MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 9, 2016
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 5, 2016

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

(a) *(5 points)* The Milky Way has been known since ancient times as a band of light stretching across the sky. We now recognize the Milky Way as the galaxy of stars in which we live, with a large collection of stars, including our sun, arranged in a giant disk. Since the individual stars are mostly too small for our eyes to resolve, we observe the collective light from these stars, concentrated in the plane of the disk. The idea that the Milky Way is actually a disk of stars was proposed by

- (i) Claudius Ptolemy, in the 2nd century AD.
- (ii) Johannes Kepler, in 1610.
- (iii) Isaac Newton, in 1695.
- (iv) **[Thomas Wright, in 1750.]**
- (v) Immanuel Kant, in 1755.
- (vi) Edwin Hubble, in 1923.

(b) *(5 points)* Once it was recognized that we live in a galaxy, it was initially assumed that ours was the only galaxy. The suggestion that some of the patches of light known as nebulae might actually be other galaxies like our own was made by

- (i) Claudius Ptolemy, in the 2nd century AD.
- (ii) Johannes Kepler, in 1610.
- (iii) Isaac Newton, in 1695.
- (iv) Thomas Wright, in 1750.
- (v) **[Immanuel Kant, in 1755.]**
- (vi) Edwin Hubble, in 1923.

(c) *(5 points)* The first firm evidence that there is more than one galaxy stemmed from the ability to observe the Andromeda Nebula with high enough resolution to distinguish its individual stars. In particular, the observation of Cepheid variable stars in Andromeda allowed a distance estimate that placed it well outside the Milky Way. The observation of Cepheid variable stars in Andromeda was first made by

(i) Johannes Kepler, in 1610.

(ii) Isaac Newton, in 1695.

(iii Thomas Wright, in 1750.

(iv) Immanuel Kant, in 1755.

(v) Henrietta Swan Leavitt and Harlow Shapley in 1915.

(vi) **Edwin Hubble, in 1923.**

(d) *(5 points)* The first hint that the universe is filled with radiation with an effective temperature near 3 K, although not recognized at the time, was an observation of absorption lines in cyanogen (CN) by Adams and McKellar in 1941. They observed dark spectral lines which they interpreted as absorption by the cyanogen of light coming from the star behind the gas cloud. Explain in a few sentences how these absorption lines can be used to make inferences about the cosmic background radiation bathing the cyanogen gas cloud.

Answer:

When an atom absorbs a photon, it is excited from its initial state to some final state, and the energy of the photon must match the energy difference betwen the two states. One of the observed cyanogen lines was associated with a transition starting in the ground state, and two other observed lines were associated with transitions starting from an excited state. By comparing the intensities of these absorption lines, the astronomers could infer the relative abundance of ground state and excited cyanogen molecules, which in turn allowed them to infer the temperature of the gas cloud. They found a temperature of 2.2 K.

(e) *(5 points)* As the universe expands, the temperature of the cosmic microwave background

(i) goes up in proportion to the scale factor $a(t)$.

(ii) stays constant.

(iii) $\boxed{\text{goes down in proportion to } 1/a(t).}$

(iv) goes down in proportion to $1/a^2(t)$.

(f) *(5 points)* When Hubble measured the value of his constant, he found $H^{-1} \approx$ 100 million years, $\boxed{\text{2 billion years,}}$ 10 billion years, or 20 billion years?

(g) *(5 points)* Explain in a few sentences what is meant by the equivalence principle?

Answer:

> Ryden states that the equivalence principle is the fact that the gravitational mass of any object is equal to its inertial mass. It would also be correct to say that the graviational mass is proportional to the inertial mass. (If they are proportional, there is always a value of G which makes them equal.) The equivalence principle can also be described more generally by saying that gravity is equivalent to acceleration, so that within a small volume the effects of gravity can be removed by describing the system in an accelerating coordinate system.

PROBLEM 2: OBSERVING A DISTANT GALAXY IN A MATTER-DOMINATED FLAT UNIVERSE *(40 points)*

Suppose that we are living in a matter-dominated flat universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. The present time is denoted by t_0.

(a) *(5 points)* If we measure time in seconds, distance in meters, and coordinate distances in notches, what are the units of b?

Answer:

$a(t)$ would be measured in meters/notch, and t would be measured in seconds. So

$$[b] = \frac{[a(t)]}{[t]^{2/3}} = \boxed{\frac{\text{m}}{\text{notch-s}^{2/3}}\ .}$$

(b) *(5 points)* Suppose that we observe a distant galaxy which is one half of a "Hubble length" away, which means that the physical distance today is $\ell_p = \frac{1}{2}cH_0^{-1}$, where c is the speed of light and H_0 is the present value of the Hubble expansion rate. What is the proper velocity $v_p \equiv \frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t}$ of this galaxy relative to us?

Answer:

By Hubble's law, the velocity of recession is equal to H_0 times the physical distance, so

$$v_p = H_0 \left[\frac{1}{2}cH_0^{-1}\right] = \boxed{\frac{1}{2}c\ .}$$

A common error in this part was to use

$$H_0 = \frac{\dot{a}}{a} = \frac{2}{3}\frac{bt_0^{-1/3}}{bt_0^{2/3}} = \frac{2}{3t_0}$$

to write

$$\ell_p = \frac{3}{4}ct_0\ ,$$

and then to differentiate this expression with respect to t_0, finding $v_p = 3c/4$. The problem with this approach is that it assumes that the relation $\ell_p = \frac{1}{2}cH^{-1}$ holds for all t, so that one can differentiate it to find the velocity. But an object that is at distance $\frac{1}{2}cH^{-1}$ does not remain at a distance $\frac{1}{2}cH^{-1}$ as time progresses. It is the coordinate distance ℓ_c, and not the physical distance measured in Hubble lengths, that remains constant as the universe expands.

(c) *(5 points)* What is the coordinate distance ℓ_c between us and the distant galaxy? Express your answer in terms of b, t_0, and c (but not H_0).

Answer:

We know that $\ell_p(t) = a(t)\ell_c$, so

$$\ell_c = \frac{\ell_p(t_0)}{a(t_0)} = \frac{c}{2bH_0 t_0^{2/3}} \ .$$

To eliminate H_0, which is not allowed in the answer, we can use

$$H_0 = \frac{1}{a(t_0)} \frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0} = \frac{1}{bt_0^{2/3}} \left[\frac{2}{3} b t_0^{-1/3} \right] = \frac{2}{3t_0} \ .$$

Inserting the result into the line above,

$$\boxed{\ell_c = \frac{3}{4} \frac{c t_0^{1/3}}{b} \ .}$$

If you did not answer the previous part, you may still continue with the following parts, using the symbol ℓ_c for the coordinate distance to the galaxy.

(d) *(5 points)* At what time t_e was the light that we are now receiving from the galaxy emitted?

Answer:

We know that the coordinate velocity of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} = \frac{c}{bt^{2/3}} .$$

We can find t_e by the requirement that the coordinate distance that light travels between t_e and t_0 must be equal to ℓ_c found in part (c):

$$\int_{t_e}^{t_0} \frac{c}{bt'^{2/3}} \, \mathrm{d}t' = \frac{3}{4} \frac{c t_0^{1/3}}{b} \ .$$

Integrating,

$$\frac{3c}{b} \left[t_0^{1/3} - t_e^{1/3} \right] = \frac{3}{4} \frac{c t_0^{1/3}}{b} \ .$$

With a little algebra we see

$$t_e^{1/3} = \frac{3}{4} t_0^{1/3} \quad \Longrightarrow \quad \boxed{t_e = \frac{27}{64} t_0 \ .}$$

(e) *(5 points)* What is the redshift z of the light that we are now receiving from the distant galaxy?

Answer:

The redshift is related to the scale factor by

$$1+z=\frac{a(t_0)}{a(t_e)}=\left(\frac{t_0}{t_e}\right)^{2/3}=\left(\frac{64}{27}\right)^{2/3}=\frac{16}{9}\ ,$$

so

$$\boxed{z=\frac{7}{9}\ .}$$

(f) *(10 points)* Consider a light pulse that leaves the distant galaxy at time t_e, as calculated in part (d), and arrives here at the present time, t_0. Calculate the physical distance $r_p(t)$ between the light pulse and us. Find $r_p(t)$ as a function of t for all t between t_e and t_0.

Answer:

We first calculate the coordinate separation $r_c(t)$ between the light pulse and us, as a function of t. At time t_e it is equal to the value of ℓ_c found in part (c), and from that time onward it is reduced by the coordinate distance that light can travel between times t_e and t. Therefore,

$$\begin{aligned}r_c(t)&=\frac{3}{4}\frac{ct_0^{1/3}}{b}-\int_{t_e}^{t}\frac{c}{b\,t'^{2/3}}\,\mathrm{d}t'\\&=\frac{3}{4}\frac{ct_0^{1/3}}{b}-\frac{3c}{b}\left[t^{1/3}-t_e^{1/3}\right]\\&=\frac{3}{4}\frac{ct_0^{1/3}}{b}-\frac{3c}{b}\left[t^{1/3}-\frac{3}{4}t_0^{1/3}\right]\\&=\frac{3c}{b}\left[t_0^{1/3}-t^{1/3}\right]\ .\end{aligned}$$

The physical distance is then

$$r_p(t)=bt^{2/3}\,r_c(t)=3c\left[t_0^{1/3}t^{2/3}-t\right]=\boxed{3ct\left[\left(\frac{t_0}{t}\right)^{1/3}-1\right]\ .}$$

(g) *(5 points)* If we send a radio message now to the distant galaxy, at what time t_r will it be received?

Answer:

We calculate the time t_r by which a light ray, starting at t_0, can travel a coordinate distance equal to the value we found in part (c):

$$\int_{t_0}^{t_r} \frac{c}{b\, t'^{2/3}}\, \mathrm{d}t' = \ell_c = \frac{3}{4} \frac{c t_0^{1/3}}{b} \ .$$

Integrating,

$$\frac{3c}{b} \left[t_r^{1/3} - t_0^{1/3} \right] = \frac{3}{4} \frac{c t_0^{1/3}}{b} \ ,$$

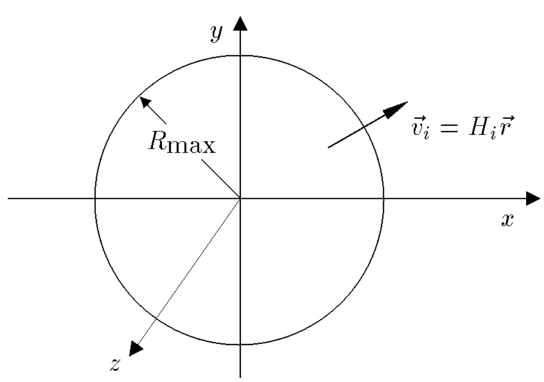
from which we find

$$t_r^{1/3} = \frac{5}{4} t_0^{1/3} \quad \Longrightarrow \quad \boxed{t_r = \frac{125}{64}\, t_0 \ .}$$

PROBLEM 3: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(25 points)*

The following problem was Problem 4 on Problem Set 3.

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i \vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} \ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} + \gamma r^n(r_i,t)\,\hat{r} \ ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i,t)} + \gamma r^n(r_i,t) \ .$$

(a) *(5 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u. *(Hint: be sure that u is the only time-dependent quantity in your equation; r, ρ, etc. must be rewritten in terms of u, ρ_i, etc.)*

(b) *(5 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

(c) *(5 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

(d) *(10 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$a(t) \equiv u(r_i, t) \ .$$

Show, for this value of n, that the differential equation for a can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on a and $\dot{a}$, but not $\ddot{a}$ or any higher derivatives.

Answer:

(a) Substituting the equation for $M(r_i)$, given on the quiz, into the differential equation for r, also given on the quiz, one finds:

$$\ddot{r} = -\frac{4\pi}{3}\frac{Gr_i^3\rho_i}{r^2} + \gamma r^n \ .$$

Dividing both sides of the equation by r_i, one has

$$\frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{Gr_i^2\rho_i}{r^2} + \gamma\frac{r^n}{r_i} \ .$$

Substituting $u = r/r_i$, this becomes

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} + \gamma u^n r_i^{n-1} \ .}$$

(b) The only dependence on r_i occurs in the last term, which is proportional to r_i^{n-1}. This dependence disappears if $\boxed{n = 1}$, since the zeroth power of any positive number is 1.

(c) This is exactly the same as the case discussed in the lecture notes, since the initial conditions do not depend on the differential equation. At $t = 0$,

$$\begin{aligned} r(r_i, 0) &= r_i && \text{(definition of } r_i\text{)} \\ \dot{r}(r_i, 0) &= H_i r_i && \text{(since } \vec{v}_i = H_i\vec{r}\text{).} \end{aligned}$$

Dividing these equations by r_i one has the intial conditions

$$\boxed{\begin{aligned} u &= 1 \\ \dot{u} &= H_i \ . \end{aligned}}$$

(d) $a(t)$ should obey the differential equation obtained in part (a) for u, for the value of n that was obtained in part (b): $n = 1$. So,

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a \ .$$

Multiplying the equation by $\dot{a} \equiv da/dt$, one finds

$$\frac{d^2a}{dt^2}\frac{da}{dt} = \left\{-\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a\right\}\frac{da}{dt} ,$$

which can be rewritten as

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2\right\} = 0 .$$

Thus the quantity inside the curly brackets must be constant. Following the lecture notes, I will call this constant E:

$$\boxed{\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 = E .}$$

Or one can use the conventionally defined quantity

$$k = -\frac{2E}{c^2} ,$$

in which case the equation can be written

$$\left\{\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2\right\} = -\frac{kc^2}{2} .$$

One can then rewrite the equation in the more standard form

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho + \gamma - \frac{kc^2}{a^2} ,}$$

where I have used $\rho(t) = \rho_i/a^3(t)$.

Additional Note: Historically, the constant γ corresponds to the "cosmological constant" which was introduced by Albert Einstein in 1917 in an effort to build a static model of the universe. The cosmological constant Λ, as defined by Einstein, is related to γ by

$$\gamma = \frac{1}{3}\Lambda c^2 .$$

The differential equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{a^2}\ ,$$

which shows that the cosmological constant contributes like a constant addition to the mass density. Modern physicists interpret the cosmological constant as a manifestation of the mass density of the vacuum. From the above equation, we can see that the mass density of the vacuum is related to Einstein's cosmological constant Λ by

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}\ .$$

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QUIZ 1 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$z = v/u$ (nonrelativistic, source moving)

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0/c$.

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2015):

$$H_0 = 67.7 \pm 0.5 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,
where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \frac{1}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with speed $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)}$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3} G\rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)} \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat $(k = 0)$: $a(t) \propto t^{2/3}$, $\Omega = 1$