theta), \]
which when combined with Eqs. (S5.2) and (S5.3) implies that
\[ \cos \theta = 0. \]

The equation \( \cos \theta = 0 \) has multiple solutions, but we know that the \( \theta \)-parameter for a closed matter-dominated universe varies between 0 and \( \pi \) during the expansion phase of the universe. Within this range, \( \cos \theta = 0 \) implies that \( \theta = \pi/2 \). Thus, the age of the universe at the time these measurements are made is given by
\[ t = \alpha c (\theta - \sin \theta) = \frac{1}{H_0} \left( \frac{\pi}{2} - 1 \right), \]
which is the total lifetime of the closed universe.

The time remaining before the universe reaches the critical density is given by
\[ \frac{\rho}{\rho_c} = \frac{\alpha}{\alpha_0}, \]
so the mass density is given by
\[ \rho = \frac{\alpha}{\alpha_0} \rho_c = \frac{3}{8\pi G}. \]

The critical density is given by
\[ \rho_c = \frac{3}{8\pi G}. \]
Problem 7: Lengths and Areas in a Two-Dimensional Signal Metric

Due only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.

Consider only half-way back to the starting point.
Note that as \( a \to \infty \), this approaches the Euclidean result, \( \theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla \).


\[
\frac{(n/d_j) \sin n}{d_j} = \theta \nabla
\]

This is the answer to the question, and then solving for \( \theta \), one has
\[
\theta \nabla \eta \nabla = d_s
\]

\[
\eta \frac{d}{dp} = \eta
\]

From part (a), one has
\[
V = \int \frac{d}{dp} \eta
\]

So
\[
V = \int \frac{d}{dp} \eta
\]

Note that as \( a \to \infty \), this approaches the Euclidean result, \( \theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla \).

To find the area, it is best to divide the region into concentric strips as shown:

\[
\eta p (\eta s) = \eta p
\]

The area is then
\[
\eta p (\eta s + 1) = \eta p
\]

The length of the strip is calculated the same way as \( S \) in part (a):
\[
\eta p (\eta s + 1) = \eta p
\]

Note that the strip has a coordinate width of \( dr \), but the distance across the width of the strip is determined by the metric to be
\[
(1 + a r)
\]

The length of the strip is calculated the same way as
\[
\theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla
\]

So
\[
\varepsilon (q + n) \frac{c}{\varepsilon} + \frac{c}{\varepsilon} \frac{c}{\varepsilon} =
\]

Note that as \( a \to \infty \), this approaches the Euclidean result, \( \theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla \).

Finally, the third segment is identical to the first, so \( S_3 = S \). The total length

Note that as \( a \to \infty \), this approaches the Euclidean result, \( \theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla \).

So
\[
\int \frac{d}{dp} \eta
\]

Note that as \( a \to \infty \), this approaches the Euclidean result, \( \theta \nabla \cdot \frac{d\mathbf{r}}{ds} = \theta \nabla \).
\[ \phi \rho \text{sin} \theta \text{cos} \phi \text{sin} r \theta d \phi + \phi \rho d \phi + \rho d \rho = \varepsilon \rho \]

Therefore, the volume of the shell is
\[ \rho d \rho \varepsilon \rho = \varepsilon \rho \]

The total volume is then obtained by integration:
\[ \rho \varepsilon \rho d \rho = \Lambda \rho \]

The radial path is at a constant value of \( \rho \), so from the center to the boundary of the sphere, the length of a path is just the integral of \( \rho \) from 0 to \( \rho \).

\( \rho \) The final path is at a constant value of \( \rho \) and \( \rho \), so from the center to the boundary of the sphere, the length of a path is just the integral of \( \rho \) from 0 to \( \rho \).

As in Problem 2 of Problem Set 4, we can integrate this formula into the volume into

\[ \frac{\phi \rho \text{sin} \theta \text{cos} \phi \text{sin} r \theta d \phi + \phi \rho d \phi + \rho d \rho}{\varepsilon \rho} = \Lambda \rho \]

The product of differential length elements corresponding to infinitesimal changes in the coordinates \( \rho \), \( \theta \), and \( \phi \) equals the differential volume element \( dV \). Therefore, the function \( ds \) has length
\[ ds = \sqrt{\left( \frac{\partial x}{\partial \rho} \right)^2 + \left( \frac{\partial x}{\partial \theta} \right)^2 + \left( \frac{\partial x}{\partial \phi} \right)^2} \]

The total volume is then obtained by integration over the range of the coordinates \( \rho \), \( \theta \), and \( \phi \), so
\[ \int_0^\rho \int_0^\pi \int_0^{2\pi} \varepsilon \rho \sin \theta \cos \phi \sin \rho \theta d \phi d \theta d \rho = \Lambda \rho \]

\[ \boxed{\varepsilon \rho \sin \theta \cos \phi \sin \rho \theta d \phi + \phi \rho d \phi + \rho d \rho = \varepsilon \rho} \]

The metric is then obtained by integrating over the range of the coordinate \( \rho \):
\[ \boxed{\phi \rho \sin \theta \cos \phi \sin \rho \theta d \phi + \phi \rho d \phi + \rho d \rho = \varepsilon \rho} \]
The total volume is then
\[ V = \int dV = a^3 (t) \int r_{\text{max}}^0 r^2 dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2 \sin \theta \sqrt{1 - kr^2}. \]

We can do the angular integrations immediately:
\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^r (t) r^2 = \pi a^3 (t) \int_0^{r_{\text{max}}} r^2 dr. \]

**Pedagogical Note:** If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:

The cell includes the volume lying between \( r \) and \( r + dr \), between \( \theta \) and \( \theta + d\theta \), and between \( \phi \) and \( \phi + d\phi \). In the limit as \( dr, d\theta, \) and \( d\phi \) all approach zero, the cell approaches a rectangular solid with sides of length:

\[ ds_1 = a (t) dr \sqrt{1 - kr^2}, \]
\[ ds_2 = a (t) r \sin \theta d\theta, \]
\[ ds_3 = a (t) rd\phi. \]

Here each \( ds \) is calculated by using the metric to find \( ds^2 \), in each case approximating the infinitesimal volume element as
\[ dV = ds_1 ds_2 ds_3, \]
resulting in the answer above. The derivation relies on the orthogonality of the \( dr, d\theta, \) and \( d\phi \) directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as \( dr d\theta \).

**Problem 11: The Schwarzschild Metric**

(a) The Schwarzschild horizon is the value of \( r \) for which the metric becomes singular. Since the metric contains the factor \( \frac{1}{\sqrt{1 - 2GM/rc^2}} \), it becomes singular at \( R_S = 2GM/c^2 \).

(b) The separation between \( A \) and \( B \) is purely in the radial direction, so the proper length of a segment along the path joining them is given by
\[ ds^2 = \frac{1}{\sqrt{1 - 2GM/rc^2}} - \frac{(t_e)^2}{(r_e)^2 - 1} dr^2, \]
so
\[ ds = dr \sqrt{1 - 2GM/rc^2}. \]

**Pedagogical Note:** If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram above:

We can do the angular integrations immediately:
\[ \frac{\tau_{\phi}}{\theta \sin \phi} \int_{\theta = \phi} \int_{\phi = \phi} (t) r^2 d\phi = \Lambda \int \int \int (t) r^2 = \Lambda. \]
The proper distance from $A$ to $B$ is obtained by adding the proper lengths of all the segments along the path, so

$$s_{AB} = \int r_B r_A \sqrt{1 - \frac{2GM}{r} c^2}.$$ 

**Extension:** The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for $s_{AB}$ as

$$s_{AB} = \int r_B r_A \sqrt{r dr} \sqrt{r - R_S}.$$ 

Then introduce the hyperbolic trigonometric substitution

$$r = R_S \cosh^2 u.$$ 

One then has

$$\sqrt{r - R_S} = \sqrt{R_S \sinh u} dr = 2R_S \cosh u \sinh u du,$$ 

and the indefinite integral becomes

$$\int \sqrt{r dr} \sqrt{r - R_S} = 2R_S \int \cosh^2 u du = R_S \left( u + \frac{1}{2} \sinh 2u \right) = R_S \left( u + \sinh u \cosh u \right).$$ 

Thus,

$$s_{AB} = R_S \left[ \sinh \left( \frac{1}{2} \sqrt{r_B R_S - 1} \right) - \sinh \left( \frac{1}{2} \sqrt{r_A R_S - 1} \right) \right].$$

**c)** A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM}{r} c^2\right)c^2 dt^2,$$ 

so

$$d\tau = \sqrt{1 - \frac{2GM}{r} c^2} dt.$$ 

The reading on the observer's clock corresponds to the proper time interval $d\tau$, so the corresponding interval of the coordinate $t$ is given by

$$\Delta t_A = \Delta \tau A \sqrt{1 - \frac{2GM}{r_A} c^2}.$$ 

**d)** Since the Schwarzschild metric does not change with time, each pulse leaving $A$ will travel the same length of time to reach $B$, thus, the pulses emitted by $A$ will arrive at $B$ with a time coordinate spacing $\Delta t_B = \Delta t_A = \Delta \tau A \sqrt{1 - \frac{2GM}{r_A} c^2}$. The clock at $B$, however, will read the proper time and not the coordinate time. Thus,

$$\Delta \tau_B = \sqrt{1 - \frac{2GM}{r_B} c^2} \Delta t_B = \sqrt{1 - \frac{2GM}{r_A} c^2} \Delta \tau_A \sqrt{1 - \frac{2GM}{r_B} c^2}.$$ 

**e)** From parts (a) and (b), the proper distance between $A$ and $B$ can be rewritten as

$$s_{AB} = \int r_B r_A \sqrt{r dr} \sqrt{r - R_S}.$$ 

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of $r = R_S$, say $R_S < r < R_S + \epsilon$. For this
\[ (\gamma/\lambda)_{\text{tan}} = \frac{(\gamma)^x}{(\gamma)^a} \quad \text{tan} = (\gamma) \theta \]

\[ I + \gamma \sqrt{A} = (\gamma)^x \sqrt{r} + (\gamma)^x \sqrt{A} = (\gamma)^x \]

The parameterization is not unique, because one can choose \( \alpha = 0 \) to represent any point along the curve. (Contrary to the desired polar coordinates, the parameterization is not unique, because one can choose \( \alpha = 0 \) to represent any point along the curve.)

The geodesic equation for a curve \( x^i(\lambda) \), where the parameter \( \lambda \) is the arc length along the curve, can be written as

\[ \frac{d}{d\lambda} \left\{ g^{ij} \frac{dx_j}{d\lambda} \right\} = \frac{1}{2} \left( \partial_i g_{k\ell} \right) \frac{dx_k}{d\lambda} \frac{dx_\ell}{d\lambda} \]

Here the indices \( j, k, \text{and } \ell \) are summed from 1 to the dimension of the space, so there is one equation for each value of \( i \).

\[ (a) \quad \text{The metric is given by} \]

\[ ds^2 = g_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2, \]

so

\[ g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{r\theta} = g_{\theta r} = 0. \]

First taking \( i = r \), the nonvanishing terms in the geodesic equation become

\[ \frac{d}{d\lambda} \left( g_{rr} \frac{dr}{d\lambda} \right) = \frac{1}{2} \left( \partial_r g_{\theta\theta} \right) \frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda}, \]

which can be written explicitly as

\[ \frac{d^2 r}{d\lambda^2} = \frac{r}{2} \left( \frac{d\theta}{d\lambda} \right)^2. \]

For \( i = \theta \), one has the simplification that \( g_{ij} \) is independent of \( \theta \) for all \( i, j \).

So

\[ \left( \frac{\partial}{\partial \theta} \right) \left( g^{rr} \right) = \frac{\partial}{\partial \theta} \left( g^{rr} \right) = 0 \]

or

\[ \left( \frac{\partial}{\partial \theta} \right) \left( g^{\theta \theta} \right) = \frac{\partial}{\partial \theta} \left( g^{\theta \theta} \right) = 0 \]

which can be written explicitly as

\[ \frac{\partial}{\partial \theta} \left( g^{\theta \theta} \right) = \frac{\partial}{\partial \theta} \left( g^{\theta \theta} \right) = 0. \]
\[ \omega = \rho \int_{\pi}^{0} \frac{(v/\lambda) + \lambda}{1/\lambda} \rho d\lambda \int_{0}^{0} \frac{d\rho}{\rho} = V \]

but \( \int_{0}^{0} \rho d\lambda \int_{0}^{0} \frac{d\rho}{\rho} = V \).

The area is then

\[ \theta \rho \frac{z(t \sin \theta + 1)}{1/\lambda} \]

which also checks. Since the metric does not contain a term in \( \theta \), the angular coordinate \( \theta \) is in the interval \( 0 \leq \theta \leq \pi \).

Since \( \theta \) runs from 0 to \( \pi \), the curve is swept out.

Since the metric does not contain a term in \( \rho \), the radial coordinate \( \rho \) is in the interval \( 0 \leq \rho \leq s \).

Since the metric does not contain a term in \( \lambda \), then the region can be treated as a rectangle. The side along which \( \rho \) varies has length \( d\theta \). The side along which \( \lambda \) varies has length \( d\rho \).

Thus, if one considers a small region in which \( \rho \) and \( \lambda \) change by \( d\lambda \) and \( d\rho \), then the region can be treated orthogonally.

Thus, if one considers a small region in which \( \lambda \) is in the interval \( 0 \leq \lambda \leq 2\lambda \), then the area can be treated as a rectangle.

The side along which \( \lambda \) varies has length \( d\rho \).

Thus, if one considers a small region in which \( \rho \) is in the interval \( 0 \leq \rho \leq s \), then the area can be treated orthogonally.

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Thus, if one considers a small region in which \( \lambda \) is in the interval \( 0 \leq \lambda \leq 2\lambda \), then the area can be treated as a rectangle.
\[ 0 = \epsilon^x \]
\[ \phi \sin \phi = \epsilon^x \]
\[ \phi \cos \phi = x = \epsilon^x \]

Thus, the equator is described by the curve \( (\phi/\epsilon)^x \) where \( \phi = \psi \) and \( \theta = \pi/2 \).

The equator is then described by the curve

\[ \sin \theta = y = x \]

The equator is brought to the right-hand side, giving

\[ \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \theta}{\partial \phi} + \frac{\partial \theta}{\partial \phi} \frac{\partial^2 \theta}{\partial \phi^2} = \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi} \]

Inserting this expansion into the boxed equation above, the first term can be

\[ \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \theta}{\partial \phi} + \frac{\partial \theta}{\partial \phi} \frac{\partial^2 \theta}{\partial \phi^2} = \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi} \]

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

\[ \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \theta}{\partial \phi} + \frac{\partial \theta}{\partial \phi} \frac{\partial^2 \theta}{\partial \phi^2} = \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi} \]

Taking the square of the right-hand side, we have

\[ \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \theta}{\partial \phi} + \frac{\partial \theta}{\partial \phi} \frac{\partial^2 \theta}{\partial \phi^2} = \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi} \]

so we substitute the above from above, we have

\[ \frac{\partial^2 \psi}{\partial \phi^2} \frac{\partial \theta}{\partial \phi} + \frac{\partial \theta}{\partial \phi} \frac{\partial^2 \theta}{\partial \phi^2} = \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi} \]

so the matrix is diagonal. For \( \mu = 1 \), the geodesic equation becomes

\[ \frac{d}{d \phi} \frac{d}{d \phi} \frac{\partial g_{ij}}{\partial \phi} \]

The nonzero metric coefficients are given by

\[ \frac{\partial g_{ij}}{\partial \phi} = \frac{\partial g_{ij}}{\partial \phi} \]

so the substitution is necessary, but it can be done by using

\[ \frac{d}{d \phi} \frac{d}{d \phi} \frac{\partial g_{ij}}{\partial \phi} = V \]

You were not asked to carry out the integration, but it can be done by using

\[ \frac{1}{(u/r)} \frac{d}{d u} \frac{d}{d u} \frac{\partial g_{ij}}{\partial \phi} = V \]

So
Now introduce a primed coordinate system that is related to the original system by a rotation in the $y$-$z$ plane by an angle $\alpha$:

\[
\begin{align*}
  x' &= x \\
y' &= y' \cos \alpha - z' \sin \alpha \\
z' &= z' \cos \alpha + y' \sin \alpha.
\end{align*}
\]

The rotated equator, which we seek to describe, is just the standard equator in the primed coordinates:

\[
\begin{align*}
x' &= r \cos \psi \\
y' &= r \sin \psi \\
z' &= 0.
\end{align*}
\]

Using the relation between the two coordinate systems given above,

\[
\begin{align*}
x &= r \cos \psi \\
y &= r \sin \psi \cos \alpha \\
z &= r \sin \psi \sin \alpha.
\end{align*}
\]

Using again the relations between polar and Cartesian coordinates,

\[
\begin{align*}
  \cos \theta &= \frac{z}{r} \\
  \tan \phi &= \frac{y}{x} \\
  \tan \psi &= \frac{x'}{z'} \\
\end{align*}
\]

(b) A segment of the equator corresponding to an interval $d\psi$ has length $a d\psi$, so the parameter $\psi$ is proportional to the arc length. Expressed in terms of the metric, this relationship becomes

\[
ds^2 = g_{ij} dx^i d\psi dx^j d\psi = a^2 d\psi^2.
\]

Thus the quantity

\[
A \equiv g_{ij} dx^i d\psi dx^j d\psi
\]

is equal to $a^2$, so the geodesic equation (5.50) reduces to the simpler form of Eq. (5.52). (Note that we are following the notation of Lecture Notes 5, except that the variable used to parameterize the path is called $\psi$ rather than $\lambda$ or $s$.

Although $A$ is not equal to 1 as we assumed in Lecture Notes 5, it is easily seen that the quantity used to parameterize the path is actually $A$, not $a$. Thus the geodesic equation remains the same.

For this problem the metric has only two nonzero components:

\[
g_{\theta \theta} = a^2, \quad g_{\phi \phi} = a^2 \sin^2 \theta.
\]

Taking $i = \theta$ in the geodesic equation,

\[
\frac{d}{d\psi} \left( g_{\theta \theta} \frac{d\theta}{d\psi} \right) = \frac{1}{2} \frac{\partial}{\partial \theta} \left( g_{\phi \phi} \frac{d\phi}{d\psi} \right) \frac{d\phi}{d\psi} \frac{d\phi}{d\psi} = \sin \theta \cos \theta \left( \frac{d\phi}{d\psi} \right)^2.
\]

Taking $i = \phi$,

\[
\frac{d}{d\psi} \left( a^2 \sin^2 \theta \frac{d\phi}{d\psi} \right) = 0 = \frac{d}{d\psi} \left( \sin^2 \theta \frac{d\phi}{d\psi} \right).
\]
\[ d\theta \left( \frac{dp}{p} \right) \left( \frac{\varepsilon^2 - 1}{(\varepsilon)^2} - 1 \right) \right) = \frac{dp}{p} = \varepsilon p. \]

Similarly
\[ \tan \phi = \tan \psi \cos \alpha = \left( \frac{dp}{p} \right) \sec \phi \frac{p}{p} = \sec \psi \frac{p}{p}. \]

To verify the geodesic equations of part (b), it is easiest to check the second one first:
\[ \sin^2 \theta \left( \frac{dp}{p} \right) = \left( 1 - \sin^2 \psi \sin^2 \alpha \right) \cos \alpha, \]
\[ \frac{d\tau}{d\psi} = \frac{1}{\varepsilon \sin \phi \sin \theta} \left( \frac{dp}{p} \right) \varepsilon^2 = \int \left( \frac{dp}{p} \right) \varepsilon^2. \]

After some straightforward algebra, one finds
\[ \frac{d^2 \theta}{d\psi^2} = \sin \psi \sin \alpha \cos^2 \alpha \left[ 1 - \sin^2 \psi \sin^2 \alpha \right]^{3/2}. \]

So the left- and right-hand sides are equal:
\[ \frac{\varepsilon}{\varepsilon} \left[ \frac{\varepsilon - 1}{\varepsilon^2} \right] = \frac{\varepsilon}{\varepsilon} \left[ \frac{dp}{p} \right] \theta \sin \phi \sin \theta. \]

\[ \frac{d}{dt} \sqrt{g} = \frac{d}{dt} \sqrt{g(\tau)} \]

This formula is true for each possible value of \( \tau \) for which \( \sqrt{g(\tau)} \) is nonzero, so \( \sqrt{g(\tau)} \) must be zero for \( \tau \) at which \( g(\tau) = 0 \). If \( g(\tau) \) is nonzero, the right-hand side in the bottom equation will be zero, and the upper equation will be satisfied. Hence, the only \( \tau \) at which \( g(\tau) = 0 \) is \( \tau = \tau_0 \). If \( \tau = \tau_0 \), there is no motion in the \( \phi \) or \( \theta \) directions. However, the
Multiply the geodesic equation by $m$, and then use the above result to rewrite it as

\[ \frac{d}{d\tau} \left( a p \sqrt{1 - r^2} \right) = ma^2 r (1 - r^2)^{2/3} \left( \frac{dr}{d\tau} \right)^2. \]

Expanding the left-hand side,

\[ \text{LHS} = \frac{d}{d\tau} \left( a p \sqrt{1 - r^2} \right) = \sqrt{1 - r^2} d \left( a p \right) + ap r (1 - r^2)^{3/2} \frac{dr}{d\tau} = ma^2 r (1 - r^2)^{2/3} \left( \frac{dr}{d\tau} \right)^2. \]

Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

\[ \sqrt{1 - r^2} d \left( a p \right) = 0. \]

Multiplying by $\sqrt{1 - r^2}$, one has the desired result:

\[ \frac{d}{d\tau} \left( a p \right) = 0 = \frac{1}{a} \frac{\partial}{\partial \vartheta} \left( \frac{\vartheta}{a} \right). \]

To find the length of the radial line shown, one must integrate this expression from the value

\[ \left. \frac{\partial}{\partial \vartheta} \right|_0 \left( \frac{\vartheta}{a} \right) = \int_0^a = \frac{\pi}{2} \sqrt{a}. \]

For $\vartheta = \text{constant}$, the expression for the metric reduces to

\[ ds^2 = u d\theta^2 \Rightarrow ds = \sqrt{u} d\theta. \]

Since $\theta$ runs from 0 to $2\pi$, and for the circumference

\[ \vartheta \vartheta \vartheta = sp \quad \Leftrightarrow \quad \vartheta \vartheta \vartheta \vartheta = \frac{n}{a} \]

you were not expected to do it, but the integral can be carried out, giving

\[ \left. \frac{\partial}{\partial \vartheta} \right|_0 \left( \frac{\vartheta}{a} \right) = \frac{\pi}{2} \sqrt{a}. \]

So at the center, which is 0, to the value of $n$ at the outer edge, which is $a$. Any terms that would be proportional to $du^2$ or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case

\[ \left. \frac{\partial}{\partial \vartheta} \right|_0 \left( \frac{\vartheta}{a} \right) = \int_0^a = \frac{\pi}{2} \sqrt{a}. \]

For $n = \vartheta = \text{constant}$, the expression for the metric reduces to

\[ ds^2 = \frac{a}{4} u \left( a - u \right) = \Rightarrow ds = \frac{1}{2} \sqrt{a} u \left( a - u \right) du. \]

Since $\theta$ runs from 0 to $2\pi$, and for the circumference

\[ \vartheta \vartheta \vartheta = 0 \]
\( I = \varepsilon^2 p \) 

\( I = \varepsilon \phi^2 \) 

\( I = \varepsilon^2 \phi^2 \) 

\( I = \varepsilon \phi^2 \) 

\( I = \varepsilon \phi^2 \) 

The metric coefficients are then just read off from this expression: 

\[ \theta \partial \theta = 0 \]

The metric was given as 

\[ I = \varepsilon^4 p \]

\[ I = \varepsilon \phi^4 \]
adapting to the rotating cylindrical coordinate system,
\[ \tau \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

Note that this equation is really just
\[ \left[ \frac{4 \partial}{\partial \tau} + (\frac{\partial}{\partial \tau} \phi \phi) \right] \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

one has
\[ \left[ \frac{4 \partial}{\partial \tau} + (\frac{\partial}{\partial \tau} \phi \phi) \right] \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

Then using
\[ \left[ \frac{4 \partial}{\partial \tau} + (\frac{\partial}{\partial \tau} \phi \phi) \right] \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

Note that one cannot recover the centrifugal force by itself since both \( \phi \) and \( \phi^2 \) are functions of \( \tau \).

Therefore, in the general expression, \( \phi \) and \( \phi^2 \) are functions of \( \tau \).

Note that the second term in the right-hand side is really the sum of the contributions
\[ \left( \frac{4 \partial}{\partial \tau} \phi \phi \right) \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

so that
\[ \left[ \frac{4 \partial}{\partial \tau} \phi \phi \right] \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

The right-hand side is similar when we sum over \( \lambda \) and \( \sigma \) without changing the order of the term proportional to \( \phi \).

Substituting (c)
\[ \phi = 0 \]

the term proportional to \( \phi \) is the centrifugal force and
\[ \frac{4 \partial}{\partial \tau} \phi \phi \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

Note that the left-hand side is really 1/2, because the
\[ \frac{4 \partial}{\partial \tau} \phi \phi \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

expression
\[ \frac{4 \partial}{\partial \tau} \phi \phi \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

includes the terms
\[ \frac{4 \partial}{\partial \tau} \phi \phi \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]

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includes the terms
\[ \frac{4 \partial}{\partial \tau} \phi \phi \frac{\partial}{\partial \tau} - I = \frac{4 \partial}{\partial \tau} \]
Solution by Barton Zwiebach.

In the notation of the problem statement, we have

\[(\frac{2p}{\phi}) \quad \frac{\partial}{\partial p} \left( \frac{2}{e^2 p} \right) - \frac{1}{e^2 p} = \frac{2}{e^2 p} \]

\[(\frac{2p}{\phi}) \quad \frac{\partial}{\partial p} \left( \frac{2}{e^2 p} \right) = \frac{2}{e^2 p} \]

This is an acceptable answer. One can simplify (S18.10) further by noting that

\[\text{left-hand side:}\]

\[\text{Using the values in (S18.4) to evaluate the right-hand side and taking the derivatives}\]

\[\text{Expanding out}\]

\[\text{We use now (S18.8) to simplify (S18.7):}\]

\[\text{We are told that the orbit has}\]

\[\text{We also know that}\]

\[\text{We are told that the orbit has}\]

\[\text{And the connection coefficients are}\]

\[\text{From the metric}\]

**Problem 18: The Stability of Schwarzschild Orbits**
This is the desired condition for stable orbits in the Schwarzschild geometry.

\[ r = c^2 \frac{\sqrt{1 - \frac{L^2}{2m^2} \left( \frac{\delta r^2}{\delta \phi^2} \right)}}{1 - \frac{L^2}{2m^2}} \]

For the orbit with radius \( r > 0 \), we use the values of \( L \) and \( \omega \) and substitute these into (S18.17). The resulting equation:

\[ \frac{d^2 \phi}{dt^2} + \omega^2 \phi = 0 \]

where \( \omega \) is the angular frequency of the orbit. The solution describes bounded oscillations. So stability requires:

\[ \omega^2 = \frac{c^2}{r} \]

The inequality then gives us

\[ \omega^2 > \frac{c^2}{r} \]

For students interested in getting the harmonic oscillator equation (S18.6), for the orbit with radius \( r \), we complete this part of the analysis below. First we evaluate the right-hand side of the function \( H \).

\[ (0)H \equiv \frac{c^2}{r} \left( \frac{\delta r^2}{\delta \phi^2} \right) \]

This is the answer to part (d) of the problem.

\[ \text{Stability Condition:} \quad \frac{d^2 \phi}{dt^2} + \omega^2 \phi = 0 \]

The quantity \( T \) is a constant of the motion, namely, it is a number independent of the problem. The geodesic equation (S18.6) takes the form:

\[ \frac{d^2 \phi}{dt^2} + \omega^2 \phi = 0 \]

where \( \omega \) is the angular frequency of the orbit. The solution describes bounded oscillations. So stability requires:

\[ \omega^2 = \frac{c^2}{r} \]

The inequality then gives us

\[ \omega^2 > \frac{c^2}{r} \]

For the orbit with radius \( r > 0 \), we use the values of \( L \) and \( \omega \) and substitute these into (S18.17). The resulting equation:

\[ \frac{d^2 \phi}{dt^2} + \omega^2 \phi = 0 \]

where \( \omega \) is the angular frequency of the orbit. The solution describes bounded oscillations. So stability requires:

\[ \omega^2 = \frac{c^2}{r} \]

The inequality then gives us

\[ \omega^2 > \frac{c^2}{r} \]
Problem 19: Pressure and Energy Density of Mystereous Stuff

(a) If $u \propto \frac{1}{\sqrt{V}}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}}.$$  

(The above expression is proportional to $\frac{1}{\sqrt{V + \Delta V}}$, and reduces to $u = u_0$ when $\Delta V = 0$.)

Expanding to first order in $\Delta V$,

$$u = u_0 \sqrt{1 + \frac{\Delta V}{V}} = u_0 \left(1 + \frac{1}{2} \frac{\Delta V}{V} \right).$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2} \frac{\Delta V}{V} \right) V \left(1 + \frac{\Delta V}{V} \right) = U_0 \left(1 + \frac{1}{2} \frac{\Delta V}{V} \right),$$

where $U_0 = u_0 V$.

(b) The work done by the agent must supply the full change in energy, so

$$\Delta W = -p \Delta V.$$  

(c) The agent must supply the full change of the work done by the gas.

$$\Delta W = \Delta U = \frac{1}{2} \frac{\Delta V}{V} U_0.$$  

Combining this with the expression for $\Delta W$ from part (b), one sees immediately

$$0 \int_0^1 \text{ where } 0 \int_0^1 \Delta V dV = MV.$$  

Then

$$\left(\frac{\Delta \vec{z}}{\Delta V I} + I\right)_{0 \Omega} = \left(\frac{\Delta \vec{z}}{\Delta V I} + I\right)_{\Omega} \left(\frac{\Delta \vec{z}}{\Delta V I} - I\right)_{\Omega} = (\Delta V + A)n = \Omega.$$  

The local energy is the energy density times the volume, so

$$\left(\frac{\Delta \vec{z}}{\Delta V I} - I\right)_{\Omega} = \frac{\Delta \vec{z}}{\Delta V I} + I = \frac{\Delta \vec{z}}{\Delta V I} \frac{I}{\Omega n} = n$$

Expanding to first order in $\Delta V$, and reduces to

$$\Delta V + A \frac{\Delta V}{A} \Omega_{\Omega} = (\Delta V + A)n.$$  

If $n = 1$, then one can write

**Problem 19:** Pressure and Energy Density of Mystereous Stuff

8.286 Quiz 2 Review Problem Solutions, Fall 2013

69