MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 21, 2013
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QUIZ 2 SOLUTIONS

Quiz Date: November 7, 2013

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity v as a function of radius R. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii?

Answer: If stars contributed most of the mass, then at large radii the mass would appear to be concentrated as a spherical lump at the center, and the orbits of the stars would be "Keplerian," i.e., orbits in a $1/r^2$ gravitational field. Then $\vec{F} = m\vec{a}$ implies that

$$\frac{1}{R^2} \propto \frac{v^2}{R} \quad \Longrightarrow \quad v \propto \frac{1}{\sqrt{R}}\ .$$

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

Answer: $v(R)$ looks nearly flat at large radii.

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

Answer: For a gravitationally bound system in equilibrium,

$$\text{Kinetic energy} = -\frac{1}{2}\,(\text{Gravitational potential energy})\ .$$

(The equality holds whenever $\ddot{I} \approx 0$, where I is the moment of inertia.)

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

Answer: When kT was large compared to 150 MeV, the excess of quarks over antiquarks was tiny: only about 3 extra quarks for every 10^9 antiquarks. But there was massive quark-antiquark annihilation as kT fell below 150 MeV, so that today we see the excess quarks, bound into baryons, and almost no sign of antiquarks.

PROBLEM 2: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

(a) The Friedmann equation in a flat universe is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

Substituting $\rho = \text{const}/a^5$ and taking the square root of both sides gives

$$\frac{\dot{a}}{a} = \alpha a^{-5/2}\ ,$$

for some constant α. Rearranging to a form we can integrate,

$$da\, a^{3/2} = \alpha dt,$$

and therefore

$$\frac{2}{5}a^{5/2} = \alpha t.$$

Notice that once again we have eliminated the arbitrary integration constant by choosing the big bang boundary conditions $a = 0$ at $t = 0$. Solving for a yields

$$\boxed{a \propto t^{2/5}.}$$

(b) The Hubble parameter is, from its definition,

$$\boxed{H = \frac{\dot{a}}{a} = \frac{2}{5t},}$$

where we have used the time dependence of $a(t)$ that we found in part (a). (Notice that we don't need to know the constant of proportionality left undetermined in part (a), as it cancels between numerator and denominator in this calculation.)

(c) Recall that the horizon distance is the physical distance traveled by a light ray since $t = 0$,

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c\,dt'}{a(t')}.$$

Using $a(t) \propto t^{2/5}$, we find

$$\ell_{p,\text{horizon}}(t) = ct^{2/5}\int_0^t dt'\, t'^{-2/5}$$

or

$$\ell_{p,\mathrm{horizon}}(t) = ct^{2/5}\left(\frac{5}{3}t^{3/5}\right) = \boxed{\frac{5}{3}ct.}$$

(d) Since we know the Hubble parameter, we can find the mass density $\rho(t)$ easily from the Friedmann equation,

$$\rho(t) = \frac{3H^2}{8\pi G}.$$

Using the result from part (b), we find

$$\boxed{\rho(t) = \frac{3}{50\pi G}\,\frac{1}{t^2}.}$$

As a check on our algebra, since we found in (a) that $a \propto t^{2/5}$, and knew at the beginning of the calculation that $\rho \propto a^{-5}$, we should find, as we do here, that $\rho \propto t^{-2}$. Notice, however, that in this case we do not leave our answer in terms of some undetermined constant of proportionality; the units of ρ are not arbitrary, and therefore we care about its normalization.

PROBLEM 3: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] ,$$

and the metric coefficients are then just read off from this expression:

$$\begin{aligned}
&g_{11} \equiv g_{rr} = 1\\
&g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = -c^2 + r^2\omega^2\\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = r^2\omega^2\\
&g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = r^2\\
&g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = 1\ .
\end{aligned}$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\,dx^\mu\,dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\mathrm{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\ .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\,g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned}
RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\ .
\end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\ .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\ ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{d}{d\tau}\left\{ g_{\phi\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\phi g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{d}{d\tau}\left(g_{\phi\phi} \frac{d\phi}{d\tau} + g_{\phi t} \frac{dt}{d\tau} \right) = \frac{d}{d\tau}\left(r^2 \frac{d\phi}{d\tau} + r^2 \omega \frac{dt}{d\tau} \right) = 0 ,$$

so

$$\boxed{\frac{d}{d\tau}\left(r^2 \frac{d\phi}{d\tau} + r^2 \omega \frac{dt}{d\tau} \right) = 0 .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $d^2\phi/d\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (P3.1) of the problem is divided by $c^2 dt^2$, one obtains

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt} + \omega\right)^2 + \left(\frac{dz}{dt}\right)^2 \right] .$$

Then using

$$\frac{dt}{d\tau} = \frac{1}{\left(\frac{d\tau}{dt}\right)} ,$$

one has

$$\boxed{\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt} + \omega\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]}} .}$$

Note that this equation is really just

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} ,$$

adapted to the rotating cylindrical coordinate system.

PROBLEM 4: PRESSURE AND ENERGY DENSITY OF IMAGINARY STUFF *(20 points)*

(a) If the energy density u as a function of the volume V satisfies $u(V) \propto 1/V^{3/2}$, then one can write

$$u(V + \Delta V) = u_0 \left(\frac{V}{V + \Delta V} \right)^{3/2} .$$

(The above expression is proportional to $1/(V + \Delta V)^{3/2}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u(V + \Delta V) = \frac{u_0}{\left(1 + \frac{\Delta V}{V}\right)^{3/2}} = \frac{u_0}{1 + \frac{3}{2}\frac{\Delta V}{V}} = \left(1 - \frac{3}{2}\frac{\Delta V}{V}\right) u_0 .$$

The total energy is the volume times the energy density, so the total energy U *after* the piston is pulled out is given by

$$\begin{aligned} U &= [V + \Delta V]\, u(V + \Delta V) \\ &= V \left(1 + \frac{\Delta V}{V}\right) \left(1 - \frac{3}{2}\frac{\Delta V}{V}\right) u_0 \\ &= \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) U_0 , \end{aligned}$$

where $U_0 \equiv V u_0$ is the total energy before the piston is pulled out. Then

$$\boxed{\Delta U \equiv U - U_0 = -\frac{1}{2}\frac{\Delta V}{V} U_0 .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = -\frac{1}{2}\frac{\Delta V}{V} U_0 .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = \frac{1}{2}\frac{U_0}{V} = \boxed{\frac{1}{2} u_0 .}$$