MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 2, 2013

Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2

Version 2*

QUIZ DATE: Thursday, November 7, 2013, during the normal class time.

COVERAGE: Lecture Notes 4 and 5, and pp. 1–10 of Lecture Notes 6; Problem Sets 4, 5, and 6; Weinberg, *The First Three Minutes,* Chapters 4 – 7; In Ryden's *Introduction to Cosmology*, we have read Chapters 4, 5, and Sec. 6.1 during this period. These chapters, however, parallel what we have done or will be doing in lecture, so you should take them as an aid to learning the lecture material; there will be no questions on this quiz explicitly based on these sections from Ryden. But we have also read Chapters 10 (*Nucleosynthesis and the Early Universe*) and 8 (*Dark Matter*) in Ryden, and these will be included on the quiz, except for Sec. 10.3 (*Deuterium Synthesis*). We will return to deuterium synthesis later in the course. Ryden's Eqs. (10.11) and (10.12) involve similar issues from statistical mechanics, so you should not worry if you do not understand these equations. (In fact, you should worry if you do understand them; as we will discuss later, they are spectacularly incorrect.) Eq. (10.13), which is obtained by dividing Eq. (10.11) by Eq. (10.12), is nonetheless correct; for this course you need not worry how to derive this formula, but you should assume it and understand its consequences, as described by Ryden and also by Weinberg. Chapters 4 and 5 of Weinberg's book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. We will not take off for the spelling of names, as long as they are vaguely recognizable. For dates before 1900, it will be sufficient for you to know when things happened to within 100 years. For dates after 1900, it will be sufficient if you can place events within 10 years. You should expect one problem based on the readings, and several calculational problems. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 4, 5, 6, 11, 13, 15, 17, and 19. There are only three reading questions, Problems 1, 2, and 3.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years.

* Version 2, November 2, 2013 (same date as original). The document date was corrected to read 2013 instead of 2011, and cross references within Problems 15, 18, 19, and the Problem 9 solution were updated.

In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, and 2011. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

REVIEW SESSION AND OFFICE HOURS: To help you study for the quiz, Tingtao Zhou will hold a review session on Monday, November 4, at 7:30 pm, in our regular lecture room, Room 34-101. I will have my usual office hour on Wednesday evening, 7:30 pm, in Room 8-308.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" for your reference. For the second quiz, this useful information will be the following:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\rm coord} = \frac{c}{a(t)} \ .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c)$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{a(t_{\rm observed})}{a(t_{\rm emitted})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:

Trailing clock reads later by an amount $\beta \ell_0/c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) , \quad \vec{p} = \gamma m_0 \vec{v} , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \quad \ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)} \rho_m(t_i) \text{ (matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)} \rho_r(t_i) \text{ (radiation)}.$$

$$\dot\rho = -3 \frac{\dot a}{a} \left(\rho + \frac{p}{c^2}\right) , \quad \Omega \equiv \rho/\rho_c , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat $(k = 0)$:

$$a(t) \propto t^{2/3}$$

$$\Omega = 1 .$$

Closed $(k > 0)$:

$$ct = \alpha(\theta - \sin\theta) , \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{k}}\right)^3 .$$

Open $(k < 0)$:

$$ct = \alpha\left(\sinh\theta - \theta\right) , \quad \frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{\kappa}}\right)^3 ,$$

$$\kappa \equiv -k > 0 .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right\} .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + \tilde a^2(t) \left\{ d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right\} ,$$

where $\tilde a(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + \tilde a^2(t) \left\{ d\psi^2 + \sinh^2\psi \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right\} ,$$

where $\tilde a(t) = a(t)/\sqrt{-k}$. Note that $\tilde a$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt'$$

$$= 3ct \quad \text{(flat, matter-dominated)}.$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{ds} \frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

PROBLEM LIST

PROBLEM 1: DID YOU DO THE READING?

Parts (a)-(c) of this problem come from Quiz 4, 2000, and parts (d) and (e) come from Quiz 3, 2002.

(a) *(5 points)* By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0.1$ MeV? You should assume the existence of the normal three species of neutrinos for your answer.

(b) *(5 points)* Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do?

(i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.

(ii) The deuterium abundance in the Earth's oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth's surface.

(iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.

(iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.

(v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.

(c) *(5 points)* Give three examples of hadrons.

(d) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg posed the question, "Why was there no systematic search for this [cosmic background] radiation, years before 1965?" In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these two elementary particles. (If you name them both correctly, you will get 3 points extra credit. However, one right and one wrong will get you 4 points for the question, compared to 6 points for just naming one particle and getting it right.)

Answer: ________________

2nd Answer (optional): ________________

(e) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

- (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.
- (ii) There was a breakdown in communication between theorists and experimentalists.
- (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.
- (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
- (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.
- (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

PROBLEM 2: DID YOU DO THE READING? *(24 points)*

The following problem was Problem 1 of Quiz 2 in 2007.

(a) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

- (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
- (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
- (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
- (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

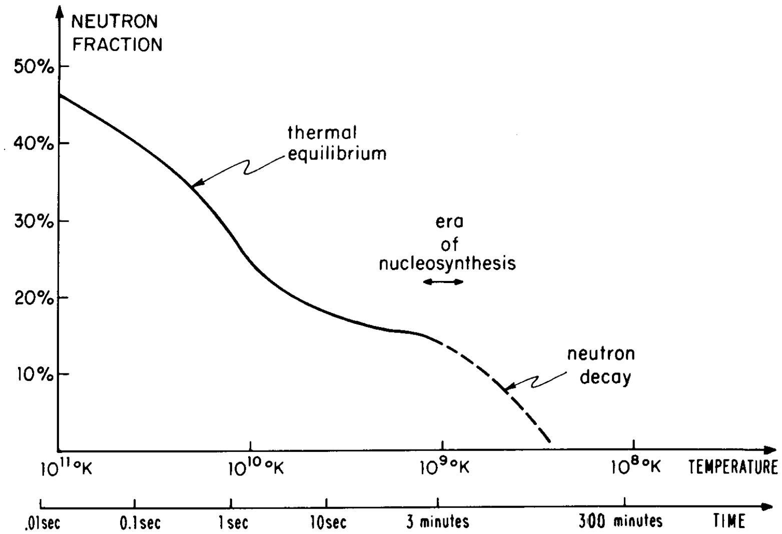
(b) *(6 points)* In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1
(iv) $\sim 10^{-6}$ (v) either zero or negligible

Baryon Number: (i) $\sim 10^{-20}$ (ii) $\sim 10^{-9}$ (iii) $\sim 10^{-6}$
(iv) ~ 1 (v) anywhere from 10^{-5} to 1

Lepton Number: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1
(iv) $\sim 10^{-6}$ (v) could be as high as ~ 1, but is assumed to be very small

(c) *(12 points)* The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance.*



(i) *(3 points)* During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(ii) *(3 points)* During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(iii) *(3 points)* The masses of the neutron and proton are not exactly equal, but instead

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

(iv) *(3 points)* During the period labeled "era of nucleosynthesis," (choose one:)

(A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.

(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.

(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.

(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

PROBLEM 3: DID YOU DO THE READING? *(20 points)*

The following problem comes from Quiz 2, 2011.

(a) *(8 points)* During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.

(i) In *The First Three Minutes,* Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium, He^4. Describe at least one of these two chains.

(ii) Explain briefly what is the *deuterium bottleneck*, and what is its role during nucleosynthesis.

(b) *(12 points)* In Chapter 4 of *The First Three Minutes*, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures.

In this part of the problem you will explore more quantitatively this statement.

(i) For a radiation-dominated universe the scale-factor $a(t) \propto t^{1/2}$. Find the cosmic time t as a function of the Hubble expansion rate H.

(ii) The mass density stored in radiation ρ_r is proportional to the temperature T to the fourth power: i.e., $\rho_r \simeq \alpha T^4$, for some constant α. For a wide range of temperatures we can take $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg} \cdot \mathrm{m}^{-3} \cdot \mathrm{K}^{-4}$. If the temperature is measured in degrees Kelvin (K), then ρ_r has the standard SI units, $[\rho_r] = \mathrm{kg} \cdot \mathrm{m}^{-3}$. Use the Friedmann equation for a flat universe ($k = 0$) with $\rho = \rho_r$ to express the Hubble expansion rate H in terms of the temperature T. You will need the SI value of the gravitational constant $G \simeq 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$. What is the Hubble expansion rate, in inverse seconds, at the start of nucleosynthesis, when $T = T_{\rm nucl} \simeq 0.9 \times 10^9\,\mathrm{K}$?

(iii) Using the results in (i) and (ii), express the cosmic time t as a function of the temperature. Your result should agree with Weinberg's claim above. What is the cosmic time, in seconds, when $T = T_{\rm nucl}$?

∗PROBLEM 4: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time t at which $a/\sqrt{\kappa} = 2\alpha$.

∗PROBLEM 5: ANTICIPATING A BIG CRUNCH

Suppose that we lived in a closed, matter-dominated universe, as described by the equations on the front of the quiz. Suppose further that we measured the mass density parameter Ω to be $\Omega_0 = 2$, and we measured the Hubble "constant" to have some value H_0. How much time would we have before our universe ended in a big crunch, at which time the scale factor $a(t)$ would collapse to 0?

∗PROBLEM 6: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance $\ell_{\rm phys}$ at time t. You should leave your answer in the form of a definite integral.

The form of $a(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.*, dominated by nonrelativistic matter), then $a(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$
$$a = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho a^3}{c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 7: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

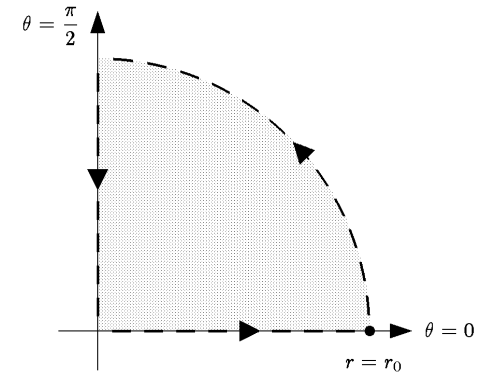
The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2\,dr^2 + r^2(1 + br)^2\,d\theta^2 ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then

moving at fixed r to $\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

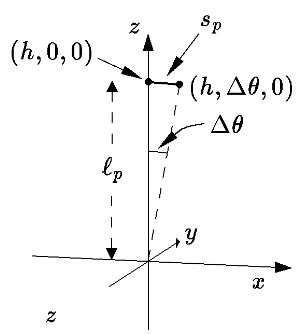
b) *(15 points)* Find the area enclosed by this path.

PROBLEM 8: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as a, dropping its explicit t-dependence.

A small rod has one end at the point ($r = h$, $\theta = 0$, $\phi = 0$) and the other end at the point ($r = h$, $\theta = \Delta\theta$, $\phi = 0$). Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin ($r = 0$) to the first end $(h, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1-r^2} = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor a, and the coordinates h and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor a.

PROBLEM 9: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2 .$$

Express the metric in terms of this new variable.

PROBLEM 10: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} .$$

Calculate the volume $V(r_{\max})$ of the sphere described by

$$r \le r_{\max} .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

∗ PROBLEM 11: THE SCHWARZSCHILD METRIC *(25 points)*

The follow problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass M is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated A and B, are located along the same radial line, with values of the coordinate r given by r_A and r_B, respectively, with $r_A < r_B$. You should assume that both observers lie outside the Schwarzschild horizon.

a) *(5 points)* Write down the expression for the Schwarzschild horizon radius R_S, expressed in terms of M and fundamental constants.

b) *(5 points)* What is the proper distance between A and B? It is okay to leave the answer to this part in the form of an integral that you do not evaluate—but be sure to clearly indicate the limits of integration.

c) *(5 points)* Observer A has a clock that emits an evenly spaced sequence of ticks, with proper time separation $\Delta\tau_A$. What will be the coordinate time separation Δt_A between these ticks?

d) *(5 points)* At each tick of A's clock, a light pulse is transmitted. Observer B receives these pulses, and measures the time separation on his own clock. What is the time interval $\Delta\tau_B$ measured by B.

e) *(5 points)* Suppose that the object creating the gravitational field is a static black hole, so the Schwarzschild metric is valid for all r. Now suppose that one considers the case in which observer A lies on the Schwarzschild horizon, so $r_A \equiv R_S$. Is the proper distance between A and B finite for this case? Does the time interval of the pulses received by B, $\Delta\tau_B$, diverge in this case?

PROBLEM 12: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\,d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$

$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

*** PROBLEM 13: AN EXERCISE IN TWO-DIMENSIONAL METRICS** *(30 points)*

(a) *(8 points)* Consider first a two-dimensional space with coordinates r and θ. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \ .$$

Consider the curve described by

$$r(\theta) = (1 + \epsilon\cos^2\theta)\,r_0 \ ,$$

where ϵ and r_0 are constants, and θ runs from θ_1 to θ_2. Write an expression, in the form of a definite integral, for the length S of this curve.

(b) *(5 points)* Now consider a two-dimensional space with the same two coordinates r and θ, but this time the metric will be

$$\mathrm{d}s^2 = \left(1 + \frac{r}{a}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \ ,$$

where a is a constant. θ is a periodic (angular) variable, with a range of 0 to 2π, with 2π identified with 0. What is the length R of the path from the origin ($r = 0$) to the point $r = r_0, \theta = 0$, along the path for which $\theta = 0$ everywhere along the path? You can leave your answer in the form of a definite integral. (Be sure, however, to specify the limits of integration.)

(c) *(7 points)* For the space described in part (b), what is the total area contained within the region $r < r_0$. Again you can leave your answer in the form of a definite integral, making sure to specify the limits of integration.

(d) *(10 points)* Again for the space described in part (b), consider a geodesic described by the usual geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s} \ .$$

The geodesic is described by functions $r(s)$ and $\theta(s)$, where s is the arc length along the curve. Write explicitly both (i.e., for i=1=r and i=2=θ) geodesic equations.

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE

In this problem we will test the geodesic equation by computing the geodesic curves on the surface of a sphere. We will describe the sphere as in Lecture Notes 5, with metric given by

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \quad .$$

(a) Clearly one geodesic on the sphere is the equator, which can be parametrized by $\theta = \pi/2$ and $\phi = \psi$, where ψ is a parameter which runs from 0 to 2π. Show that if the equator is rotated by an angle α about the x-axis, then the equations become:

$$\cos\theta = \sin\psi\sin\alpha$$

$$\tan\phi = \tan\psi\cos\alpha \quad .$$

(b) Using the generic form of the geodesic equation on the front of the exam, derive the differential equation which describes geodesics in this space.

(c) Show that the expressions in (a) satisfy the differential equation for the geodesic. Hint: The algebra on this can be messy, but I found things were reasonably simple if I wrote the derivatives in the following way:

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}} \quad , \qquad \frac{d\phi}{d\psi} = \frac{\cos\alpha}{1-\sin^2\psi\sin^2\alpha} \quad .$$

*PROBLEM 15: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \quad .$$

We will assume that this metric is given, and that $a(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt}dt \; .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed $v_{\rm phys}$ of the object, since it is the speed that would be measured by a comoving observer. Write an expression for $v_{\rm phys}$ as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A\frac{dr}{d\tau}\right] = B\left(\frac{dt}{d\tau}\right)^2 + C\left(\frac{dr}{d\tau}\right)^2 + D\left(\frac{d\theta}{d\tau}\right)^2 + E\left(\frac{d\phi}{d\tau}\right)^2 \quad ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

(f) *(5 points EXTRA CREDIT)* On Problem 1 of Problem Set 6 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

$$p = \frac{m v_{\rm phys}}{\sqrt{1-\frac{v_{\rm phys}^2}{c^2}}} \quad ,$$

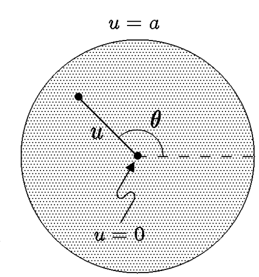
falls off as $1/a(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

PROBLEM 16: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*

The following problem was Problem 3, Quiz 2, 2002.

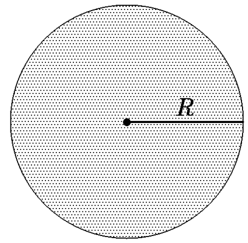
Consider a two-dimensional curved space described by polar coordinates u and θ, where $0 \le u \le a$ and $0 \le \theta \le 2\pi$, and $\theta = 2\pi$ is as usual identified with $\theta = 0$. The metric is given by

$$\mathrm{d}s^2 = \frac{a\,\mathrm{d}u^2}{4u(a-u)} + u\,\mathrm{d}\theta^2 \; .$$

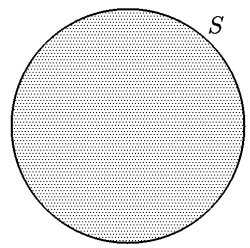


A diagram of the space is shown at the right, but you should of course keep in mind that the diagram does not accurately reflect the distances defined by the metric.

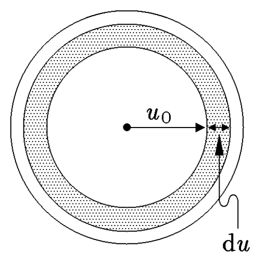
(a) *(6 points)* Find the radius R of the space, defined as the length of a radial (i.e., $\theta = constant$) line. You may express your answer as a definite integral, which you need not evaluate. Be sure, however, to specify the limits of integration.



(b) *(6 points)* Find the circumference S of the space, defined as the length of the boundary of the space at $u = a$.



(c) *(7 points)* Consider an annular region as shown, consisting of all points with a u-coordinate in the range $u_0 \le u \le u_0 + \mathrm{d}u$. Find the physical area $\mathrm{d}A$ of this region, to first order in $\mathrm{d}u$.



(d) *(3 points)* Using your answer to part (c), write an expression for the total area of the space.

(e) *(10 points)* Consider a geodesic curve in this space, described by the functions $u(s)$ and $\theta(s)$, where the parameter s is chosen to be the arc length along the curve. Find the geodesic equation for $u(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[F(u,\theta)\,\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \ldots \ ,$$

where $F(u,\theta)$ is a function that you will find. (Note that by writing F as a function of u and θ, we are saying that it *could* depend on either or both of them, but we are not saying that it *necessarily* depends on them.) You need not simplify the left-hand side of the equation.

(f) *(8 points)* Similarly, find the geodesic equation for $\theta(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[G(u,\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \ldots \ ,$$

where $G(u,\theta)$ is a function that you will find. Again, you need not simplify the left-hand side of the equation.

∗PROBLEM 17: ROTATING FRAMES OF REFERENCE *(35 points)*

The following problem was Problem 3, Quiz 2, 2004.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] \ , \qquad \text{(P17.1)}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (P17.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, *and* $\bar{z}$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] \ ,$$

and then introducing new coordinates t, r, ϕ, *and* z *that are related by*

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z \ ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, *and* $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu \ ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = \,? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \,? \\ g_{22} &\equiv g_{\phi\phi} = \,? \\ g_{33} &\equiv g_{zz} = \,? \end{aligned} \tag{P17.2}$$

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{P17.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for $\mathrm{d}\tau/\mathrm{d}t$, in terms of the quantities indicated, and then ask yourself how this result can be used to find $\mathrm{d}t/\mathrm{d}\tau$.)*

PROBLEM 18: THE STABILITY OF SCHWARZSCHILD ORBITS *(30 points)*

This problem was Problem 4, Quiz 2 in 2007. I have modified the reference to the homework problem to correspond to the current (2013) context, where it is Problem 3 of Problem Set 6. In 2007 it had also been a homework problem prior to the quiz.

This problem is an elaboration of the Problem 3 of Problem Set 6, for which both the statement and the solution are reproduced at the end of this quiz. This material is reproduced for your reference, but you should be aware that the solution to the present problem has important differences. You can copy from this material, but to allow the grader to assess your understanding, you are expected to present a logical, self-contained answer to this question.

In the solution to that homework problem, it was stated that further analysis of the orbits in a Schwarzschild geometry shows that the smallest *stable* circular orbit occurs for $r = 3R_S$. Circular orbits are possible for $\frac{3}{2}R_S < r < 3R_S$, but they are not stable. In this problem we will explore the calculations behind this statement.

We will consider a body which undergoes small oscillations about a circular orbit at $r(t) = r_0$, $\theta = \pi/2$, where r_0 is a constant. The coordinate θ will therefore be fixed, but all the other coordinates will vary as the body follows its orbit.

(a) *(12 points)* The first step, since $r(\tau)$ will not be a constant in this solution, will be to derive the equation of motion for $r(\tau)$. That is, for the Schwarzschild metric

$$ds^2 = -c^2 d\tau^2 = -h(r)c^2 dt^2 + h(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ , \tag{P18.1}$$

where

$$h(r) \equiv 1 - \frac{R_S}{r} \ ,$$

work out the explicit form of the geodesic equation

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \ , \tag{P18.2}$$

for the case $\mu = r$. You should use this result to find an explicit expression for

$$\frac{d^2 r}{d\tau^2} \ .$$

You may allow your answer to contain $h(r)$, its derivative $h'(r)$ with respect to r, and the derivative with respect to τ of any coordinate, including $dt/d\tau$.

(b) *(6 points)* It is useful to consider r and ϕ to be the independent variables, while treating t as a dependent variable. Find an expression for

$$\left(\frac{dt}{d\tau}\right)^2$$

in terms of r, $dr/d\tau$, $d\phi/d\tau$, $h(r)$, and c. Use this equation to simplify the expression for $d^2r/d\tau^2$ obtained in part (a). The goal is to obtain an expression of the form

$$\frac{d^2r}{d\tau^2} = f_0(r) + f_1(r)\left(\frac{d\phi}{d\tau}\right)^2 . \qquad \text{(P18.3)}$$

where the functions $f_0(r)$ and $f_1(r)$ might depend on R_S or c, and might be positive, negative, or zero. Note that the intermediate steps in the calculation involve a term proportional to $(dr/d\tau)^2$, but the net coefficient for this term vanishes.

(c) *(7 points)* To understand the orbit we will also need the equation of motion for ϕ. Evaluate the geodesic equation (P18.2) for $\mu = \phi$, and write the result in terms of the quantity L, defined by

$$L \equiv r^2 \frac{d\phi}{d\tau} . \qquad \text{(P18.4)}$$

(d) *(5 points)* Finally, we come to the question of stability. Substituting Eq. (P18.4) into Eq. (P18.3), the equation of motion for r can be written as

$$\frac{d^2r}{d\tau^2} = f_0(r) + f_1(r)\frac{L^2}{r^4} .$$

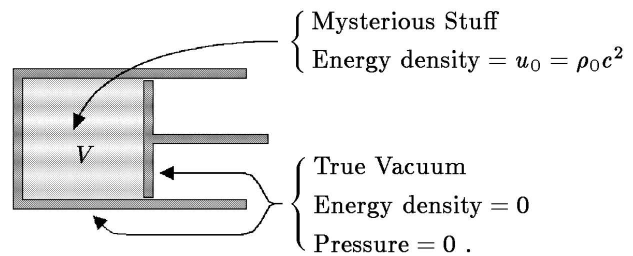
Now consider a small perturbation about the circular orbit at $r = r_0$, and write an equation that determines the stability of the orbit. (That is, if some external force gives the orbiting body a small kick in the radial direction, how can you determine whether the perturbation will lead to stable oscillations, or whether it will start to grow?) You should express the stability requirement in terms of the unspecified functions $f_0(r)$ and $f_1(r)$. You are NOT asked to carry out the algebra of inserting the explicit forms that you have found for these functions.

* PROBLEM 19: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

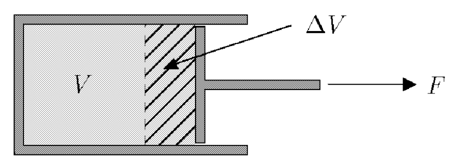
The following problem was Problem 3, Quiz 3, 2002.

In Lecture Notes 6, with further calculations in Problem 4 of Problem Set 6, a thought experiment involving a piston was used to show that $p = \frac{1}{3}\rho c^2$ for radiation. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

(a) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(b) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(c) The most obvious answers would be proton, neutron, and pi meson. However, there are many other possibilities, including many that were not mentioned by Weinberg. See Weinberg chapter 7, pages 136-8.

(d) The correct answers were the $\boxed{\text{neutrino}}$ and the $\boxed{\text{antiproton.}}$ The neutrino was first hypothesized by Wolfgang Pauli in 1932 (in order to explain the kinematics of beta decay), and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.

(e) The correct answers were $\boxed{\text{(ii), (v) and (vi).}}$ The others were incorrect for the following reasons:

(i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe".

PROBLEM 2: DID YOU DO THE READING? *(24 points)*

(a) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

$\boxed{\text{(ii)}}$ In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

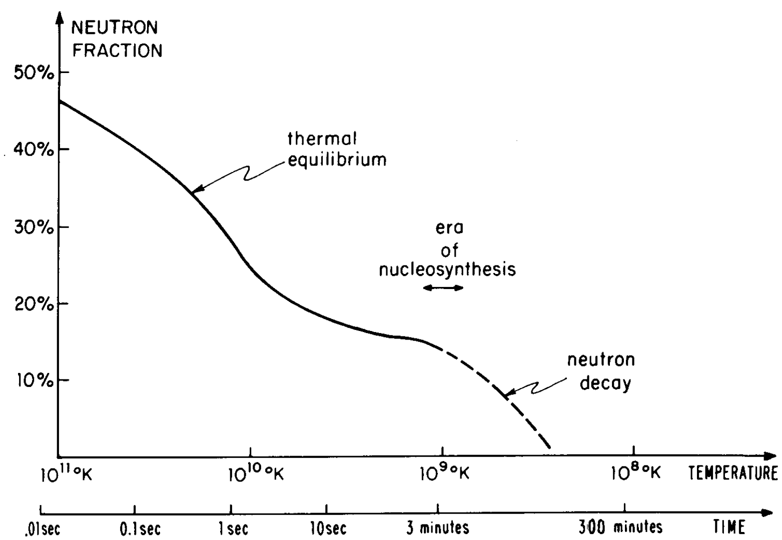
$\boxed{\text{(iv)}}$ In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

(b) *(6 points)* In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge:	(i) $\sim 10^9$	(ii) ~ 1000	(iii) ~ 1
	(iv) $\sim 10^{-6}$	$\boxed{\text{(v) either zero or negligible}}$	
Baryon Number:	(i) $\sim 10^{-20}$	$\boxed{\text{(ii)} \sim 10^{-9}}$	(iii) $\sim 10^{-6}$
	(iv) ~ 1	(v) anywhere from 10^{-5} to 1	
Lepton Number:	(i) $\sim 10^9$	(ii) ~ 1000	(iii) ~ 1
	(iv) $\sim 10^{-6}$	$\boxed{\text{(v) could be as high as} \sim 1\text{, but is assumed to be very small}}$	

(c) *(12 points)* The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance.*



(i) *(3 points)* During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

$\boxed{\text{(E)}}$ Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(ii) *(3 points)* During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

$\boxed{\text{(C)}}$ The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(iii) *(3 points)* The masses of the neutron and proton are not exactly equal, but instead

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

$\boxed{\text{(B)}}$ The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

(iv) *(3 points)* During the period labeled "era of nucleosynthesis," (choose one:)

$\boxed{\text{(A)}}$ Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.

(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.

(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.

(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

PROBLEM 3: DID YOU DO THE READING? *(20 points)*†

(a) *(8 points)*

(i) *(4 points)* We will use the notation X^A to indicate a nucleus,* where X is the symbol for the element which indicates the number of protons, while A is the mass number, namely the total number of protons and neutrons. With this notation H^1, H^2, H^3, He^3 and He^4 stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + p \to He^4 + \gamma,}$$

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + n \to He^4 + \gamma.}$$

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place (in general with different probabilities). For example:

$$p + n \to H^2 + \gamma \qquad H^2 + H^2 \to He^4 + \gamma,$$

$$p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + H^2 \to He^4 + n,$$

$$p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + H^2 \to He^4 + p,$$

$$\ldots$$

Students who described chains different from those of Weinberg, but that can still take place, got full credit for this part. Also, notice that photons in the reactions above carry the additional energy released. However, since the main point was to describe the nuclear reactions, students who didn't include the photons still received full credit.

(ii) *(4 points)* The *deuterium bottleneck* is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium (H^2) plays a crucial role in

* Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.

nucleosynthesis, since it is the starting point for all the chains. However, the deuterium nucleus is extremely loosely bound compared to H^3, He^3, or especially He^4. So, there will be a range of temperatures which are low enough for H^3, He^3, and He^4 nuclei to be bound, but too high to allow the deuterium nucleus to be stable. This is the temperature range where the *deuterium bottleneck* is in action: even if H^3, He^3, and He^4 nuclei could in principle be stable at those temperatures, they do not form because deuterium, which is the starting point for their formation, cannot be formed yet. Nucleosynthesis cannot proceed at a significant rate until the temperature is low enough so that deuterium nuclei are stable; at this point the deuterium bottleneck has been passed.

(b) *(12 points)*

(i) *(3 points)* If we take $a(t) = bt^{1/2}$, for some constant b, we get for the Hubble expansion rate:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \quad \Longrightarrow \quad \boxed{t = \frac{1}{2H}.}$$

(ii) *(6 points)* By using the Friedmann equation with $k = 0$ and $\rho = \rho_r = \alpha T^4$, we find:

$$H^2 = \frac{8\pi}{3} G\rho_r = \frac{8\pi}{3} G\alpha T^4 \quad \Longrightarrow \quad \boxed{H = T^2\sqrt{\frac{8\pi}{3}G\alpha}\ .}$$

If we substitute the given numerical values $G \simeq 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$ and $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg \cdot m^{-3} \cdot K^{-4}}$ we get:

$$\boxed{H \simeq T^2 \times 5.03 \times 10^{-21}\,\mathrm{s^{-1} \cdot K^{-2}}\ .}$$

Notice that the units correctly combine to give H in units of s^{-1} if the temperature is expressed in degrees Kelvin (K). In detail, we see:

$$[G\alpha]^{1/2} = (\mathrm{N \cdot m^2 \cdot kg^{-2} \cdot kg \cdot m^{-3} \cdot K^{-4}})^{1/2} = \mathrm{s^{-1} \cdot K^{-2}}\ ,$$

where we used the fact that $1\,\mathrm{N} = 1\,\mathrm{kg \cdot m \cdot s^{-2}}$. At $T = T_{\rm nucl} \simeq 0.9 \times 10^9\,\mathrm{K}$ we get:

$$\boxed{H \simeq 4.07 \times 10^{-3}\mathrm{s}^{-1}.}$$

(iii) *(3 points)* Using the results in parts (i) and (ii), we get

$$\boxed{t = \frac{1}{2H} \simeq \left(\frac{9.95 \times 10^{19}}{T^2}\right)\mathrm{s \cdot K^2}\ .}$$

To good accuracy, the numerator in the expression above can be rounded to 10^{20}. The above equation agrees with Weinberg's claim that, for a radiation dominated universe, time is proportional to the inverse square of the temperature. In particular for $T = T_{\rm nucl}$ we get:

$$\boxed{t_{\rm nucl} \simeq 123\ \mathrm{s} \approx 2\ \mathrm{min.}}$$

†Solution written by Daniele Bertolini.

PROBLEM 4: EVOLUTION OF AN OPEN UNIVERSE

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh\theta - \theta)$$

$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)\ .$$

Evaluating the second of these equations at $a/\sqrt{\kappa} = 2\alpha$ yields a solution for θ:

$$2\alpha = \alpha(\cosh\theta - 1) \quad \Longrightarrow \quad \cosh\theta = 3 \quad \Longrightarrow \quad \theta = \cosh^{-1}(3)\ .$$

We can use these results in the first equation to solve for t. Noting that

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{8} = 2\sqrt{2}\ ,$$

we have

$$\boxed{t = \frac{\alpha}{c}\left[2\sqrt{2} - \cosh^{-1}(3)\right]\ .}$$

Numerically, $t \approx 1.06567\,\alpha/c$.

PROBLEM 5: ANTICIPATING A BIG CRUNCH

The critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \ ,$$

so the mass density is given by

$$\rho = \Omega_0 \rho_c = 2\rho_c = \frac{3H_0^2}{4\pi G} \ . \tag{S5.1}$$

Substituting this relation into

$$H_0^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ ,$$

we find

$$H_0^2 = 2H_0^2 - \frac{kc^2}{a^2} \ ,$$

from which it follows that

$$\frac{a}{\sqrt{k}} = \frac{c}{H_0} \ . \tag{S5.2}$$

Now use

$$\alpha = \frac{4\pi}{3} \frac{G\rho a^3}{k^{3/2} c^2} \ .$$

Substituting the values we have from Eqs. (S5.1) and (S5.2) for ρ and $a/\sqrt{k}$, we have

$$\alpha = \frac{c}{H_0} \ . \tag{S5.3}$$

To determine the value of the parameter θ, use

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

which when combined with Eqs. (S5.2) and (S5.3) implies that $\cos\theta = 0$. The equation $\cos\theta = 0$ has multiple solutions, but we know that the θ-parameter for a closed matter-dominated universe varies between 0 and π during the expansion phase of the universe. Within this range, $\cos\theta = 0$ implies that $\theta = \pi/2$. Thus, the age of the universe at the time these measurements are made is given by

$$\begin{aligned} t &= \frac{\alpha}{c}(\theta - \sin\theta) \\ &= \frac{1}{H_0}\left(\frac{\pi}{2} - 1\right) \ . \end{aligned}$$

The total lifetime of the closed universe corresponds to $\theta = 2\pi$, or

$$t_{\rm final} = \frac{2\pi\alpha}{c} = \frac{2\pi}{H_0} \ ,$$

so the time remaining before the big crunch is given by

$$t_{\rm final} - t = \frac{1}{H_0}\left[2\pi - \left(\frac{\pi}{2} - 1\right)\right] = \boxed{\left(\frac{3\pi}{2} + 1\right)\frac{1}{H_0} \ .}$$

PROBLEM 6: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\,dt^2 + a^2(t)d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{a^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{a(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{a(t')}dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} a(t)\,d\psi = \boxed{a(t)\int_0^t \frac{c}{a(t')}dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)} .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{a(t)} .$$

Then using $a = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $a = \alpha(1 - \cos\theta)$, one can see that a returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 5. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 7: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1 + ar)^2\, dr^2$, or $ds = (1 + ar)\, dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\, dr = r_0 + \frac{1}{2} a r_0^2 .$$

Along the second segment $dr = 0$, so $ds = r(1 + br)\, d\theta$, where $r = r_0$. So the length of the second segment is

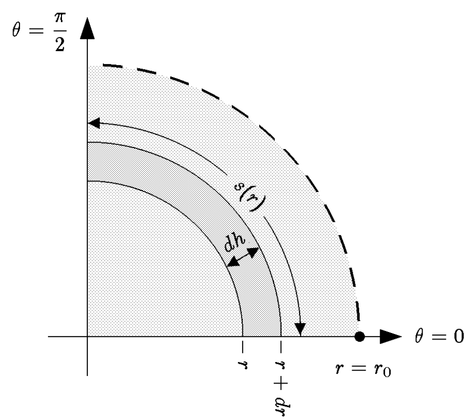
$$S_2 = \int_0^{\pi/2} r_0(1 + br_0)\, d\theta = \frac{\pi}{2} r_0(1 + br_0) .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$S = 2S_1 + S_2 = 2\left(r_0 + \frac{1}{2}ar_0^2\right) + \frac{\pi}{2}r_0(1+br_0)$$

$$= \boxed{\left(2+\frac{\pi}{2}\right)r_0 + \frac{1}{2}(2a+\pi b)r_0^2\ .}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1+ar)\,dr\ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2}r(1+br)\ .$$

The area is then

$$dA = s(r)\,dh\ ,$$

so

$$A = \int_0^{r_0} s(r)\,dh$$

$$= \int_0^{r_0} \frac{\pi}{2}r(1+br)(1+ar)\,dr$$

$$= \frac{\pi}{2}\int_0^{r_0}\left[r + (a+b)r^2 + abr^3\right]dr$$

$$= \boxed{\frac{\pi}{2}\left[\frac{1}{2}r_0^2 + \frac{1}{3}(a+b)r_0^3 + \frac{1}{4}abr_0^4\right]}$$

PROBLEM 8: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(h,0,0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{a\,dr}{\sqrt{1-r^2}}\ .$$

So

$$\boxed{\ell_p = \int_0^h \frac{a\,dr}{\sqrt{1-r^2}} = a\sin^{-1}h\ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = ar\,d\theta\ ,$$

so

$$\boxed{s_p = ah\,\Delta\theta\ .}$$

(c) From part (a), one has

$$h = \sin(\ell_p/a)\ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{a\sin(\ell_p/a)}\ .}$$

Note that as $a \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p/\ell_p$.

PROBLEM 9: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from}\\ \text{origin to } r_0}} ds \ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0 \ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\, d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0)\sin\theta\, d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\, d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0)\sin\theta\, d\theta\, d\phi \ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0)\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \\ &= \rho^2(r_0)(2\pi)\left(-\cos\theta\Big|_0^{\pi}\right) \\ \Longrightarrow &\quad \boxed{A = 4\pi\rho^2(r_0) \ .} \end{aligned}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 2 of Problem Set 5, we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\, dr \ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi\int_0^{r_0} \rho^2(r)\, dr \ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma} \ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma} \ .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma}\,)$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}\,)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .}$$

PROBLEM 10: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = a(t)\frac{dr}{\sqrt{1 - kr^2}} \times a(t)r d\theta \times a(t) r\sin\theta d\phi$$

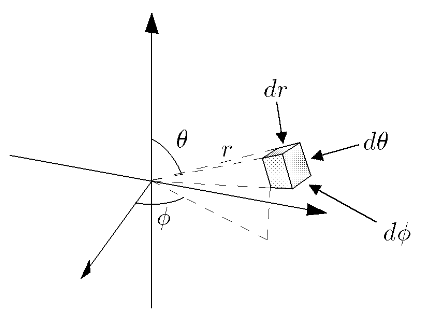
The total volume is then

$$V = \int dV = a^3(t) \int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi a^3(t) \int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}} \ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r+dr$, between θ and $\theta+d\theta$, and between ϕ and $\phi+d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$ds_1 = a(t)\frac{dr}{\sqrt{1-kr^2}}$$

$$ds_2 = a(t) r\, d\theta$$

$$ds_3 = a(t) r \sin\theta\, d\theta \ .$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation

relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\, d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\sqrt{k}\, r = \sin\psi \quad (\text{if } k > 0)$$

$$\sqrt{-k}\, r = \sinh\psi \quad (\text{if } k > 0).$$

The answer is

$$V = \begin{cases} 2\pi a^3(t) \left[\dfrac{\sin^{-1}\left(\sqrt{k}\, r_{\max}\right)}{k^{3/2}} - \dfrac{\sqrt{1-kr_{\max}^2}}{k} \right] & (\text{if } k > 0) \\ 2\pi a^3(t) \left[\dfrac{\sqrt{1-kr_{\max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\, r_{\max}\right)}{(-k)^{3/2}} \right] & (\text{if } k < 0) \quad .] \end{cases}$$

PROBLEM 11: THE SCHWARZSCHILD METRIC

a) The Schwarzschild horizon is the value of r for which the metric becomes singular. Since the metric contains the factor

$$\left(1 - \frac{2GM}{rc^2}\right) \ ,$$

it becomes singular at

$$\boxed{R_S = \frac{2GM}{c^2} \ .}$$

b) The separation between A and B is purely in the radial direction, so the proper length of a segment along the path joining them is given by

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 \ ,$$

so

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \ .$$

The proper distance from A to B is obtained by adding the proper lengths of all the segments along the path, so

$$\boxed{s_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}\ .}$$

EXTENSION: The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for s_{AB} as

$$s_{AB} = \int_{r_A}^{r_B} \frac{\sqrt{r}\,dr}{\sqrt{r - R_S}}\ .$$

Then introduce the hyperbolic trigonometric substitution

$$r = R_S \cosh^2 u\ .$$

One then has

$$\sqrt{r - R_S} = \sqrt{R_S}\,\sinh u$$

$$dr = 2R_S \cosh u \sinh u\, du\ ,$$

and the indefinite integral becomes

$$\begin{aligned}\int \frac{\sqrt{r}\,dr}{\sqrt{r - R_S}} &= 2R_S \int \cosh^2 u\, du \\ &= R_S \int (1 + \cosh 2u) du \\ &= R_S\left(u + \frac{1}{2}\sinh 2u\right) \\ &= R_S(u + \sinh u\, \cosh u) \\ &= R_S \sinh^{-1}\left(\sqrt{\frac{r}{R_S} - 1}\right) + \sqrt{r(r - R_S)}\ .\end{aligned}$$

Thus,

$$\begin{aligned}s_{AB} = R_S &\left[\sinh^{-1}\left(\sqrt{\frac{r_B}{R_S} - 1}\right) - \sinh^{-1}\left(\sqrt{\frac{r_A}{R_S} - 1}\right)\right] \\ &+ \sqrt{r_B(r_B - R_S)} - \sqrt{r_A(r_A - R_S)}\ .\end{aligned}$$

c) A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2\ ,$$

so

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}}\, dt\ .$$

The reading on the observer's clock corresponds to the proper time interval $d\tau$, so the corresponding interval of the coordinate t is given by

$$\boxed{\Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}}\ .}$$

d) Since the Schwarzschild metric does not change with time, each pulse leaving A will take the same length of time to reach B. Thus, the pulses emitted by A will arrive at B with a time coordinate spacing

$$\Delta t_B = \Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}}\ .$$

The clock at B, however, will read the proper time and not the coordinate time. Thus,

$$\Delta\tau_B = \sqrt{1 - \frac{2GM}{r_B c^2}}\, \Delta t_B$$

$$= \boxed{\sqrt{\frac{1 - \frac{2GM}{r_B c^2}}{1 - \frac{2GM}{r_A c^2}}}\, \Delta\tau_A\ .}$$

e) From parts (a) and (b), the proper distance between A and B can be rewritten as

$$s_{AB} = \int_{R_S}^{r_B} \frac{\sqrt{r}dr}{\sqrt{r - R_S}}\ .$$

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of $r = R_S$, say $R_S < r < R_S + \epsilon$. For this range the quantity $\sqrt{r}$ in the numerator can be approximated by $\sqrt{R_S}$, so the contribution has the form

$$\sqrt{R_S}\int_{R_S}^{R_S+\epsilon}\frac{dr}{\sqrt{r-R_S}}\ .$$

Changing the integration variable to $u \equiv r - R_S$, the contribution can be easily evaluated:

$$\sqrt{R_S}\int_{R_S}^{R_S+\epsilon}\frac{dr}{\sqrt{r-R_S}} = \sqrt{R_S}\int_0^{\epsilon}\frac{du}{\sqrt{u}} = 2\sqrt{R_S\epsilon} < \infty\ .$$

So, although the integrand is infinite at $r = R_S$, the integral is still finite.

$$\boxed{\text{The proper distance between } A \text{ and } B \text{ does not diverge.}}$$

Looking at the answer to part (d), however, one can see that when $r_A = R_S$,

$$\boxed{\text{The time interval } \Delta\tau_B \text{ diverges.}}$$

PROBLEM 12: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda}\ .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2\,d\theta^2\ ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2\ , \qquad g_{r\theta} = g_{\theta r} = 0\ .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{g_{rr}\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r g_{\theta\theta}\right)\frac{d\theta}{d\lambda}\,\frac{d\theta}{d\lambda}\ ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r r^2\right)\left(\frac{d\theta}{d\lambda}\right)^2\ ,$$

or

$$\boxed{\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2\ .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i,j). So

$$\boxed{\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0\ .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda\ , \qquad y(\lambda) = 1\ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1}\ ,$$

$$\theta(\lambda) = \tan^{-1}\frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda)\ .$$

Calculating the needed derivatives,*

$$\begin{aligned}\frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2+1}}\\ \frac{d^2r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3}\\ \frac{d\theta}{d\lambda} &= -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2}\frac{1}{\lambda^2} = -\frac{1}{r^2}\ .\end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \Longleftrightarrow \frac{1}{r^3} = r\left(-\frac{1}{r^2}\right)^2\ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \Longleftrightarrow \frac{d}{d\lambda}\left\{r^2\left(-\frac{1}{r^2}\right)\right\} = 0\ ,$$

which also checks.

PROBLEM 13: AN EXERCISE IN TWO-DIMENSIONAL METRICS

(30 points)

(a) Since

$$r(\theta) = (1+\epsilon\cos^2\theta)\, r_0\ ,$$

as the angular coordinate θ changes by $\mathrm{d}\theta$, r changes by

$$\mathrm{d}r = \frac{\mathrm{d}r}{\mathrm{d}\theta}\,\mathrm{d}\theta = -2\epsilon r_0\cos\theta\sin\theta\,\mathrm{d}\theta\ .$$

---

* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi/\cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = (1+\tan^2\phi)d\phi\ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2}\ .$$

$\mathrm{d}s^2$ is then given by

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2\\ &= 4\epsilon^2 r_0^2\cos^2\theta\sin^2\theta\,\mathrm{d}\theta^2 + (1+\epsilon\cos^2\theta)^2\, r_0^2\,\mathrm{d}\theta^2\\ &= \left[4\epsilon^2\cos^2\theta\sin^2\theta + (1+\epsilon\cos^2\theta)^2\right] r_0^2\,\mathrm{d}\theta^2\ ,\end{aligned}$$

so

$$\mathrm{d}s = r_0\sqrt{4\epsilon^2\cos^2\theta\sin^2\theta + (1+\epsilon\cos^2\theta)^2}\,\mathrm{d}\theta\ .$$

Since θ runs from θ_1 to θ_2 as the curve is swept out,

$$\boxed{S = r_0\int_{\theta_1}^{\theta_2}\sqrt{4\epsilon^2\cos^2\theta\sin^2\theta + (1+\epsilon\cos^2\theta)^2}\,d\theta\ .}$$

(b) Since θ does not vary along this path,

$$\mathrm{d}s = \sqrt{1+\frac{r}{a}}\,\mathrm{d}r\ ,$$

and so

$$\boxed{R = \int_0^{r_0}\sqrt{1+\frac{r}{a}}\,\mathrm{d}r\ .}$$

(c) Since the metric does not contain a term in $\mathrm{d}r\,\mathrm{d}\theta$, the r and θ directions are orthogonal. Thus, if one considers a small region in which r is in the interval r' to $r'+\mathrm{d}r'$, and θ is in the interval θ' to $\theta'+\mathrm{d}\theta'$, then the region can be treated as a rectangle. The side along which r varies has length $\mathrm{d}s_r = \sqrt{1+(r'/a)}\,\mathrm{d}r'$, while the side along which θ varies has length $\mathrm{d}s_\theta = r'\,\mathrm{d}\theta'$. The area is then

$$\mathrm{d}A = \mathrm{d}s_r\,\mathrm{d}s_\theta = r'\sqrt{1+(r'/a)}\,\mathrm{d}r'\,\mathrm{d}\theta'\ .$$

To cover the area for which $r < r_0$, r' must be integrated from 0 to r_0, and θ' must be integrated from 0 to 2π:

$$A = \int_0^{r_0}\mathrm{d}r'\int_0^{2\pi}\mathrm{d}\theta'\, r'\sqrt{1+(r'/a)}\ .$$

But

$$\int_0^{2\pi}\mathrm{d}\theta' = 2\pi\ ,$$

so

$$\boxed{A = 2\pi \int_0^{r_0} \mathrm{d}r'\, r' \sqrt{1 + (r'/a)}\ .}$$

You were not asked to carry out the integration, but it can be done by using the substitution $u = 1 + (r'/a)$, so $\mathrm{d}u = (1/a)\,\mathrm{d}r'$, and $r' = a(u-1)$. The result is

$$A = \frac{4\pi a^2}{15}\left[2 + \left(\frac{3r_0^2}{a^2} + \frac{r_0}{a} - 2\right)\sqrt{1 + \frac{r_0}{a}}\,\right] .$$

(d) The nonzero metric coefficients are given by

$$g_{rr} = 1 + \frac{r}{a}\ , \qquad g_{\theta\theta} = r^2\ ,$$

so the metric is diagonal. For $i = 1 = r$, the geodesic equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{rr}\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r}\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s}\ ,$$

so if we substitute the values from above, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial}{\partial r}\left(1 + \frac{r}{a}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial r^2}{\partial r}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 .$$

Simplifying slightly,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{2a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 .}$$

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}r}{\mathrm{d}s}\right\} = \frac{1}{a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \left(1 + \frac{r}{a}\right)\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} .$$

Inserting this expansion into the boxed equation above, the first term can be brought to the right-hand side, giving

$$\boxed{\left(1 + \frac{r}{a}\right)\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = -\frac{1}{2a}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 .}$$

The $i = 2 = \theta$ equation is simpler, because none of the g_{ij} coefficients depend on θ, so the right-hand side of the geodesic equation vanishes. One has simply

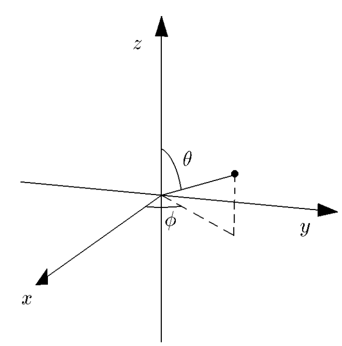
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = 0\ .}$$

For most purposes this is the best way to write the equation, since it leads immediately to $r^2(\mathrm{d}\theta/\mathrm{d}s) = const$. However, it is possible to expand the derivative, giving the alternative form

$$r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0\ .$$

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE

(a) Rotations are easy to understand in Cartesian coordinates. The relationship between the polar and Cartesian coordinates is given by

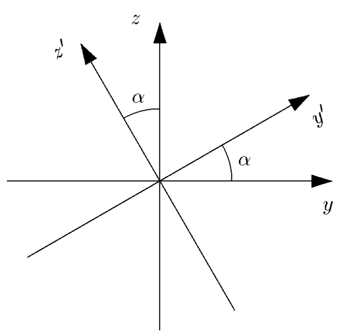


$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\ . \end{aligned}$$

The equator is then described by $\theta = \pi/2$, and $\phi = \psi$, where ψ is a parameter running from 0 to 2π. Thus, the equator is described by the curve $x^i(\psi)$, where

$$\begin{aligned} x^1 &= x = r\cos\psi \\ x^2 &= y = r\sin\psi \\ x^3 &= z = 0\ . \end{aligned}$$

Now introduce a primed coordinate system that is related to the original system by a rotation in the y-z plane by an angle α:



$$x = x'$$
$$y = y' \cos\alpha - z' \sin\alpha$$
$$z = z' \cos\alpha + y' \sin\alpha \ .$$

The rotated equator, which we seek to describe, is just the standard equator in the primed coordinates:

$$x' = r\cos\psi \ , \qquad y' = r\sin\psi \ , \qquad z' = 0 \ .$$

Using the relation between the two coordinate systems given above,

$$x = r\cos\psi$$
$$y = r\sin\psi\cos\alpha$$
$$z = r\sin\psi\sin\alpha \ .$$

Using again the relations between polar and Cartesian coordinates,

$$\boxed{\begin{aligned} \cos\theta &= \frac{z}{r} = \sin\psi\sin\alpha \\ \tan\phi &= \frac{y}{x} = \tan\psi\cos\alpha \ . \end{aligned}}$$

(b) A segment of the equator corresponding to an interval $d\psi$ has length $a\,d\psi$, so the parameter ψ is proportional to the arc length. Expressed in terms of the metric, this relationship becomes

$$ds^2 = g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}d\psi^2 = a^2 d\psi^2 \ .$$

Thus the quantity

$$A \equiv g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}$$

is equal to a^2, so the geodesic equation (5.50) reduces to the simpler form of Eq. (5.52). (Note that we are following the notation of Lecture Notes 5, except that the variable used to parameterize the path is called ψ, rather than λ or s. Although A is not equal to 1 as we assumed in Lecture Notes 5, it is easily seen that Eq. (5.52) follows from (5.50) provided only that $A = constant$.) Thus,

$$\frac{d}{d\psi}\left\{g_{ij}\frac{dx^j}{d\psi}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\psi}\frac{dx^\ell}{d\psi} \ .$$

For this problem the metric has only two nonzero components:

$$g_{\theta\theta} = a^2 \ , \qquad g_{\phi\phi} = a^2\sin^2\theta \ .$$

Taking $i = \theta$ in the geodesic equation,

$$\frac{d}{d\psi}\left\{g_{\theta\theta}\frac{d\theta}{d\psi}\right\} = \frac{1}{2}\partial_\theta g_{\phi\phi}\frac{d\phi}{d\psi}\frac{d\phi}{d\psi} \quad \Longrightarrow$$

$$\boxed{\frac{d^2\theta}{d\psi^2} = \sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 \ .}$$

Taking $i = \phi$,

$$\frac{d}{d\psi}\left\{a^2\sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \quad \Longrightarrow$$

$$\boxed{\frac{d}{d\psi}\left\{\sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \ .}$$

(c) This part is mainly algebra. Taking the derivative of

$$\cos\theta = \sin\psi\sin\alpha$$

implies

$$-\sin\theta\, d\theta = \cos\psi\sin\alpha\, d\psi \ .$$

Then, using the trigonometric identity $\sin\theta = \sqrt{1-\cos^2\theta}$, one finds

$$\sin\theta = \sqrt{1-\sin^2\psi\sin^2\alpha} \ ,$$

so

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}} \ .$$

Similarly

$$\tan\phi = \tan\psi\cos\alpha \quad \Longrightarrow \quad \sec^2\phi\, d\phi = \sec^2\psi\, d\psi\cos\alpha \ .$$

Then

$$\begin{aligned}\sec^2\phi &= \tan^2\phi + 1 = \tan^2\psi\cos^2\alpha + 1\\ &= \frac{1}{\cos^2\psi}[\sin^2\psi\cos^2\alpha + \cos^2\psi]\\ &= \sec^2\psi[\sin^2\psi(1-\sin^2\alpha)+\cos^2\psi]\\ &= \sec^2\psi[1-\sin^2\psi\sin^2\alpha] \ ,\end{aligned}$$

So

$$\frac{d\phi}{d\psi} = \frac{\cos\alpha}{1-\sin^2\psi\sin^2\alpha} \ .$$

To verify the geodesic equations of part (b), it is easiest to check the second one first:

$$\begin{aligned}\sin^2\theta\frac{d\phi}{d\psi} &= (1-\sin^2\psi\sin^2\alpha)\frac{\cos\alpha}{1-\sin^2\psi\sin^2\alpha}\\ &= \cos\alpha \ ,\end{aligned}$$

so clearly

$$\frac{d}{d\psi}\left\{\sin^2\theta\frac{d\phi}{d\psi}\right\} = \frac{d}{d\psi}(\cos\alpha) = 0 \ .$$

To verify the first geodesic equation from part (b), first calculate the left-hand side, $d^2\theta/d\psi^2$, using our result for $d\theta/d\psi$:

$$\frac{d^2\theta}{d\psi^2} = \frac{d}{d\psi}\left(\frac{d\theta}{d\psi}\right) = \frac{d}{d\psi}\left\{-\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}}\right\} \ .$$

After some straightforward algebra, one finds

$$\frac{d^2\theta}{d\psi^2} = \frac{\sin\psi\sin\alpha\cos^2\alpha}{\left[1-\sin^2\psi\sin^2\alpha\right]^{3/2}} \ .$$

The right-hand side of the first geodesic equation can be evaluated using the expression found above for $d\phi/d\psi$, giving

$$\begin{aligned}\sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 &= \sqrt{1-\sin^2\psi\sin^2\alpha}\,\sin\psi\sin\alpha\,\frac{\cos^2\alpha}{\left[1-\sin^2\psi\sin^2\alpha\right]^2}\\ &= \frac{\sin\psi\sin\alpha\cos^2\alpha}{\left[1-\sin^2\psi\sin^2\alpha\right]^{3/2}} \ .\end{aligned}$$

So the left- and right-hand sides are equal.

PROBLEM 15: GEODESICS IN A CLOSED UNIVERSE

(a) *(7 points)* For purely radial motion, $d\theta = d\phi = 0$, so the line element reduces do

$$-c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \ .$$

Dividing by dt^2,

$$-c^2\left(\frac{d\tau}{dt}\right)^2 = -c^2 + \frac{a^2(t)}{1-r^2}\left(\frac{dr}{dt}\right)^2 \ .$$

Rearranging,

$$\boxed{\frac{d\tau}{dt} = \sqrt{1-\frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2} \ .}$$

(b) *(3 points)*

$$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}} = \boxed{\frac{1}{\sqrt{1-\frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}} \ .}$$

(c) *(10 points)* During any interval of clock time dt, the proper time that would be measured by a clock moving with the object is given by $d\tau$, as given by the metric. Using the answer from part (a),

$$d\tau = \frac{d\tau}{dt}\, dt = \sqrt{1-\frac{a^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\, dt \ .$$

Integrating to find the total proper time,

$$\boxed{\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)} \left(\frac{dr_p}{dt}\right)^2}\, dt\ .}$$

(d) *(10 points)* The physical distance $d\ell$ that the object moves during a given time interval is related to the coordinate distance dr by the spatial part of the metric:

$$d\ell^2 = ds^2 = a^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \quad \Longrightarrow \quad d\ell = \frac{a(t)}{\sqrt{1-r^2}}\, dr\ .$$

Thus

$$\boxed{v_{\rm phys} = \frac{d\ell}{dt} = \frac{a(t)}{\sqrt{1-r^2}}\frac{dr}{dt}\ .}$$

Discussion: A common mistake was to include $-c^2\, dt^2$ in the expression for $d\ell^2$. To understand why this is not correct, we should think about how an observer would measure $d\ell$, the distance to be used in calculating the velocity of a passing object. The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval $dt_{\rm meas}$. Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the ***same*** time t. Thus, when we compute the distance between the two marks, we set $dt = 0$. To compute the speed she would then divide the distance by $dt_{\rm meas}$, which is nonzero.

(e) *(10 points)* We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\ .$$

This formula is true for each possible value of μ, while the Einstein summation convention implies that the indices ν, λ, and σ are summed. We are trying to derive the equation for r, so we set $\mu = r$. Since the metric is diagonal, the only contribution on the left-hand side will be $\nu = r$. On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when $\lambda = \sigma$. The term will vanish unless $dx^\lambda/d\tau$ is nonzero, so λ must be

either r or t (i.e., there is no motion in the θ or ϕ directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r}\ .$$

Since $g_{tt} = -c^2$, the derivative with respect to r will vanish. Thus, the only nonzero contribution on the right-hand side arises from $\lambda = \sigma = r$. Using

$$g_{rr} = \frac{a^2(t)}{1-r^2}\ ,$$

the geodesic equation becomes

$$\frac{d}{d\tau}\left\{g_{rr}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left(\partial_r g_{rr}\right)\frac{dr}{d\tau}\frac{dr}{d\tau}\ ,$$

or

$$\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left[\partial_r\left(\frac{a^2}{1-r^2}\right)\right]\frac{dr}{d\tau}\frac{dr}{d\tau}\ ,$$

or finally

$$\boxed{\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = a^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2\ .}$$

This matches the form shown in the question, with

$$A = \frac{a^2}{1-r^2}\ ,\ \text{and } C = a^2\frac{r}{(1-r^2)^2}\ ,$$

with $B = D = E = 0$.

(f) *(5 points EXTRA CREDIT)* The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for p. Using the answer from (d),

$$p = \frac{m v_{\rm phys}}{\sqrt{1 - \frac{v_{\rm phys}^2}{c^2}}} = \frac{m\frac{a(t)}{\sqrt{1-r^2}}\frac{dr}{dt}}{\sqrt{1 - \frac{a^2}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}}\ .$$

Using the answer from (b), this simplifies to

$$p = m\frac{a(t)}{\sqrt{1-r^2}}\frac{dr}{dt}\frac{dt}{d\tau} = m\frac{a(t)}{\sqrt{1-r^2}}\frac{dr}{d\tau}\ .$$

Multiply the geodesic equation by m, and then use the above result to rewrite it as

$$\frac{d}{d\tau}\left\{\frac{ap}{\sqrt{1-r^2}}\right\} = ma^2\,\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2 .$$

Expanding the left-hand side,

$$LHS = \frac{d}{d\tau}\left\{\frac{ap}{\sqrt{1-r^2}}\right\} = \frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{ap\} + ap\frac{r}{(1-r^2)^{3/2}}\frac{dr}{d\tau}$$
$$= \frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{ap\} + ma^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2 .$$

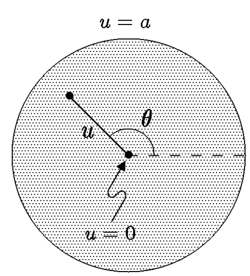
Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{ap\} = 0 .$$

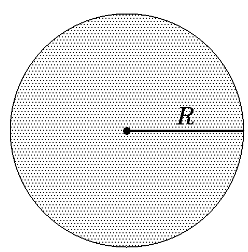
Multiplying by $\sqrt{1-r^2}$, one has the desired result:

$$\frac{d}{d\tau}\{ap\} = 0 \quad\Longrightarrow\quad \boxed{p \propto \frac{1}{a(t)}\ .}$$

PROBLEM 16: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*



(a) For $\theta = constant$, the expression for the metric reduces to



$$ds^2 = \frac{a\,du^2}{4u(a-u)} \quad\Longrightarrow$$
$$ds = \frac{1}{2}\sqrt{\frac{a}{u(a-u)}}\,du .$$

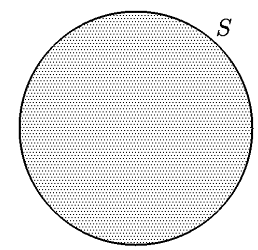
To find the length of the radial line shown, one must integrate this expression from the value

of u at the center, which is 0, to the value of u at the outer edge, which is a. So

$$\boxed{R = \frac{1}{2}\int_0^a \sqrt{\frac{a}{u(a-u)}}\,du\ .}$$

You were not expected to do it, but the integral can be carried out, giving $R = (\pi/2)\sqrt{a}$.

(b) For $u = constant$, the expression for the metric reduces to

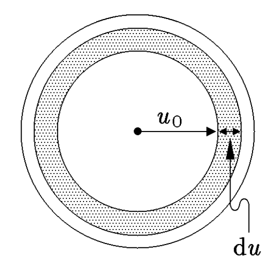


$$ds^2 = u\,d\theta^2 \quad\Longrightarrow\quad ds = \sqrt{u}\,d\theta .$$

Since θ runs from 0 to 2π, and $u = a$ for the circumference of the space,

$$\boxed{S = \int_0^{2\pi}\sqrt{a}\,d\theta = 2\pi\sqrt{a}\ .}$$

(c) To evaluate the answer to first order in du means to neglect any terms that would be proportional to du^2 or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case we can imagine bending it into a rectangle without changing its area. The area is then equal to the circumference times the width. Both the circumference and the width must be calculated by using the metric:



$$\begin{aligned} \mathrm{d}A &= \text{circumference} \times \text{width} \\ &= [2\pi\sqrt{u_0}\,] \times \left[\frac{1}{2}\sqrt{\frac{a}{u_0(a-u_0)}}\,\mathrm{d}u\right] \\ &= \boxed{\pi\sqrt{\frac{a}{(a-u_0)}}\,\mathrm{d}u\ .} \end{aligned}$$

(d) We can find the total area by imagining that it is broken up into annuluses, where a single annulus starts at radial coordinate u and extends to $u + \mathrm{d}u$. As in part (a), this expression must be integrated from the value of u at the center, which is 0, to the value of u at the outer edge, which is a.

$$\boxed{A = \pi \int_0^a \sqrt{\frac{a}{(a-u)}}\,\mathrm{d}u\ .}$$

You did not need to carry out this integration, but the answer would be $A = 2\pi a$.

(e) From the list at the front of the exam, the general formula for a geodesic is written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{k\ell}}{\partial x^i}\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^\ell}{\mathrm{d}s}\ .$$

The metric components g_{ij} are related to $\mathrm{d}s^2$ by

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ ,$$

where the Einstein summation convention (sum over repeated indices) is assumed. In this case

$$\begin{aligned} g_{11} &\equiv g_{uu} = \frac{a}{4u(a-u)} \\ g_{22} &\equiv g_{\theta\theta} = u \\ g_{12} &= g_{21} = 0\ , \end{aligned}$$

where I have chosen $x^1 = u$ and $x^2 = \theta$. The equation with $\mathrm{d}u/\mathrm{d}s$ on the left-hand side is found by looking at the geodesic equations for $i = 1$. Of course j, k, and ℓ must all be summed, but the only nonzero contributions arise when $j = 1$, and k and ℓ are either both equal to 1 or both equal to 2:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{uu}\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial u}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2\ .$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a}{4u(a-u)}\frac{\mathrm{d}u}{\mathrm{d}s}\right] &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{a}{4u(a-u)}\right)\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}(u)\right]\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \frac{1}{2}\left[\frac{a}{4u(a-u)^2} - \frac{a}{4u^2(a-u)}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \boxed{\frac{1}{8}\frac{a(2u-a)}{u^2(a-u)^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2\ .} \end{aligned}$$

(f) This part is solved by the same method, but it is simpler. Here we consider the geodesic equation with $i = 2$. The only term that contributes on the left-hand side is $j = 2$. On the right-hand side one finds nontrivial expressions when k and ℓ are either both equal to 1 or both equal to 2. However, the terms on the right-hand side both involve the derivative of the metric with respect to $x^2 = \theta$, and these derivatives all vanish. So

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2\ ,$$

which reduces to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left[u\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = 0\ .}$$

PROBLEM 17: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right]\ ,$$

and the metric coefficients are then just read off from this expression:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = -c^2 + r^2\omega^2 \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = r^2\omega^2 \\ g_{22} &\equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = r^2 \\ g_{33} &\equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = 1\ . \end{aligned}$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\mathrm{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\ .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\, g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned}
RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\ .
\end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\ .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\ ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0\ ,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0\ .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (P17.1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\ .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)}\ ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}}\ .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\ ,$$

adapted to the rotating cylindrical coordinate system.

PROBLEM 18: THE STABILITY OF SCHWARZSCHILD ORBITS* *(30 points)*

From the metric:

$$ds^2 = -c^2 d\tau^2 = -h(r)\, c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \tag{S18.1}$$

and the convention $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ we read the nonvanishing metric components:

$$g_{tt} = -h(r)c^2\,, \quad g_{rr} = \frac{1}{h(r)}\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\phi\phi} = r^2 \sin^2\theta\,. \tag{S18.2}$$

We are told that the orbit has $\theta = \pi/2$, so on the orbit $d\theta = 0$ and the relevant metric and metric components are:

$$ds^2 = -c^2 d\tau^2 = -h(r)\, c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\phi^2\,, \tag{S18.3}$$

$$g_{tt} = -h(r)c^2\,, \quad g_{rr} = \frac{1}{h(r)}\,, \quad g_{\phi\phi} = r^2\,. \tag{S18.4}$$

We also know that

$$h(r) = 1 - \frac{R_S}{r}\,. \tag{S18.5}$$

(a) The geodesic equation

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\,, \tag{S18.6}$$

for the index value $\mu = r$ takes the form

$$\frac{d}{d\tau}\left[g_{rr}\frac{dr}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial r}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\quad.$$

Expanding out

$$\frac{d}{d\tau}\left[\frac{1}{h}\frac{dr}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2\,.$$

Using the values in (S18.4) to evaluate the right-hand side and taking the derivatives on the left-hand side:

$$\underline{-\frac{h'}{h^2}\left(\frac{dr}{d\tau}\right)^2} + \frac{1}{h}\frac{d^2r}{d\tau^2} = -\frac{1}{2}c^2 h'\left(\frac{dt}{d\tau}\right)^2 \underline{-\frac{1}{2}\frac{h'}{h^2}\left(\frac{dr}{d\tau}\right)^2} + r\left(\frac{d\phi}{d\tau}\right)^2\,.$$

* Solution by Barton Zwiebach.

Here $h' \equiv \frac{dh}{dr}$ and we have supressed the arguments of h and h' to avoid clutter. Collecting the underlined terms to the right and multiplying by h, we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}h' hc^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{h'}{h}\left(\frac{dr}{d\tau}\right)^2 + rh\left(\frac{d\phi}{d\tau}\right)^2\,. \tag{S18.7}$$

(b) Dividing the expression (S18.3) for the metric by $d\tau^2$ we readily find

$$-c^2 = -hc^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\,,$$

and rearranging,

$$hc^2\left(\frac{dt}{d\tau}\right)^2 = c^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\,. \tag{S18.8}$$

This is the most useful form of the answer. Of course, we also have

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{h} + \frac{1}{h^2c^2}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{hc^2}\left(\frac{d\phi}{d\tau}\right)^2\,. \tag{S18.9}$$

We use now (S18.8) to simplify (S18.7):

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}h'\left(c^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) + \frac{1}{2}\frac{h'}{h}\left(\frac{dr}{d\tau}\right)^2 + rh\left(\frac{d\phi}{d\tau}\right)^2\,.$$

Expanding out, the terms with $(\frac{dr}{d\tau})^2$ cancel and we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}h' c^2 + \left(rh - \frac{1}{2}h' r^2\right)\left(\frac{d\phi}{d\tau}\right)^2\,. \tag{S18.10}$$

This is an acceptable answer. One can simplify (S18.10) further by noting that $h' = R_S/r^2$ and $rh = r - R_S$:

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{R_S c^2}{r^2} + \left(r - \frac{3}{2}R_S\right)\left(\frac{d\phi}{d\tau}\right)^2\,. \tag{S18.11}$$

In the notation of the problem statement, we have

$$f_0(r) = -\frac{1}{2}\frac{R_S c^2}{r^2}\,, \quad f_1(r) = r - \frac{3}{2}R_S\,. \tag{S18.12}$$

(c) The geodesic equation (S18.6) for $\mu = \phi$ gives

$$\frac{d}{d\tau}\left[g_{\phi\phi}\,\frac{d\phi}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial\phi}\,\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\,.$$

Since no metric component depends on ϕ, the right-hand side vanishes and we get:

$$\frac{d}{d\tau}\left[r^2\,\frac{d\phi}{d\tau}\right] = 0 \quad\rightarrow\quad \frac{d}{d\tau}L = 0\,, \quad\text{where}\quad L \equiv r^2\,\frac{d\phi}{d\tau}\,. \tag{S18.13}$$

The quantity L is a constant of the motion, namely, it is a number independent of τ.

(d) Using (S18.13) the second-order differential equation (S18.11) for $r(\tau)$ takes the form stated in the problem:

$$\frac{d^2r}{d\tau^2} = f_0(r) + \frac{f_1(r)}{r^4}L^2 \equiv H(r)\,, \tag{S18.14}$$

where we have introduced the function $H(r)$ (recall that L is a constant!). The differential equation then takes the form

$$\frac{d^2r}{d\tau^2} = H(r)\,. \tag{S18.15}$$

Since we are told that a circular orbit with radius r_0 exists, the function $r(\tau) = r_0$ must solve this equation. Being the constant function, the left-hand side vanishes and, consequently, the right-hand side must also vanish:

$$H(r_0) = f_0(r_0) + \frac{f_1(r_0)}{r_0^4}L^2 = 0\,. \tag{S18.16}$$

To investigate stability we consider a small perturbation $\delta r(\tau)$ of the orbit:

$$r(\tau) = r_0 + \delta r(\tau)\,, \quad\text{with}\quad \delta r(\tau) \ll r_0 \ \text{ at some initial } \tau\,.$$

Substituting this into (S18.15) we get, to first nontrivial approximation

$$\frac{d^2\delta r}{d\tau^2} = H(r_0 + \delta r) \simeq H(r_0) + \delta r H'(r_0) = \delta r\, H'(r_0)\,,$$

where $H'(r) = \frac{dH(r)}{dr}$ and we used $H(r_0) = 0$ from (S18.16). The resulting equation

$$\frac{d^2\delta r(\tau)}{d\tau^2} = H'(r_0)\,\delta r(\tau)\,, \tag{S18.17}$$

is familiar because $H'(r_0)$ is just a number. The condition of stability is that this number is negative: $H'(r_0) < 0$. Indeed, in this case (S18.17) is the harmonic oscillator equation

$$\frac{d^2x}{dt^2} = -\omega^2 x\,, \quad\text{with replacements}\quad x \leftrightarrow \delta r,\quad t \leftrightarrow \tau\,,\quad -\omega^2 \leftrightarrow H'(r_0)\,,$$

and the solution describes bounded oscillations. So stability requires:

$$\text{Stability Condition:}\ \ H'(r_0) = \frac{d}{dr}\left[f_0(r) + \frac{f_1(r)}{r^4}L^2\right]_{r=r_0} < 0\,. \tag{S18.18}$$

This is the answer to part (d).

– –

For students interested in getting the famous result that orbits are stable for $r > 3R_S$ we complete this part of the analysis below. First we evaluate $H'(r_0)$ in (S18.18) using the values of f_0 and f_1 in (S18.12):

$$H'(r_0) = \frac{d}{dr}\left[-\frac{1}{2}\frac{R_Sc^2}{r^2} + \left(\frac{1}{r^3} - \frac{3R_S}{2r^4}\right)L^2\right]_{r=r_0} = \frac{R_Sc^2}{r_0^3} - \frac{3L^2}{r_0^5}(r_0 - 2R_S)\,.$$

The inequality in (S18.18) then gives us

$$R_Sc^2 - \frac{3L^2}{r_0^2}(r_0 - 2R_S) < 0\,, \tag{S18.19}$$

where we multiplied by $r_0^3 > 0$. To complete the calculation we need the value of L^2 for the orbit with radius r_0. This value is determined by the vanishing of $H(r_0)$:

$$-\frac{1}{2}\frac{R_Sc^2}{r_0^2} + (r_0 - \frac{3}{2}R_S)\frac{L^2}{r_0^4} = 0 \quad\rightarrow\quad \frac{L^2}{r_0^2} = \frac{1}{2}\frac{R_Sc^2}{(r_0 - \frac{3}{2}R_S)}\,.$$

Note, incidentally, that the equality to the right demands that for a circular orbit $r_0 > \frac{3}{2}R_S$. Substituting the above value of L^2/r_0^2 in (S18.19) we get:

$$R_Sc^2 - \frac{3}{2}\frac{R_Sc^2}{(r_0 - \frac{3}{2}R_S)}(r_0 - 2R_S) < 0\,.$$

Cancelling the common factors of R_Sc^2 we find

$$1 - \frac{3}{2}\frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2}R_S)} < 0\,,$$

which is equivalent to

$$\frac{3}{2}\frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2}R_S)} > 1\,.$$

For $r_0 > \frac{3}{2}R_S$, we get

$$3(r_0 - 2R_S) > 2(r_0 - \frac{3}{2}R_S) \quad\rightarrow\quad r_0 > 3R_S\,. \tag{S18.20}$$

This is the desired condition for stable orbits in the Schwarzschild geometry.

PROBLEM 19: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V+\Delta V) = u_0 \sqrt{\frac{V}{V+\Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V+\Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1+\frac{\Delta V}{V}}} = \frac{u_0}{1+\frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \ .$$

The total energy is the energy density times the volume, so

$$U = u(V+\Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 \ .}$$