MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department
Physics 8.286: The Early Universe
November 2, 2013
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2
Version 2

QUIZ DATE: Thursday, November 7, 2013, during the normal class time.

COVERAGE:
Lecture Notes 4 and 5, and pp. 1–10 of Lecture Notes 6; Problem Sets 4, 5, and 6; Weinberg, The First Three Minutes, Chapters 4–7; in Ryden's Introduction to Cosmology, we have read Chapters 4, 5, and Sec. 6.1 during this period. These chapters, however, parallel what we have done or will be doing in lecture, so you should take them as an aid to learning the lecture material; there will be no questions on this quiz explicitly based on these sections from Ryden. But we have also read Chapters 10 (Nucleosynthesis and the Early Universe) and 8 (Dark Matter) in Ryden, and these will be included on the quiz, except for Sec. 10.3 (Deuterium Synthesis). We will return to deuterium synthesis later in the course. Ryden's Eqs. (10.11) and (10.12) involve similar issues from statistical mechanics, so you should not worry if you do not understand these equations. (In fact, you should worry if you do understand them; as we will discuss later, they are spectacularly incorrect.) Eq. (10.13), which is obtained by dividing Eq. (10.11) by Eq. (10.12), is nonetheless correct; for this course you need not worry how to derive this formula, but you should assume it and understand its consequences, as described by Ryden and also by Weinberg. Chapters 4 and 5 of Weinberg's book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. We will not take off for the spelling of names, as long as they are vaguely recognizable. For dates before 1900, it will be sufficient for you to know when things happened to within 100 years. For dates after 1900, it will be sufficient if you can place events within 10 years. You should expect one problem based on the readings, and several calculational problems. One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 4, 5, 6, 11, 13, 15, 17, and 19. There are only three reading questions, Problems 1, 2, and 3.

The starred problems are not to be handed in, but are being made available to help you study. Try some multiple from quizzes in previous years.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. Try some multiple from quizzes in previous years.

PHYSICAL CONSTANTS:

COSMOLOGICAL REDSHIFT:

\[
\frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}
\]

DOPPLER SHIFT (for motion along a line):

\[
\frac{\nu}{c} = \frac{v}{u} (\text{nonrelativistic, observer moving})
\]

\[
\frac{\nu}{c} = \frac{v}{u} \left(1 - \frac{v}{u}\right)^{-1} (\text{nonrelativistic, source moving})
\]

\[
\frac{\nu}{c} = \sqrt{1 + \beta (1 - \beta)} (\text{special relativity, with } \beta = \frac{v}{c})
\]

SPEED OF LIGHT IN COMOVING COORDINATES:

\[
\frac{\nu}{\lambda} = c a(t)
\]

INFORMATION TO BE GIVEN ON QUIZ:

These review problems are not to be handed in, but are being made available to help you study. Try some multiple from quizzes in previous years.

INFORMATION TO BE GIVEN ON QUIZ:

WEDNESDAY, NOVEMBER 6, 2013, 4:30 PM, ROOM 8-306

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COSMOLOGICAL REDSHIFT:

\[
\frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}
\]

INFORMATION TO BE GIVEN ON QUIZ:

These review problems are not to be handed in, but are being made available to help you study. Try some multiple from quizzes in previous years.
\[ \begin{align*} \frac{dp}{d\tau} \left( \frac{p^2}{E} \right)^{\frac{1}{2}} &= \frac{1}{\sqrt{1 - \frac{p^2}{c^2}}} \frac{dp}{d\tau} \\ \frac{sp}{s\tau} \left( \frac{sp}{s\tau} \right)^{\frac{1}{2}} &= \left\{ \frac{sp}{s\tau} \right\}^{\frac{1}{2}} \frac{sp}{s\tau} \end{align*} \]

CEPTIC EQUATION:

\[ \begin{align*} t^\phi \theta e^{\mu} \sigma + \frac{\phi}{\tau} e^{\mu} \sigma + \frac{\phi}{\tau} e^{\mu} \sigma &= t^\phi \theta e^{\mu} \sigma - \frac{\phi}{\tau} e^{\mu} \sigma = t^\phi \theta e^{\mu} \sigma \end{align*} \]

SCHWARZSCHILD METRIC:

\[ \begin{align*} \left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} &= \left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} \frac{\phi}{\tau} e^{\mu} \sigma - \frac{\phi}{\tau} e^{\mu} \sigma = \phi \end{align*} \]

ROBERTSON-WALKER METRIC:

\[ \begin{align*} t^\phi \theta e^{\mu} \sigma + \frac{\phi}{\tau} e^{\mu} \sigma &= t^\phi \theta e^{\mu} \sigma - \frac{\phi}{\tau} e^{\mu} \sigma = t^\phi \theta e^{\mu} \sigma \end{align*} \]

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

\[ \begin{align*} \frac{\phi}{\tau} &= \phi \\
\left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} &= \left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} \frac{\phi}{\tau} e^{\mu} \sigma - \frac{\phi}{\tau} e^{\mu} \sigma = \phi \end{align*} \]

ENERGY-MOMENTUM FOUR-VECTOR:

\[ \begin{align*} \frac{\phi}{\tau} &= \phi \\
\left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} &= \left( \frac{\phi}{\tau} \right)^{\frac{1}{2}} \frac{\phi}{\tau} e^{\mu} \sigma - \frac{\phi}{\tau} e^{\mu} \sigma = \phi \end{align*} \]

SPECIAL RELATIVITY:
Problem 1: Did You Do the Reading?

(a) (5 points) By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0$ MeV? You should assume the existence of the normal three species of neutrinos for your answer.

(b) (5 points) Measurements of the primordial deuterium abundance in the Sun's photosphere indicate that there is no deuterium in the Sun's photosphere. If this is true, what does this imply about the formation of structures in the early universe?

(c) (5 points) Give three examples of hadrons.

(d) (6 points) In Chapter 6 of "The First Three Minutes," Steven Weinberg posed the question, "Why wasn't systematic search for this [cosmic background] radiation, years before 1965?" In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted to exist but have not been discovered. Write a short paragraph discussing the implications of this question for the development of theoretical physics.

(e) (5 points) Give three examples of interactions that can be easily damaged by astrophysical processes other than primordial nucleosynthesis.

(f) (5 points) Give three examples of interactions that are most likely to affect the spectra of primordial nucleosynthesis.

(g) (5 points) Give three examples of interactions that tend to destroy light by converting it into something else.

(h) (5 points) Give three examples of interactions that are most likely to affect the spin of primordial nucleosynthesis.

(i) (5 points) Give three examples of interactions that are most likely to affect the temperature of primordial nucleosynthesis.

(j) (5 points) Give three examples of interactions that are most likely to affect the velocity of primordial nucleosynthesis.

(k) (5 points) Give three examples of interactions that are most likely to affect the pressure of primordial nucleosynthesis.

(l) (5 points) Give three examples of interactions that are most likely to affect the density of primordial nucleosynthesis.

Problem List:

1. Did You Do the Reading (2000)?
2. Did You Do the Reading (2002)?
3. Did You Do the Reading (2007)?
4. Evolution of an Open Universe
5. Anticipating a Big Crunch
6. Geodesics
7. Geodesics in a Closed Universe
8. Geometry in a Closed Universe
9. The General Spherically Symmetric Metric
10. Volumes in a Robertson-Walker Universe
11. The Schwarzschild Metric
12. Geodesics on the Surface of a Sphere
13. Geodesics in a Closed Universe
14. A Two-Dimensional Curved Space
15. Rotating Frames of Reference
16. The Stability of Schwarzschild Orbits
17. Pressure and Energy Density of Mysterious Stuff
18. Lengths and Areas in a Two-Dimensional Metric
19. Pressure and Energy Density of Mysterious Stuff
20. A Two-Dimensional Curved Space
21. Rotating Frames of Reference
22. The Schwarzschild Metric
23. Geodesics in a Closed Universe
24. Geodesics on the Surface of a Sphere
In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3 °K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer; circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1 °K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3 °K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously any theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

**Problem 2: Did You Do the Reading?**

(24 points)

The following problem was Problem 1 of Quiz 2 in 2007.

(a) (6 points)

In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutron density decreased, and the number density of neutrons was to zero at the time of nucleosynthesis. This model successfully explained the observed present abundances of light elements, the ratio of photons to nuclear particles being about $10^9$. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed.

(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about $10^3$, not $10^9$ as Alpher and Herman concluded.

(b) (6 points)

In Weinberg’s “Recipe for a Hot Universe,” he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge:

(i) $\sim 10^9$

(ii) $\sim 1000$

(iii) $\sim 1$

(iv) $\sim 10^{-6}$

(v) either zero or negligible

Baryon Number:

(i) $\sim 10^{-20}$

(ii) $\sim 10^{-9}$

(iii) $\sim 10^{-6}$

(iv) $\sim 1$

(v) anywhere from $10^{-5}$ to 1

Lepton Number:

(i) $\sim 10^9$

(ii) $\sim 1000$

(iii) $\sim 1$

(iv) $\sim 10^{-6}$

(v) could be as high as $\sim 1$, but is assumed to be very small.
The figure below comes from Weinberg's Chapter 5, and is labeled "The Shifting Neutron-Proton Balance." During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\text{antineutrino} + \text{proton} \rightarrow \text{electron} + \text{neutron}
\]
\[\text{neutrino} + \text{neutron} \rightarrow \text{positron} + \text{proton} \]

(E) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\text{antineutrino} + \text{proton} \rightarrow \text{positron} + \text{neutron}
\]
\[\text{neutrino} + \text{neutron} \rightarrow \text{electron} + \text{proton} \]

(F) Neutrons and protons can be created and destroyed by reactions such as:

\[
\text{proton} + \text{neutrino} \rightarrow \text{positron} + \text{antineutrino}
\]
\[\text{neutron} + \text{antineutrino} \rightarrow \text{electron} + \text{positron} \]


During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\text{antineutrino} + \text{proton} \rightarrow \text{electron} + \text{neutron}
\]
\[\text{neutrino} + \text{neutron} \rightarrow \text{positron} + \text{proton} \]

(E) Neutrons and protons can be converted from one into the other through reactions such as:

\[
\text{antineutrino} + \text{proton} \rightarrow \text{positron} + \text{neutron}
\]
\[\text{neutrino} + \text{neutron} \rightarrow \text{electron} + \text{proton} \]

(F) Neutrons and protons can be created and destroyed by reactions such as:

\[
\text{proton} + \text{neutrino} \rightarrow \text{positron} + \text{antineutrino}
\]
\[\text{neutron} + \text{antineutrino} \rightarrow \text{electron} + \text{positron} \]

The masses of the neutron and proton are not exactly equal:

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.
\[ \frac{dp}{d\phi} = \frac{c^2 - 1}{c^2} \]

Then

\[ \phi = \frac{t^2}{t^2} \]

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The metric simplifies to
\[ ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a(t)^2 \left( \frac{d\psi}{\sin \psi} \right)^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \].

(a) (7 points)
A light pulse travels on a null trajectory, which means that \( d\tau = 0 \) for each segment of the trajectory. Consider a light pulse that moves along a radial line, so \( \theta = \phi = \text{constant} \). Find an expression for \( d\psi/dt \) in terms of quantities that appear in the metric.

(b) (8 points)
Write an expression for the physical horizon distance \( \ell_{\text{phys}} \) at time \( t \). You should leave your answer in the form of a definite integral.

The form of \( a(t) \) depends on the content of the universe. If the universe is matter-dominated (i.e., dominated by nonrelativistic matter), then \( a(t) \) is described by the parametric equations
\[ c t = \alpha(\theta - \sin \theta), \quad a = \alpha(1 - \cos \theta) \]
where \( \alpha \equiv \frac{4\pi G \rho}{3c^2} \).

These equations are identical to those on the front of the exam, except that I have chosen \( k = 1 \).

(c) (10 points)
Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for \( d\psi/d\theta \), where \( \theta \) is the parameter used to describe the evolution.

(d) (5 points)
Suppose that a photon leaves the origin of the coordinate system \( (\psi = 0) \) at \( t = 0 \). How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 7: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

(a) (10 points)
Consider a two dimensional space, described in polar coordinates \( (r, \theta) \), with a metric given by
\[ ds^2 = (1 + ar) dr^2 + r^2 (1 + br)^2 d\theta^2 \].

Consider the path in this space which is formed by starting at the origin, moving along the \( \theta = 0 \) line to \( r = r_0 \), then moving at fixed \( r \) to \( \theta = \pi/2 \), and then moving back to the origin at fixed \( \theta \). The path is shown below:

The following problem was Problem 4, Quiz 2, 1994:

(b) (15 points)
Find the area enclosed by this path.

PROBLEM 8: GEOMETRY IN A CLOSED UNIVERSE

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with \( k = 1 \). The questions below all pertain to some fixed time \( t \), so the scale factor can be written simply as \( a \), dropping its explicit \( t \)-dependence.

A small rod has one end at the point \( (r = h, \theta = 0, \phi = 0) \) and the other end at the point \( (r = h, \theta = \Delta \theta, \phi = 0) \). Assume that \( \Delta \theta \ll 1 \).

The following problem was Problem 3, Quiz 2, 1994:
(a) Find the physical distance $\ell_p$ from the origin $(r = 0)$ to the first end $(h, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int dr \sqrt{1 - r^2} = \sin^{-1} r$$

$$\int dr \frac{1}{\sqrt{1 - r^2}} = \frac{1}{2} \ln \left(\frac{1 + r}{1 - r}\right)$$

(b) Find the physical length $s_p$ of the rod. Express your answer in terms of the scale factor $a$, and the coordinates $h$ and $\Delta \theta$.

(c) Note that $\Delta \theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance $\ell_p$, the physical length $s_p$, and the scale factor $a$.

**Problem 9: The General Spherically Symmetric Metric**

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three-dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the space outside a spherically symmetric mass $M$ is described by the following metric:

$$ds^2 = dr^2 + \rho^2(r) \left(d\theta^2 + \sin^2 \theta d\phi^2\right)$$

for some function $\rho(r)$.

(a) Find the physical radius $a$ of the sphere. (By "radius," I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate $\sigma$ is introduced, where $\sigma$ is related to $r$ by

$$\sigma = r^2$$

Express the metric in terms of this new variable.

**Problem 10: Volumes in a Robertson-Walker Universe**

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t) \left\{ dr^2 + \frac{k}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \right\}$$

Calculate the volume $V(r_{\text{max}})$ of the sphere described by

$$\left\{\left(\frac{d\phi}{\sin \theta} \sin \phi \right) + e^{\phi} - 1\right\} e^\phi = \varepsilon^p$$

Where

$$\varepsilon = \frac{\kappa}{A^2}$$

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t) \left\{ dr^2 + \frac{k}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \right\}$$

(a) Assume that the scale factor $a$ and the scale factor $a'$ are different, and that the spacetime elements $d\tau$ and $d\tau'$ are related by

$$d\tau = \sqrt{\frac{k}{1 - kr^2}} d\tau'$$

(b) Find the physical length $s_p$ of the rod. Express your answer in terms of the spatial coordinates $r$, $\theta$, and $\phi$.

(c) Find the physical distance $\ell_p$ from the origin $(r = 0)$ to the first end $(\theta = 0, \phi = 0)$ of the rod. You may find one of the following integrals useful:

$$\int \sqrt{\frac{k}{1 - kr^2}} = \frac{1}{2} \ln \left(\frac{1 + \sqrt{1 - kr^2}}{1 - \sqrt{1 - kr^2}}\right)$$

(d) Express your answer in terms of the scale factor $a$, and the coordinates $r$, $\theta$, and $\phi$.
Let your answer to (a) to show that the line $y = x$ is a geodesic curve.

> \theta \sin \theta = \ell

> \theta \cos \theta = x

(b) Now introduce the usual Cartesian coordinates, defined by

\[ \lambda = \theta, \quad r = x \]

The geodesic is described by functions $\theta$ and $r$.

Consider the curve described by the usual geodesic equation, and that is the arc length measured along the curve. Use the general formula on

\[ s(\theta) \quad \text{and} \quad s(r) \]

where $s$ is the arc length.

Use your answer to (a) to show that the line $y = x$ is a geodesic curve.

The metric is given by

\[ g_{\lambda \lambda} = \delta \]

\[ g_{rr} = \delta \]

where $\delta$ is the Kronecker delta.

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ g_{\ell \ell} = \delta \quad \text{and} \quad g_{\theta \theta} = \delta \]

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider the curve described by

\[ t \quad \phi^t + t \phi^p = \delta \]

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]

Consider first a two-dimensional space with coordinates $\ell$ and $\theta$.

The metric is given by

\[ t \quad \phi^t + t \phi^p = \delta \]
Problem 16: A 2-Dimensional Curved Space

The problem was Problem 3, Quiz 2, 2002.

The following problem was Problem 3, Quiz 2, 2000, where it was worth 70 points.

Problem 16: Geodesics in a Closed Universe

Consider the case of closed Robertson-Walker universe. Taking the same is true in a closed universe.

The metric is given by

\[ ds^2 = \left(\frac{r}{a} + \frac{\theta^2}{\sin^2 \theta} \right) dt^2 + \left(\frac{r}{a} - \frac{\theta^2}{\sin^2 \theta} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

We will assume that this metric is given, and that \( a \) is as usual identified with \( \pi \).

\[ 2 \leq \theta \leq \pi \]

\[ \frac{1}{2} \leq u \leq 1, \quad \theta \leq 0, \quad \phi \leq 0 \]

**Problem 16.2: Geodesics**

Consider the case of closed Robertson-Walker universe. Take the same is true in a closed universe.

The metric is given by

\[ ds^2 = \left(\frac{r}{a} + \frac{\theta^2}{\sin^2 \theta} \right) dt^2 + \left(\frac{r}{a} - \frac{\theta^2}{\sin^2 \theta} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

We will assume that this metric is given, and that \( a \) is as usual identified with \( \pi \).

\[ 2 \leq \theta \leq \pi \]

\[ \frac{1}{2} \leq u \leq 1, \quad \theta \leq 0, \quad \phi \leq 0 \]

**Problem 16.3: Geodesics**

Consider the case of closed Robertson-Walker universe. Take the same is true in a closed universe.

The metric is given by

\[ ds^2 = \left(\frac{r}{a} + \frac{\theta^2}{\sin^2 \theta} \right) dt^2 + \left(\frac{r}{a} - \frac{\theta^2}{\sin^2 \theta} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

We will assume that this metric is given, and that \( a \) is as usual identified with \( \pi \).

\[ 2 \leq \theta \leq \pi \]

\[ \frac{1}{2} \leq u \leq 1, \quad \theta \leq 0, \quad \phi \leq 0 \]

**Problem 16.4: Geodesics**

Consider the case of closed Robertson-Walker universe. Take the same is true in a closed universe.

The metric is given by

\[ ds^2 = \left(\frac{r}{a} + \frac{\theta^2}{\sin^2 \theta} \right) dt^2 + \left(\frac{r}{a} - \frac{\theta^2}{\sin^2 \theta} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

We will assume that this metric is given, and that \( a \) is as usual identified with \( \pi \).

\[ 2 \leq \theta \leq \pi \]

\[ \frac{1}{2} \leq u \leq 1, \quad \theta \leq 0, \quad \phi \leq 0 \]

**Problem 16.5: Geodesics**

Consider the case of closed Robertson-Walker universe. Take the same is true in a closed universe.

The metric is given by

\[ ds^2 = \left(\frac{r}{a} + \frac{\theta^2}{\sin^2 \theta} \right) dt^2 + \left(\frac{r}{a} - \frac{\theta^2}{\sin^2 \theta} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

We will assume that this metric is given, and that \( a \) is as usual identified with \( \pi \).

\[ 2 \leq \theta \leq \pi \]

\[ \frac{1}{2} \leq u \leq 1, \quad \theta \leq 0, \quad \phi \leq 0 \]
where $g$ is equal to $1$. Find explicit expressions to complete the list:

\[ \begin{align*}
(\text{a}) & \quad g_{ij} = e^{\phi} + \phi, \\
(\text{b}) & \quad g_{ij} = e^{\phi} + \phi, \\
(\text{c}) & \quad g_{ij} = e^{\phi} + \phi,
\end{align*} \]

and then introducing new coordinates $s, t, \theta, \phi$ that are related by

\[ \begin{align*}
(\text{a}) & \quad z = z, \\
(\text{b}) & \quad \phi = \phi, \\
(\text{c}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The metric can be written in matrix form by using the standard coordinate functions.

And then introducing new coordinates $s, t, \theta, \phi$ that are related by

\[ \begin{align*}
(\text{a}) & \quad z = z, \\
(\text{b}) & \quad \phi = \phi, \\
(\text{c}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

To work the problem, you do not need to know anything about this.

\[ \begin{align*}
(\text{a}) & \quad \phi = \phi, \\
(\text{b}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

In this problem we will use the framework of general relativity and geodesics to determine the geodesic equation for a geodesic curve in this space, described by the functions $u, \theta, \phi, z$.

\[ \begin{align*}
(\text{a}) & \quad \phi = \phi, \\
(\text{b}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

In this problem we will use the framework of general relativity and geodesics to determine the geodesic equation for a geodesic curve in this space, described by the functions $u, \theta, \phi, z$.

\[ \begin{align*}
(\text{a}) & \quad \phi = \phi, \\
(\text{b}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

In this problem we will use the framework of general relativity and geodesics to determine the geodesic equation for a geodesic curve in this space, described by the functions $u, \theta, \phi, z$.

\[ \begin{align*}
(\text{a}) & \quad \phi = \phi, \\
(\text{b}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

In this problem we will use the framework of general relativity and geodesics to determine the geodesic equation for a geodesic curve in this space, described by the functions $u, \theta, \phi, z$.

\[ \begin{align*}
(\text{a}) & \quad \phi = \phi, \\
(\text{b}) & \quad t = t, \quad \theta = \theta.
\end{align*} \]

The problem will concern the consequences of the metric and how they affect the behavior of geodesic curves in this space.

In this problem we will use the framework of general relativity and geodesics to determine the geodesic equation for a geodesic curve in this space, described by the functions $u, \theta, \phi, z$.
\[
\begin{align*}
\frac{\partial}{\partial r} & = \frac{\partial}{\partial r} \\
\frac{\partial}{\partial \phi} & = \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \theta} & = \frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial t} & = \frac{\partial}{\partial t}
\end{align*}
\]

You may allow your answer to contain \( \eta \), its derivative \( \frac{d \eta}{d \tau} \), and the derivative with respect to \( \tau \). If you find an expression for \( \frac{d \eta}{d \tau} \), in terms of \( \phi \), \( t \), and \( r \), and use the metric to find an expression for \( d \phi / d \tau \) in terms of \( d r / d \tau \), then ask yourself how this result can be used to work out the explicit form of the geodesic equation.

(a) \( \text{(10 points)} \) Use the metric to find an expression for \( d \tau / d \phi \).

(b) \( \text{(10 points)} \) Use the metric to find an expression for \( d \phi / d t \).

(c) \( \text{(10 points)} \) Explicitly write the equation that results when the free index \( \mu \) is contracted to \( \tau \) corresponding to the coordinate \( \tau \).

(d) \( \text{(10 points)} \) Explicitly write the equation that results when the free index \( \mu \) is contracted to \( \phi \).

If you cannot answer part (a), you can introduce unspecified functions \( f(\theta, r) \), \( f(t, r) \), and \( f(\phi, r) \) to the metric of the problem 3 of Problem Set 6. In 2007 it had also been a homework problem prior to this quiz. If you made modifications to the coordinate system in the exam, then contract to the coordinate \( \phi \) to the exam coordinate system as an explicit check on your work.

**Problem 18, The Stability of Schwarzschild Orbits**

(a) \( \text{(10 points)} \) From the geodesic equations and the Schwarzschild metric, find an expression for \( \frac{d^2 \phi}{d \tau^2} \). If you haven't yet found an expression for \( d \phi / d \tau \) in terms of \( \phi \), \( t \), and \( r \), you should depend on the equation of \( \phi \), \( t \), and \( r \). Use this expression to find an expression for \( \frac{d^2 \phi}{d \tau^2} \) in terms of \( \phi \), \( t \), and \( r \).

(b) \( \text{(10 points)} \) From the geodesic equations and the Schwarzschild metric, find an expression for \( \frac{d^2 \phi}{d t^2} \). If you haven't yet found an expression for \( d \phi / d t \) in terms of \( \phi \), \( t \), and \( r \), you should depend on the equation of \( \phi \), \( t \), and \( r \). Use this expression to find an expression for \( \frac{d^2 \phi}{d t^2} \) in terms of \( \phi \), \( t \), and \( r \).

(c) \( \text{(10 points)} \) From the geodesic equations and the Schwarzschild metric, find an expression for \( \frac{d^2 \phi}{d r^2} \). If you haven't yet found an expression for \( d \phi / d r \) in terms of \( \phi \), \( t \), and \( r \), you should depend on the equation of \( \phi \), \( t \), and \( r \). Use this expression to find an expression for \( \frac{d^2 \phi}{d r^2} \) in terms of \( \phi \), \( t \), and \( r \).

(d) \( \text{(10 points)} \) From the geodesic equations and the Schwarzschild metric, find an expression for \( \frac{d^2 \phi}{d \tau^2} \). If you haven't yet found an expression for \( d \phi / d \tau \) in terms of \( \phi \), \( t \), and \( r \), you should depend on the equation of \( \phi \), \( t \), and \( r \). Use this expression to find an expression for \( \frac{d^2 \phi}{d \tau^2} \) in terms of \( \phi \), \( t \), and \( r \).

If you cannot answer part (a), you can introduce unspecified functions \( f(\theta, r) \), \( f(t, r) \), and \( f(\phi, r) \) to the metric of the problem 3 of Problem Set 6. In 2007 it had also been a homework problem prior to this quiz. If you made modifications to the coordinate system in the exam, then contract to the coordinate \( \phi \) to the exam coordinate system as an explicit check on your work.
It is useful to consider \( r \) and \( \phi \) to be the independent variables, while \( t \) is a dependent variable. Find an expression for the geodesic equation (P18.2) for \( \mu = \phi \), and write the result in terms of the quantity \( L = r^2 \frac{d\phi}{d\tau} \).

Finally, we come to the question of stability. Substituting Eq. (P18.4) into Eq. (P18.3), the equation of motion for \( r \) can be written as

\[
\frac{d^2 r}{d\tau^2} = f_0(r) + f_1(r) \left( \frac{d\phi}{d\tau} \right)^2 L^2 r^4.
\]

Now consider a small perturbation about the circular orbit at \( r = r_0 \), and write an equation that determines the stability of the orbit. (That is, if some external force gives the orbiting body a small kick in the radial direction, how can you determine whether the perturbation will lead to stable oscillations, or whether it will start to grow?) You should express the stability requirement in terms of the unspecified functions \( f_0(r) \) and \( f_1(r) \). You are NOT asked to carry out the algebra of inserting the explicit forms that you have found for these functions.

These functions can only be plugged into the explicit forms that you have found for \( f_0(r) \) and \( f_1(r) \) in terms of the unspecified functions \( f \) and \( g \). You are NOT asked to express the stability condition as a specific equation for \( L \). You should simply express the stability condition in terms of \( f \) and \( g \), and explain as best you can how you would determine the pressure if you knew the answers to these two questions.

PROBLEM 19: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) (15 points) Using the fact that the energy density of mysterious stuff falls off as \( \frac{1}{\sqrt{V}} \), find the amount \( \Delta U \) by which the energy inside the piston changes when the volume is enlarged by \( \Delta V \). Define \( \Delta U \) to be positive if the energy increases.

(b) (5 points) If the (unknown) pressure of the mysterious stuff is called \( p \), how much work \( \Delta W \) is done by the agent that pulls out the piston?

(c) (5 points) Use your results from (a) and (b) to express the pressure \( p \) of the mysterious stuff in terms of its energy density \( u \). (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)
PROBLEM 1: DID YOU DO THE READING?

(a) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(b) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(c) The most obvious answers would be proton, neutron, and pion meson. However, there are many other possibilities, including many that were not mentioned by Weinberg. See Weinberg chapter 7, pages 136-8.

(d) The correct answers were the neutrino and the antiproton. The neutrino was first hypothesized by Wolfgang Pauli in 1932 in order to explain the kinematics of beta decay, and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.

(e) The correct answers were (ii), (v) and (vi). The others were incorrect for the following reasons:

(i) The earliest prediction of the CMB temperature, by Alpher and Herman, was 5 degrees, not 0.1 degrees.

(ii) Weinberg quotes his experimental colleagues as saying that the 3° Kelvindiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iii) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously any theory of the early universe."
Electric Charge: 

1. $\sim 10^9$
2. $\sim 1000$
3. $\sim 1$
4. $\sim 10^{-6}$
5. Either zero or negligible

Baryon Number: 

1. $\sim 10^{-20}$
2. $\sim 10^{-9}$
3. $\sim 10^{-6}$
4. $\sim 1$
5. Anywhere from $10^{-5}$ to $1$

Lepton Number: 

1. $\sim 10^9$
2. $\sim 1000$
3. $\sim 1$
4. $\sim 10^{-6}$
5. Could be as high as $\sim 1$, but is assumed to be very small

---

The figure below comes from Weinberg's Chapter 5, and is labeled “The Shifting Neutron-Proton Balance.”

(i) (3 points) During the period labeled „thermal equilibrium,” the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

\[
\text{antineutrino} + \text{proton} \leftrightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} \leftrightarrow \text{positron} + \text{proton} \
\]

(E) Neutrons and protons can be created and destroyed by reactions such as

\[
\text{proton} + \text{neutrino} \leftrightarrow \text{positron} + \text{antineutrino} \\
\text{neutron} + \text{antineutrino} \leftrightarrow \text{electron} + \text{positron} \
\]

(ii) (3 points) During the period labeled „neutron decay,” the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

\[
\text{antineutrino} + \text{proton} \leftrightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} \leftrightarrow \text{positron} + \text{proton} \
\]

(E) Neutrons and protons can be converted from one into the other through reactions such as

\[
\text{antineutrino} + \text{proton} \leftrightarrow \text{positron} + \text{neutron} \\
\text{neutrino} + \text{neutron} \leftrightarrow \text{electron} + \text{proton} \
\]

(F) Neutrons and protons can be converted from one into the other through reactions such as

\[
\text{antineutrino} + \text{proton} \leftrightarrow \text{electron} + \text{neutron} \\ 
\text{neutrino} + \text{neutron} \leftrightarrow \text{positron} + \text{proton} \
\]
The masses of the neutron and proton are not exactly equal, but instead
(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).
(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).
(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).
(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.
(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.
(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

During the period labeled “era of nucleosynthesis,” (choose one):
(A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.
(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.
(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.
(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

We will use the notation \( X^A \) to indicate a nucleus,* where \( X \) is the symbol for the element which indicates the number of protons, while \( A \) is the mass number, namely the total number of protons and neutrons. With this notation \( H^1, H^2, H^3, He^3 \) and \( He^4 \) stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

\[
\begin{align*}
\text{p} + \text{n} & \rightarrow \text{H}^2 + \gamma \text{H}^2 \\
\text{p} & \rightarrow \text{He}^3 + \gamma \text{He}^3 \\
\text{n} & \rightarrow \text{He}^4 + \gamma \\
\end{align*}
\]

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place in general with different products. Students who described chains different from those of Weinberg, but that still may take place (in general with different products), got full credit. For example:

\[
\begin{align*}
\text{n} + \text{p}^3 & \rightarrow \text{H}^2 + \gamma \text{H}^2 \\
\text{H}^2 & \rightarrow \text{He}^4 + \gamma \\
\text{n} & \rightarrow \text{He}^4 + \gamma \\
\end{align*}
\]

Notice that some students labeled about atoms, while we are talking about nuclei, which mostly survive until the present time.

The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 GeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 MeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 KeV.

Problem 3: Did you do the reading?

(a) (8 points)

(i) (4 points)

The deuterium bottleneck is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium \( (\text{H}^2) \) plays a crucial role in the chain of reactions that produce helium, starting from protons and neutrons. We will use the notation \( X^A \) to indicate a nucleus, where \( X \) is the symbol for the element which indicates the number of protons, while \( A \) is the mass number, namely the total number of protons and neutrons. With this notation \( \text{H}^1, \text{H}^2, \text{H}^3, \text{He}^3 \) and \( \text{He}^4 \) stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

\[
\begin{align*}
\text{p} + \text{n} & \rightarrow \text{H}^2 + \gamma \text{H}^2 \\
\text{p} & \rightarrow \text{He}^3 + \gamma \text{He}^3 \\
\text{n} & \rightarrow \text{He}^4 + \gamma \\
\end{align*}
\]

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place in general with different products. Students who described chains different from those of Weinberg, but that still may take place (in general with different products), got full credit. For example:

\[
\begin{align*}
\text{n} + \text{p}^3 & \rightarrow \text{H}^2 + \gamma \text{H}^2 \\
\text{H}^2 & \rightarrow \text{He}^4 + \gamma \\
\text{n} & \rightarrow \text{He}^4 + \gamma \\
\end{align*}
\]

Notice that some students labeled about atoms, while we are talking about nuclei, which mostly survive until the present time.

The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 GeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 MeV.

The neutron is more massive than a proton with a rest energy difference of 1.293 KeV.

Problem 3: Did you do the reading?

(a) (8 points)

(i) (4 points)

The deuterium bottleneck is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium \( (\text{H}^2) \) plays a crucial role in

...
Numerically, $t \approx 1.06977 a/c$.

We have

$$\left(\frac{\mathcal{E}_1}{1 - \text{cosh}^2} \right)^{\frac{1}{2}} = \frac{\mathcal{E}}{\mathcal{V}}.$$  

With $\mathcal{E} = \mathcal{V} = \theta$,

We can use these results in the first equation to solve for $\alpha$. Nothing that

$$\theta = \frac{\mathcal{V}}{\mathcal{E}} \left(1 - \frac{\mathcal{E}}{\mathcal{V}} \text{cosh} \theta \right) = \frac{\mathcal{V}}{\mathcal{E}} \left(1 - \theta \text{cosh} \theta \right) = \theta$$

Solving the equation for the second equation at $\theta = \mathcal{V}$ yields a solution for $\theta$.

$$\theta \approx 3 = \theta \text{cosh} \left(1 - \theta \text{cosh} \theta \right) = \frac{\mathcal{V}}{\mathcal{E}} \left(1 - \theta \text{cosh} \theta \right) \approx 2$$

The evolution of the universe can be described by the following parameter equations:

**Problem 4: Evolution of an Open Universe**

Solution written by Pauule Bollottt

$$\tau \approx 3 \approx \text{m}.$$

To good accuracy, the numerator in the expression above can be rounded off to $3L.10^{-6} \approx \text{m}$. Form $L = \mathcal{L}_{\text{cd}}$. The above expression agrees with the results obtained for a Newtonian universe described by the formula

$$(\text{He} + \text{He}) \approx \frac{\mathcal{L}}{\mathcal{E}} + \frac{\mathcal{L}}{\mathcal{E}} = \frac{\mathcal{L}}{\mathcal{E}} + \frac{\mathcal{L}}{\mathcal{E}} = \frac{\mathcal{L}}{\mathcal{E}}$$

We get

$$\mathcal{V} = \frac{\mathcal{V}}{\mathcal{E}} \left(1 - \frac{\mathcal{E}}{\mathcal{V}} \text{cosh} \theta \right) = \frac{\mathcal{V}}{\mathcal{E}} \left(1 - \theta \text{cosh} \theta \right) = \theta$$

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The evolution of the universe can be described by the following parameter equations:

$$\tau \approx 3 \approx \text{m}.$$
PROBLEM 5: ANTICIPATING A BIG CRUNCH

The critical density is given by
\[ \rho_c = \frac{3}{8\pi G}, \]
so the mass density is given by
\[ \rho = \Omega_0 \rho_c = 2 \rho_c = \frac{3}{4\pi G}. \]
(S5.1)

Substituting this relation into
\[ H^2_0 = \frac{8\pi}{3} G \rho - \frac{k}{a^2}, \]
we find
\[ H^2_0 = 2 H^2_0 - \frac{k}{a^2}, \]
from which it follows that
\[ a \sqrt{k} = \frac{c}{H_0}. \]
(S5.2)

Now use
\[ \alpha = \frac{4\pi}{3} G \rho a^3 \frac{k^3}{2c^2}. \]
Substituting the values we have from Eqs. (S5.1) and (S5.2) for \( \rho \) and \( a/\sqrt{k} \), we have
\[ \alpha = \frac{c}{H_0}. \]
(S5.3)

To determine the value of the parameter \( \theta \), use
\[ a \sqrt{k} = \alpha (1 - \cos \theta) \]
which when combined with Eqs. (S5.2) and (S5.3) implies that \( \cos \theta = 0 \).
The equation \( \cos \theta = 0 \) has multiple solutions, but we know that the \( \theta \)-parameter for a closed matter-dominated universe varies between 0 and \( \pi \) during the expansion phase of the universe. Within this range, \( \cos \theta = 0 \) implies that \( \theta = \frac{\pi}{2} \). Thus, the age of the universe at the time these measurements are made is given by
\[ t = \frac{\alpha c}{(\frac{\pi}{2} - \sin \theta)}. \]

The total lifetime of the closed universe corresponds to
\[ \theta = 2\pi, \]
or \( t_{\text{final}} = 2\pi \alpha c = 2\pi H_0, \]
so the time remaining before the crunch is given by
\[ \frac{\theta}{\pi} = \frac{\alpha c}{2\pi H_0} = \text{time remaining}. \]

PROBLEM 6: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since \( \psi = \phi = \text{constant} \), \( d\psi = d\phi = 0 \), and for light rays one always has \( d\tau = 0 \). The line element therefore reduces to
\[ 0 = -c^2 dt^2 + a(t)^2 d\psi^2. \]
Rearranging gives
\[ \left( \frac{d\psi}{dt} \right)^2 = \frac{c^2}{a(t)} \]
The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the \( \psi \) coordinate that can be reached by time \( t \) is found by integrating its rate of change:
\[ \psi_{\text{hor}} = \int_0^t \frac{c}{a(t')} dt'. \]
The physical horizon distance is the proper length of the shortest line drawn at a closed matter-dominated universe where between \( t \) and \( t' \), the equation \( \cos \theta = 0 \) holds multiple solutions, but we know that the \( \theta \)-parameter for a closed universe varies between 0 and \( \pi \). Thus, the condition \( \cos \theta = 0 \) implies that \( \theta = \frac{\pi}{2} \). Thus, the age of the universe at the time these measurements are made is given by
\[ t = \frac{\alpha c}{\pi - \theta}. \]

To determine the value of the parameter \( \theta \), use
\[ \frac{\theta}{\pi} = \frac{\alpha c}{2\pi H_0} = \text{time remaining}. \]

The equation \( \cos \theta = 0 \) has multiple solutions, but we know that the \( \theta \)-parameter for a closed matter-dominated universe varies between 0 and \( \pi \) during the expansion phase of the universe. Within this range, \( \cos \theta = 0 \) implies that \( \theta = \frac{\pi}{2} \). Thus, the age of the universe at the time these measurements are made is given by
\[ t = \frac{\alpha c}{(\frac{\pi}{2} - \sin \theta)}. \]

The total lifetime of the closed universe corresponds to
\[ \theta = 2\pi, \]
or \( t_{\text{final}} = 2\pi \alpha c = 2\pi H_0, \]
so the time remaining before the crunch is given by
\[ \frac{\theta}{\pi} = \frac{\alpha c}{2\pi H_0} = \text{time remaining}. \]
SOLVED PROBLEM 7: LENGTHS AND AREAS IN A TWO-DIMENSIONAL CURVATURE METRIC

Problem Statement:

In the Robertson-Walker coordinate system, the closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates $x, y, z, w$. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 0)$ as the center of the coordinate system. If we define the north pole to be $(0, 0, 0, 1)$ and the south pole to be $(0, 0, 0, -1)$, this means that the south pole is at a distance of $a$ from the origin.

The Robertson-Walker coordinate system is one of the coordinate systems that can be used to describe the structure of the universe. In this system, the universe is modeled as a 3-dimensional sphere embedded in a 4-dimensional Euclidean space.

Discussion:

Some students answered that the photon would return to its starting point. However, this is not correct. The photon would not return to its starting point because the universe is curved, and the path of the photon would be affected by the curvature of the space.

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\[
\text{Finally, the third segment is identical to the first, so } S_3 = S_1. \text{ The total length is then }
\]
\[
S = 2S_1 + S_2 = 2\left(\frac{r_0 + 1}{2} + \pi r_0 \left(1 + b r_0\right)\right) + \pi r_0 (1 + b r_0).
\]

\[
\text{To find the area, it is best to divide the region into concentric strips as shown:}
\]
\[
\text{The area is then }
\]
\[
\varphi p (\varphi + 1) \frac{d\varphi}{2} = \varphi p
\]
The function \( \frac{\partial \rho}{\partial \rho} = \frac{\partial r}{\partial \rho} \) is replaced by a new coordinate \( \sigma \equiv \rho \). By the previous calculation, the area of such a shell is just the product of the infinitesimal variances of the two coordinates:

\[
A = (0, \sigma) \varepsilon d \sigma = V
\]

Thus, the area is then obtained by integrating over the range of the coordinate \( \sigma \).

The total volume is then obtained by integrating over the range of the coordinate \( \rho \).

\[
V = \int_{\rho_0}^{\rho_1} 2\pi \rho^2 d\rho = \int_{\rho_0}^{\rho_1} 2\pi \rho^2 d\rho = \pi \rho_0^2 (\rho_1^2 - \rho_0^2)
\]

This is precisely the well-known formula for the area of a Euclidean sphere. Hence, the answer above becomes the well-known formula in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere.

As a check, notice that if \( \frac{d\rho}{d\rho} \neq 1 \), then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, \( 4\pi \).

\[
\int_{\rho_0}^{\rho_1} \frac{d\rho}{d\rho} = V
\]

Therefore, notice that if \( \frac{d\rho}{d\rho} \neq 1 \), then the answer above becomes the well-known formula for the area of a Euclidean sphere.
The total volume is then
\[ V = \int dV = a^3 \left( \frac{t}{r_{\text{max}}} \right) \int_0^{r_{\text{max}}} r^2 \sin \theta \sqrt{1 - kr^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi r^2 \sin \theta. \]

We can do the angular integrations immediately:
\[ V = 4\pi a^3 \left( \frac{t}{r_{\text{max}}} \right) \int_0^{r_{\text{max}}} r^2 \sin \theta \sqrt{1 - kr^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi = 4\pi a^3 \left( \frac{t}{r_{\text{max}}} \right) \int_0^{r_{\text{max}}} r^2 \sin \theta \sqrt{1 - kr^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi. \]

**Pedagogical Note:** If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:

The cell includes the volume lying between \( r \) and \( r + dr \), between \( \theta \) and \( \theta + d\theta \), and between \( \phi \) and \( \phi + d\phi \). In the limit as \( dr, d\theta, \) and \( d\phi \) all approach zero, the cell approaches a rectangular solid with sides of length:

\[ ds_1 = a \left( \frac{t}{r_{\text{max}}} \right) dr \sqrt{1 - kr^2}, \]
\[ ds_2 = a \left( \frac{t}{r_{\text{max}}} \right) r d\theta, \]
\[ ds_3 = a \left( \frac{t}{r_{\text{max}}} \right) r \sin \theta d\theta. \]

Here each is calculated by using the metric to find \( ds_i \) in each case according to:
\[ \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \left( \frac{t}{r_{\text{max}}} \right) = \varepsilon_{sp} \]

so
\[ e^{4p} \left( \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \right) = \varepsilon^{sp}. \]

The separation between \( A \) and \( B \) is purely in the radial direction, so the proper time along the path from point \( A \) to \( B \) is
\[ \frac{\varepsilon^{\alpha \beta} - 1}{4p_{\alpha \beta}} = \varepsilon^{\theta \rho}. \]

It becomes singular at
\[ \left( \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \right) = \varepsilon_{sp}. \]

Further, since the metric contains the factor
\[ (1 - \frac{2GM}{rc^2}) \]

The Schwarzschild horizon is the value of \( r \) for which the metric becomes singular:

\[ (0 > \frac{r}{r_{\text{H}}}) \left[ \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \right] = \Lambda \]

\[ (0 < \frac{r}{r_{\text{H}}}) \left[ \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \right] = \Lambda \]

The answer is
\[ \phi = \sin \frac{1}{r_{\text{H}}} \]
\[ \phi = \frac{1}{r_{\text{H}}} \]

**Extension:** The integral can then be evaluated using the substitution
\[ \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \theta \sin \theta d\theta d\phi = \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \theta \sin \theta d\theta d\phi \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} = \Lambda. \]

**Problem 11: The Schwarzschild Metric**

**a)** The Schwarzschild horizon is the value of \( r \) for which the metric becomes singular. Since the metric contains the factor
\[ (1 - \frac{2GM}{rc^2}) \]

it becomes singular at
\[ R_{\text{S}} = 2GMc^2. \]

**b)** The separation between \( A \) and \( B \) is purely in the radial direction, so the proper length of a segment along the path joining them is given by
\[ ds^2 = \left( \frac{t}{r_{\text{max}}} \right) - 1 \int_0^{r_{\text{max}}} r^2 \sin \theta \sqrt{1 - kr^2} d\theta d\phi. \]

Here each is calculated by using the metric to find \( ds^2 \) in each case according to:
\[ \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} = \varepsilon^{\theta \rho}. \]

We can do the angular integrations immediately:
\[ \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \left( \frac{t}{r_{\text{max}}} \right) \int_0^{r_{\text{max}}} r^2 \sin \theta \sqrt{1 - kr^2} d\theta d\phi = \Lambda \]

so
\[ e^{4p} \left( \frac{\varepsilon_{\alpha \beta} - 1}{4p_{\alpha \beta}} \right) = \varepsilon^{sp}. \]
The proper distance from A to B is obtained by adding the proper lengths of all the segments along the path, so
\[ s_{AB} = \int_{r_A}^{r_B} \sqrt{1 - \frac{2GM}{rc^2}} \, dr. \]

EXTENSION: The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for \( s_{AB} \) as
\[ s_{AB} = \int_{r_A}^{r_B} \sqrt{r} \, dr \sqrt{r - R_S}. \]

Then introduce the hyperbolic trigonometric substitution
\[ r = R_S \cosh^2 u. \]

One then has
\[ \sqrt{r - R_S} = \sqrt{R_S \sinh u} \, du = 2R_S \cosh u \sinh u \, du, \]
and the indefinite integral becomes
\[ \int \sqrt{r} \, dr \sqrt{r - R_S} = 2R_S \int \cosh^2 u \, du = \frac{R_S}{2} (u + \sinh 2u) \]
\[ = R_S \left( u + \sinh u \cosh u \right) \]
\[ = R_S \sinh \left( \sqrt{\frac{r}{R_S} - 1} \right). \]

Thus,
\[ s_{AB} = R_S \left[ \sinh \left( \sqrt{\frac{r_B}{R_S} - 1} \right) - \sinh \left( \sqrt{\frac{r_A}{R_S} - 1} \right) \right] + \sqrt{r_B} (r_B - R_S) - \sqrt{r_A} (r_A - R_S). \]

c) A tick of the clock and the following tick are two events that differ only in their time coordinate. Thus, the metric reduces to
\[ -c^2 d\tau^2 = -\left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2, \]
so
\[ d\tau = \sqrt{1 - \frac{2GM}{rc^2}} \, dt. \]

The reading on the observer's clock corresponds to the proper time interval \( d\tau \), so the corresponding interval of the coordinate \( t \) is given by
\[ \Delta t_A = \Delta \tau_A \sqrt{1 - \frac{2GM}{r_A c^2}}. \]

d) Since the Schwarzschild metric does not change with time, each pulse leaving A will arrive at B with the same length of time to reach B, thus the proper emitted by A will take the same length of time to reach B. Thus, the proper events by A and B will all arrive at B with a time coordinate spacing \( \Delta t_B = \Delta t_A = \Delta \tau_A \sqrt{1 - \frac{2GM}{r_A c^2}}. \)

The clock at B, however, will read the proper time and not the coordinate time. Thus,
\[ \Delta \tau_B = \sqrt{1 - \frac{2GM}{r_B c^2}} \Delta t_B = \left[ \left( 1 - \frac{2GM}{r_B c^2} \right)^{\frac{1}{2}} - \left( 1 - \frac{2GM}{r_A c^2} \right)^{\frac{1}{2}} \right] \Delta \tau_A. \]

e) From parts (a) and (b), the proper distance between A and B can be rewritten as
\[ s_{AB} = R_S \int_{r_A}^{r_B} \sqrt{r} \, dr \sqrt{r - R_S}. \]

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of \( r = R_S \), say \( R_S < r < R_S + \epsilon \).
range the quantity \( \sqrt{r} \) in the numerator can be approximated by \( \sqrt{R} \), so the contribution has the form

\[
\int_{R}^{\infty} \frac{\sqrt{r}}{r - R} \, dr
\]

Changing the integration variable to \( u \equiv r - R \), the contribution can be easily evaluated:

\[
\int_{0}^{\infty} \frac{\sqrt{u}}{\sqrt{u} - \epsilon} \, du = 2 \sqrt{R} \epsilon < \infty.
\]

So, although the integrand is infinite at \( r = R \), the integral is still finite. The proper distance between \( A \) and \( B \) does not diverge.

Looking at the answer to part (d), however, one can see that when \( r = 0 \):

\[
\frac{\partial}{\partial \theta} \left( \frac{\sqrt{r}}{r - R} \right) = \frac{\sqrt{r}}{\sqrt{r} - \epsilon}.
\]

For part (a), the metric is given by

\[
\frac{ds^2}{2} = \frac{\sqrt{r}}{r - R} \, dr + \frac{r}{r - R} \, d\theta.
\]

First taking \( i = r \), the nonvanishing terms in the geodesic equation become

\[
0 = \frac{\partial}{\partial \theta} \left( \frac{\sqrt{r}}{r - R} \right) \quad \text{and} \quad \frac{\sqrt{r}}{\sqrt{r} - \epsilon} \quad \text{such that}
\]

\[
\frac{\partial^2}{\partial \theta^2} \left( \frac{\sqrt{r}}{r - R} \right) = \frac{\sqrt{r}}{\sqrt{r} - \epsilon}.
\]

For \( i = \theta \), one has the simplification that \( g_{ij} \) is independent of \( \theta \) for all \( (i,j) \).

\[
\frac{d}{d\lambda} \left( \frac{\sqrt{r}}{r - R} \right) = \frac{\sqrt{r}}{\sqrt{r} - \epsilon}.
\]

The geodesic equation for a curve \( x^i(\lambda) \), where the parameter \( \lambda \) is the arc length along the curve, can be written as

\[
\frac{d}{d\lambda} \left\{ g_{ij} \frac{dx^j}{d\lambda} \right\} = \frac{1}{2} \left( \frac{\partial^2}{\partial \lambda^2} g_{ij} \right) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}.
\]

Here the indices \( j, k \), and \( \ell \) are summed from 1 to the dimension of the space, so there is one equation for each value of \( i \).

\[
\frac{d}{d\lambda} \left( \frac{g_{rr}}{r - R} \right) = \frac{1}{2} \left( \frac{\partial^2}{\partial \lambda^2} g_{rr} \right) \frac{1}{r - R}.
\]

or

\[
\frac{d^2}{d\lambda^2} \left( \frac{g_{rr}}{r - R} \right) = \frac{1}{2} \left( \frac{\partial^2}{\partial \lambda^2} g_{rr} \right) \frac{1}{r - R}.
\]

The proper distance between \( A \) and \( B \) does not diverge.

The geodesic equation for a curve \( x^i(\lambda) \) where the parameter \( \lambda \) is the arc length along the curve can be written as

\[
\frac{d}{d\lambda} \left( \frac{\sqrt{r}}{r - R} \right) = \frac{\sqrt{r}}{\sqrt{r} - \epsilon}.
\]

The time interval \( \Delta \tau \) between

\[
\frac{\partial}{\partial \theta} \left( \frac{\sqrt{r}}{r - R} \right) = 0 \quad \text{and} \quad \frac{\sqrt{r}}{\sqrt{r} - \epsilon},
\]

therefore the indices \( j, k \), and \( \ell \) are summed from 1 to the dimension of the space, so

\[
\frac{d}{d\lambda} \left( \frac{g_{rr}}{r - R} \right) = \frac{1}{2} \left( \frac{\partial^2}{\partial \lambda^2} g_{rr} \right) \frac{1}{r - R}.
\]

So, although the integrand is infinite at \( r = R \), the integral is still finite. The proper distance between \( A \) and \( B \) does not diverge.
\[
\begin{align*}
\int 2^\phi dz = \rho P \\
\int \left(\frac{v}{d} + 1\right)^\phi dP + \int 0^\phi dP = V
\end{align*}
\]

To cover the area for which \( r > 0 \) and \( L = \phi \) must be integrated from 0 to \( \phi = \pi \) and \( \rho = \sqrt{0} \) must be integrated from 0 to \( r = d \) and \( \phi = \sqrt{0} \).

The area is then

\[
\int \frac{\sqrt{0}}{d} + 1 \phi \right)^\phi dP = V dP
\]

Since the metric does not contain a term in \( dP \) of the form \( \rho \) and \( \theta \) directions are

\[
\int \frac{\sqrt{0}}{d} + 1 \phi \right)^\phi dP = V dP
\]

Thus, if one considers a small region in which \( r \) is the integral orthogonal, then the area covered does not contain a term in \( dP \) of the form \( \rho \) and \( \theta \) directions are

\[
\int \frac{\sqrt{0}}{d} + 1 \phi \right)^\phi dP = V dP
\]

Since the metric does not contain a term in \( dP \) of the form \( \rho \) and \( \theta \) directions are

\[
\int \frac{\sqrt{0}}{d} + 1 \phi \right)^\phi dP = V dP
\]

Since the metric does not contain a term in \( dP \) of the form \( \rho \) and \( \theta \) directions are

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\]

Since the metric does not contain a term in \( dP \) of the form \( \rho \) and \( \theta \) directions are

\[
\int \frac{\sqrt{0}}{d} + 1 \phi \right)^\phi dP = V dP
\]
\[ A = 2 \pi \int_0^r \int_0^{r'} \sqrt{1 + \left( \frac{r'}{a} \right)^2} \, dr \, dr' \]

You were not asked to carry out the integration, but it can be done by using the substitution

\[ u = 1 + \left( \frac{r'}{a} \right)^2, \quad du = \left( \frac{1}{a^2} \right) dr' \]

The result is

\[ A = 4 \pi a^2 \left[ 2 + \left( \frac{3r_0^2}{a^2} + r_0 - 2 \right) \sqrt{1 + \frac{r_0}{a}} \right] \]

(d) The nonzero metric coefficients are given by

\[ g_{rr} = 1 + \frac{r}{a}, \quad g_{\theta \theta} = r^2 \]

so the metric is diagonal. For \( i = 1 = r \), the geodesic equation becomes

\[ \frac{d}{ds} \left( g_{rr} \frac{dr}{ds} \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{1}{g_{rr}} \right) \frac{d}{ds} \left( g_{\theta \theta} \frac{d\theta}{ds} \right) = 0 \]

Simplifying slightly,

\[ \frac{d}{ds} \left( \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{1}{1 + \frac{r}{a}} \right) \frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \]

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

\[ \frac{d}{ds} \left( \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{1}{1 + \frac{r}{a}} \right) \frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \]

The equation is simpler, because none of the coefficients depend on \( \theta \), so the right-hand side of the geodesic equation vanishes. One thus simply

\[ 0 = \zeta = \zeta x \]

For most purposes this is the best way to write the equation, since it reads-in

\[ 0 = \left( \frac{s\theta}{\theta \theta} \right) \frac{\zeta}{\theta \theta} + \frac{z\theta}{\theta \theta} \frac{\zeta}{\theta \theta} \]

so the right-hand side of the geodesic equation vanishes. One thus simply

\[ 0 = \left( \frac{s\theta}{\theta \theta} \right) \frac{\zeta}{\theta \theta} \]

The nonzero metric coefficients are given by

\[ \frac{d}{ds} \left( \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{1}{1 + \frac{r}{a}} \right) \frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \]

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

\[ \frac{d}{ds} \left( \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{1}{1 + \frac{r}{a}} \right) \frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \]

The equation is simpler, because none of the coefficients depend on \( \theta \), so the right-hand side of the geodesic equation vanishes. One thus simply

\[ 0 = \zeta = \zeta x \]

For most purposes this is the best way to write the equation, since it reads-in

\[ 0 = \left( \frac{s\theta}{\theta \theta} \right) \frac{\zeta}{\theta \theta} + \frac{z\theta}{\theta \theta} \frac{\zeta}{\theta \theta} \]
Now introduce a primed coordinate system that is related to the original system by a rotation in the $y$-$z$ plane by an angle $\alpha$:

\begin{align*}
  x' &= x \\
  y' &= y' \cos \alpha - z' \sin \alpha \\
  z' &= z' \cos \alpha + y' \sin \alpha.
\end{align*}

The rotated equator, which we seek to describe, is just the standard equator in the primed coordinates:

\begin{align*}
  x' &= r \cos \psi, \\
  y' &= r \sin \psi, \\
  z' &= 0.
\end{align*}

Using the relation between the two coordinate systems given above,

\begin{align*}
  x &= r \cos \psi \\
  y &= r \sin \psi \cos \alpha \\
  z &= r \sin \psi \sin \alpha.
\end{align*}

Using again the relations between polar and Cartesian coordinates,

\begin{align*}
  \cos \theta &= \frac{z}{r} \sin \psi \sin \alpha \\
  \tan \phi &= \frac{y}{x} \tan \psi \cos \alpha.
\end{align*}

(b) A segment of the equator corresponding to an interval $d\psi$ has length $a d\psi$, so the parameter $\psi$ is proportional to the arc length. Expressed in terms of the metric, this relationship becomes

\[ ds^2 = g_{ij} dx^i d\psi dx^j d\psi = a^2 d\psi^2. \]

Thus the quantity

\[ A = g_{ij} dx^i d\psi dx^j d\psi \]

is equal to $a^2$, so the geodesic equation (5.50) reduces to the simpler form of Eq. (5.52). (Note that we are following the notation of Lecture Notes 5, except that the variable used to parameterize the path is called $\psi$ rather than $\lambda$ or $s$.)

Although $A$ is not equal to 1 as we assumed in Lecture Notes 5, it is easily seen that the prime used to emphasize the path is indeed $A$, not a.

Thus, the geodesic equation

\[ \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \psi} \right) \theta \sin \theta = \frac{\theta}{\psi} \frac{\partial}{\partial \psi} \]

is equal to $\theta / \psi$, so the geodesic equation reduces to the simpler form of

\[ \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \psi} \right) \theta \sin \theta = \frac{\theta}{\psi} \frac{\partial}{\partial \psi} \]

in the primed coordinates, while we seek to describe, is just the standard equation:

\[ \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \psi} \right) \theta \sin \theta = \frac{\theta}{\psi} \frac{\partial}{\partial \psi} \]

in the primed coordinates. The rotated equator, which we seek to describe, is just the standard equator:

\begin{align*}
  \theta &= \psi \\
  \phi &= \psi \phi.
\end{align*}

(c) This part is mainly algebra. Taking the derivative of

\[ \cos \theta = \sin \psi \sin \alpha \]

implies

\[ -\sin \theta d\theta = \cos \psi \sin \alpha d\psi. \]

Then, using the trigonometric identity $\sin^2 \theta = 1 - \cos^2 \theta$, one finds

\[ \sin \theta = \sqrt{1 - \sin^2 \psi \sin^2 \alpha}. \]
\[ d\theta = -\cos\psi \sin\alpha \sqrt{1 - \sin^2\psi \sin^2\alpha}. \]

Similarly
\[ \tan\phi = \tan\psi \cos\alpha = \Rightarrow \sec^2\phi d\phi = \sec^2\psi d\psi \cos\alpha. \]

Then
\[ \sec^2\phi = \tan^2\phi + 1 = \tan^2\psi \cos^2\alpha + 1 = 1 \cos^2\psi \left[ \sin^2\psi \cos^2\alpha + \cos^2\psi \right] = \sec^2\psi \left[ \sin^2\psi (1 - \sin^2\alpha) + \cos^2\psi \right]. \]

So
\[ d\phi = d\psi \cos\alpha \sqrt{1 - \sin^2\psi \sin^2\alpha}. \]

To verify the geodesic equations of part (b), it is easiest to check the second one first:
\[ \sin^2\theta d\phi d\psi = (1 - \sin^2\psi \sin^2\alpha) \cos\alpha \sqrt{1 - \sin^2\psi \sin^2\alpha} = \cos\alpha, \]
so clearly
\[ \frac{d}{d\psi} \left( \sin^2\theta d\phi d\psi \right) = \frac{d}{d\psi} \left( \cos\alpha \right) = 0. \]

To verify the first geodesic equation from part (b), first calculate the left-hand side,
\[ \frac{d^2\theta}{d\psi^2}, \]
using our result for \( \frac{d\theta}{d\psi} \):
\[ \frac{d^2\theta}{d\psi^2} = \frac{d}{d\psi} \left( \frac{d\theta}{d\psi} \right) = \frac{d}{d\psi} \left( -\cos\psi \sin\alpha \sqrt{1 - \sin^2\psi \sin^2\alpha} \right). \]

After some straightforward algebra, one finds
\[ \frac{d^2\theta}{d\psi^2} = \sin\psi \sin\alpha \cos^2\alpha \left[ 1 - \sin^2\psi \sin^2\alpha \right]^{3/2}. \]

The right-hand side of the first geodesic equation can be evaluated using the expression found above for \( \frac{d\phi}{d\psi} \), giving
\[ \sin\theta \cos\theta \left( \frac{d\phi}{d\psi} \right)^2 = \sqrt{1 - \sin^2\psi \sin^2\alpha} \sin\psi \sin\alpha \cos^2\alpha \left[ 1 - \sin^2\psi \sin^2\alpha \right]^{3/2}. \]

So the left- and right-hand sides are equal.

**Problem 15: Geodesics in a Closed Universe**

(a) (7 points)

For purely radial motion,
\[ d\theta = d\phi = 0, \]
so the line element reduces
\[ -c^2 dt^2 = a^2 (t) \left( \frac{dr}{c^2} \right)^2 \left[ 1 - \frac{r^2}{a^2} \right]. \]

Dividing by \( \frac{dt}{c} \),
\[ -c^2 \left( \frac{d\tau}{dt} \right)^2 = -c^2 + a^2 \left( \frac{t}{c} \right)^2 \left( \frac{dr}{c^2} \right)^2 \left[ 1 - \frac{r^2}{a^2} \right]. \]

Rearranging,
\[ \frac{d\tau}{dt} = \frac{c}{\sqrt{1 - \frac{a^2}{c^2} \left( \frac{t^2}{c^2} \right) \left( \frac{1 - r^2}{a^2} \right)}} \left( \frac{dr}{c^2} \right). \]

(b) (3 points)

\[ \frac{dt}{d\tau} = 1 \]
\[ \frac{d\tau}{dt} = \frac{1}{\sqrt{1 - \frac{a^2}{c^2} \left( \frac{t^2}{c^2} \right) \left( \frac{1 - r^2}{a^2} \right)}} \left( \frac{dr}{c^2} \right). \]

(c) (10 points)

During any interval of clock time \( dt \), the proper time that would be measured by a clock moving with the object is given by \( dt' \), as given by the line element above.

The right-hand side of the first geodesic equation can be evaluated using the expression found above for \( \frac{d\phi}{d\psi} \), giving
\[ \frac{d}{d\psi} \left( \frac{\cos\phi \sin\psi - 1}{\cos\theta} \right) \sin\theta = \frac{d\phi}{d\psi}, \]

where some straightforward algebra, one finds
\[ \frac{d}{d\psi} \left( \frac{\cos\phi \sin\psi - 1}{\cos\theta} \right) \sin\theta = \frac{d\phi}{d\psi}. \]
\[ \frac{dP}{dt} \left( x^4 - 1 \right) \frac{v}{v^4} u = \frac{dP}{dx} \frac{v}{v^4} \left( x^4 - 1 \right) u \]

Lest the answer from (e) simplify the expression for \( p \), this translates to

\[ \frac{dP}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} = \frac{d}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} = d \]

This formula is true for any possible value of \( p \) (i.e., \( dx = d \)), and

\[ \frac{x^4 - 1}{v^4} = C \]

or finally

\[ \frac{x^4 - 1}{v^4} = \lambda \]

This makes the form shown in the question with

\[ \left\{ \frac{2P}{x} \left( x^4 - 1 \right) \frac{v}{v^4} \right\} = \left\{ \frac{2P}{x} \left( x^4 - 1 \right) \frac{v}{v^4} \right\} \frac{2P}{p} \]

The geodesic equation becomes

\[ \frac{x^4 - 1}{v^4} = \lambda \]

The only nonzero contribution on the right-hand side arises from \( \lambda = 1 \). Likewise, the derivative will require to be calculated. Thus, the only right-hand side is proportional to

\[ \frac{d}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} \]

If \( \lambda = 0 \), there is no motion in the direction \( \phi \), or direction. However, the

\[ \begin{aligned} \frac{dP}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} &= \frac{dP}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} \\ \frac{d}{dx} \left( x^4 - 1 \right) \frac{v}{v^4} &= \lambda \]

Integrate to find the total proper time.
Multiply the geodesic equation by \( m \), and then use the above result to rewrite it as

\[
\frac{d}{d\tau} \left\{ a p \sqrt{1 - r^2} \right\} = ma^2 r (1 - r^2)^{3/2} \left( \frac{dr}{d\tau} \right)^2.
\]

Expanding the left-hand side,

\[
\text{LHS} = \frac{d}{d\tau} \left\{ a p \sqrt{1 - r^2} \right\} = \frac{1}{\sqrt{1 - r^2}} \frac{d}{d\tau} \left\{ a p \right\} + apr \frac{1 - r^2}{2} \left( \frac{dr}{d\tau} \right)^2 = \frac{1}{\sqrt{1 - r^2}} \frac{d}{d\tau} \left\{ a p \right\} + ma^2 r (1 - r^2)^{3/2} \left( \frac{dr}{d\tau} \right)^2.
\]

Inserting this expression back into the left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

\[
\frac{d}{d\tau} \left\{ a p \sqrt{1 - r^2} \right\} = 0.
\]

Multiplying by \( \sqrt{1 - r^2} \), one has the desired result:

\[
\frac{d}{d\tau} \left\{ a p \right\} = 0 = \frac{dp}{dt}.
\]

To find the length of the radial line shown, one must integrate this expression from the value of \( u \) at the center, which is 0, to the value of \( u \) at the outer edge, which is \( a \). So at the center, which is 0, the value of \( p \) at the outer edge, which is \( a \),

\[
\int_0^a 2\pi \sqrt{a u (a - u)} \, du = \pi a^2.
\]

(b) For \( u = \) constant, the expression for the metric reduces to

\[
ds^2 = u \, d\theta^2 = \frac{\pi a}{2} \sqrt{u} \, d\theta.
\]

Since \( \theta \) runs from 0 to \( 2\pi \), and for the circumference,

\[
\theta \, d\theta = s_p \iff \theta \, d\theta = \frac{\pi a}{2} \, d\theta
\]

one has the desired result:

\[
\int_0^{2\pi} \frac{\pi a}{2} \, d\theta = \pi a^2.
\]

(c) To evaluate the answer to first order in \( du \), means to neglect any terms that would be proportional to \( du^2 \) or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case the metric reduces to

\[
d s^2 = 2\pi \frac{n}{n - 1} \, d\theta^2 + \left( \frac{a}{n - 1} \right)^{1/2} \, du^2.
\]

One must integrate the expression from the value of \( \theta \) at the center, which is 0, to the value of \( \theta \) at the outer edge, which is \( \frac{\pi a}{2} \),

\[
\int_0^{\frac{\pi a}{2}} 2\pi \frac{n}{n - 1} \, d\theta + \int_0^{\frac{\pi a}{2}} \left( \frac{a}{n - 1} \right)^{1/2} \, du = \pi a^2.
\]

Expanding the left-hand side:

\[
\int_0^{\frac{\pi a}{2}} 2\pi \frac{n}{n - 1} \, d\theta + \int_0^{\frac{\pi a}{2}} \left( \frac{a}{n - 1} \right)^{1/2} \, du = \left( \frac{\pi a}{2} \right)^2.
\]

Thus the geodesic equation by \( m \), and then use the above result to rewrite...
The metric components are then just read of from this expression:

\[ \left[ \epsilon^2 p + \epsilon(p\theta + \phi) \epsilon^2 + \epsilon^2 p \right] \epsilon^2 - = \epsilon^2 p \epsilon^2 - \]

The metric was given as

\[ (v) \]

**Problem 17: Rotating Frames of Reference**

The metric components are either both equal to 1 or both equal to 2. However, the terms on the right-hand side both involve the determinant of the metric tensor on the left-hand side, which is always \( \epsilon x \). Then, only terms that contribute on the left-hand side and those of equal magnitude and equal magnitude of both tensors at \( \epsilon x \) are allowed. So, the right-hand side must be evaluated for a geodesic sum over repeated indices (as written as (c)). From the list at the front of the exam, the general formula for a geodesic is

\[ \gamma = V \]

You do not need to carry out this integration, but the answer would be

\[ \gamma = V \]

**Note:**
- In part (a), the expression must be integrated from the value of \( \epsilon x \) at the edge of the annulus to the value at the center. Since a single annulus starts at the center coordinate, one edge is not included into the summation.
adapted to the rotating cylindrical coordinate system.

\[
\frac{z^3}{r^2} \cdot \frac{\partial}{\partial \tau} = \frac{2P}{lP}
\]

Note that this equation is really just

\[
\begin{pmatrix}
\left( \frac{\partial}{\partial \tau} \right) + \left( \frac{\partial}{\partial \rho} \phi \right) \varepsilon + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^2 + \frac{\partial}{\partial \tau} - I \end{pmatrix} \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

Then using

\[
\cdot \left[ \left( \frac{\partial}{\partial \rho} \right) + \left( \frac{\partial}{\partial \phi} \phi \right) \varepsilon + \left( \frac{\partial}{\partial \phi} \varepsilon^2 \right) + \frac{\partial}{\partial \tau} - I \right] = \left( \frac{\partial}{\partial \tau} \right)
\]

Note that this one cannot just be identified as the Coriolis force. There is no term proportional to \( \omega \), since none of the metric coefficients depend on \( \tau \). If one expands the RHS

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

One has

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

Thus

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

Note that the final term in the last line is really the sum of the contributions

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

So

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

Substituting (c)

\[
\cdot \left( \frac{\partial}{\partial \rho} \phi \right) \cdot \frac{2P}{lP} + \left( \frac{\partial}{\partial \phi} \right) \varepsilon^m \cdot \frac{2P}{lP} = \frac{2P}{lP}
\]

\( \phi = \rho \) since the term proportional to \( \phi \) in the cylindrical coordinate system is the centrifugal force, and then one can identify the term proportional to \( \phi \) as the centrifugal force.
(S18.2) $\cdot \frac{dy}{c} - s = (\phi) \frac{c}{D}$, $\frac{c}{e^{\phi} \frac{dy}{c}} = (\phi) \frac{c}{D}$

In the notation of the problem statement, we have

(S18.2) $\cdot \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) + \frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

(S18.10) Notice that $S_{\phi} = S_{\phi}$, we can simplify (S18.10) further by noting that

(S18.10) $\cdot \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) + \frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

Expanding out the terms with $\left( \frac{\partial p}{\partial \phi} \right)$, we find

(S18.1) $\cdot \frac{c}{e^{\phi} \frac{dy}{c}} + \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

We use now (S18.8) to simplify (S18.1)

(S18.8) $\cdot \frac{c}{e^{\phi} \frac{dy}{c}} + \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

This is the most useful form of the answer. Of course, we also have

(S18.8) $\cdot \left( \frac{\partial p}{\partial \phi} \right) \left( \frac{\partial p}{\partial \phi} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

and rearranging, we find

(S18.3) $\cdot \frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

DIVIDING THE EXPRESSION (S18.3) BY $\frac{c}{e^{\phi} \frac{dy}{c}}$ FOR THE metric $\frac{c}{e^{\phi} \frac{dy}{c}}$ we readily find

(S18.3) $\cdot \frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

We also know that

(S18.4) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

and rearranging, we find

(S18.4) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

We are told that the metric has $\frac{c}{e^{\phi} \frac{dy}{c}}$ on the outside, $\frac{c}{e^{\phi} \frac{dy}{c}}$ in the middle, and the determinant

(S18.2) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

We are told that the metric has $\frac{c}{e^{\phi} \frac{dy}{c}}$ on the outside, $\frac{c}{e^{\phi} \frac{dy}{c}}$ in the middle, and the determinant

(S18.2) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

and the contraction $\frac{c}{e^{\phi} \frac{dy}{c}}$ with $\frac{c}{e^{\phi} \frac{dy}{c}}$ gives $\frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

Solving the equations

(S18.4) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

we find

(S18.4) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

and the contraction $\frac{c}{e^{\phi} \frac{dy}{c}}$ with $\frac{c}{e^{\phi} \frac{dy}{c}}$ gives $\frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

From the metric

(S18.2) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

we find

(S18.2) $\cdot \left( \frac{c}{e^{\phi} \frac{dy}{c}} \right) = \frac{c}{e^{\phi} \frac{dy}{c}}$

and the contraction $\frac{c}{e^{\phi} \frac{dy}{c}}$ with $\frac{c}{e^{\phi} \frac{dy}{c}}$ gives $\frac{c}{e^{\phi} \frac{dy}{c}} = \frac{c}{e^{\phi} \frac{dy}{c}}$

\textbf{PROBLEM 18: THE STABILITY OF SCHWARZSCHILD ORBITS}
\[ (S18.17) \]

where we recall from (S18.16) that \( \tau \geq 0 \) and we used \( \frac{dp}{d\phi} = \frac{\dot{\phi}}{\sin \theta} \).

\[ (S18.18) \]

Substituting this into (S18.15) we get, to first nontrivial approximation,

\[ (S18.19) \]

To investigate stability we consider a small perturbation \( \delta \) of the orbit.

\[ (S18.20) \]

Since we found that a circular orbit with radius \( R \) exists, the function \( H \) is defined for the orbit with radius \( R \) the value of \( H \) in (S18.16) is constant, and consequently, the right-hand side must also multiply the differential equation that takes the form

\[ (S18.21) \]

where we have introduced the function \( H \) that is a constant. The value of this function is defined in (S18.16). The second-order differential equation (S18.17) for the motion \( \tau \) depends on the initial conditions, the value \( \theta(0) \), and the angular momentum \( J \), and \( J \) is a constant for the circular orbit.

The quantity \( \tau \) is a constant of the motion, namely, it is a number independent of time.

\[ (S18.22) \]

Since no metric component depends on \( \phi \), the right-hand side vanishes and we get:

\[ (S18.23) \]

where \( \phi \) is the angular coordinate on the sphere and \( \dot{\phi} \) is the angular velocity.

For students interested in getting the famous result that orbits are stable for small values of \( \mu \), see (S18.24).

\[ (S18.24) \]

This is the answer to part (d).
PROBLEM 19: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If \( u \propto \frac{1}{\sqrt{V}} \), then one can write

\[
\frac{u(V + \Delta V)}{u_0 \sqrt{V}} = \frac{u_0 \sqrt{V + \Delta V}}{u_0 \sqrt{V}} = \frac{u_0 \sqrt{1 + \frac{\Delta V}{V}}}{u_0 \sqrt{1}} = \frac{u_0}{u_0} \left(1 - \frac{\Delta V}{2V}\right).
\]

The total energy is the energy density times the volume, so

\[
U = u(V + \Delta V) = u_0 \left(1 - \frac{\Delta V}{2V}\right)V(1 + \Delta V) = U_0 \left(1 + \frac{\Delta V}{2V}\right),
\]

where \( U_0 = u_0 V \).

(b) The work done by the agent must be the negative of the work done by the gas, so

\[
\Delta W = -p \Delta V.
\]

(c) The work done by the agent must be the negative of the work done by the gas, so

\[
\Delta W = \Delta U = \frac{1}{2} \Delta V V U_0.
\]

Combining this with the expression for \( \Delta W \) from part (b), one sees immediately

\[
\left(\frac{A}{A \nabla I} + I\right) = \left(\frac{A}{A \nabla I} + I\right) A \left(\frac{A}{A \nabla I} + I\right) = (A \nabla + A)n = \Omega.
\]

The local energy is the energy density times the volume, so

\[
\left(\frac{A}{A \nabla I} - I\right) = \frac{A \nabla I + I}{A \nabla I - I} = \frac{\nabla + I}{\nabla - I} = n
\]

when \( \nabla = n \Delta V \), where \( \nabla = n \) is the order of the expression proportional to \( \Delta V + A \nabla I \), and reduces to

\[
\frac{\Delta \nabla + A \nabla I}{A} = (A \nabla + A)n
\]

If \( n \equiv 1 \), then one can write

PROBLEM 19: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF