MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 8, 2018
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 3, 2018

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) *(5 points)* After telescopes became available, more and more extended objects in the sky, called nebulae, were discovered, but those were thought as members of our galaxy. Who is the person who first proposed that some of the nebulae are galaxies like our own located outside our galaxy?

(i) Isaac Newton

(ii) **[Immanuel Kant]**

(iii) Edwin Hubble

(iv) Albert Einstein

(b) *(5 points)* Before 1923, questions of the nature of the spiral and elliptical nebulae could not be settled without some reliable method of determining how far away they are. In 1923, Edwin Hubble was for the first time able to resolve the Andromeda Nebula (galaxy) into separate stars and estimated the distance to the Andromeda Nebula. What observational quantity did he measure to estimate the distance?

(i) the radial velocity of individual stars in the Adromeda Nebula

(ii) the radial velocity of the Andromeda Nebula itself

(iii) **[the periods of variation of a class of stars in the Andromeda Nebula]**

(iv) the parallax of bright stars in the Adromeda Nebula

[Comment: Hubble used Cepheid variables to estimate the distance to the Andromeda Nebula (galaxy) with a tight relation between the observed periods of variation of the Cepheids and their absolute luminosities provided by Henrietta Swan Leavitt and Harlow Shapley.]

— Problem 1 continues on next page. —

(c) *(5 points)* In 1917, a year after the completion of Einstein's general theory of relativity, __________ looked specifically for a solution that would be homogeneous, isotropic, and static, and thus was forced to mutilate the equations by introducing a term, the so-called cosmological constant. In the same year, another solution of the modified theory was found by the Dutch astronomer __________. Although this solution appeared to be static, it had the remarkable property of predicting a redshift proportional to the distance. In 1922, the general homogeneous and isotropic solution of the original Einstein equations was found by the Russian mathematician __________, which provides a mathematical background for the most modern cosmological theories. Which is the right answer to fill in the blanks in turn?

(i) Friedmann — Einstein — de Sitter

(ii) Friedmann — de Sitter — Einstein

(iii) Einstein — Friedmann — de Sitter

(iv) **Einstein — de Sitter — Friedmann**

(v) de Sitter — Einstein — Friedmann

(vi) de Sitter — Friedmann — Einstein

(d) *(5 points)* After radio noises with the equivalent temperature of about 3.5° K were detected, Penzias, Wilson, Dicke, Peebles, Roll, and Wilkinson decided to publish a pair of companion letters in the Astrophysical Journal, in which Penzias and Wilson would announce their observations, and Dicke, Peebles, Roll, and Wilkinson would explain the cosmological interpretation. What is the title of the paper written by Penzias and Wilson?

(i) **"A Measurement of Excess Antenna Temperature at 4,080 Mc/s"**

(ii) "Cosmic Black-Body Radiation"

(iii) "Origin of the Microwave Radio Background"

(iv) "Three Degrees Above Zero: Bell Labs in the Information Age"

[Comment: (ii) is the title of the companion letter by Dicke, Peebles, Roll, and Wilkinson; (iii) is the title of a paper written by Peebles and Dicke in 1966, in which they refuted a suggestion by Michele Kaufman that the background radiation was emitted by ionized intergalactic hydrogen; and (iv) is the title of a book written by Jeremy Bernstein.]

— Problem 1 continues on next page. —

(e) *(5 points)* The universe contains different types of particles. Which of the following statements is NOT true?

(i) A baryon is defined as a particle made of three quarks.

(ii) Electrons and neutrinos are leptons.

(iii) There are three types of neutrinos and they all have zero charge.

(iv) The component of the universe made of ions, atoms, and molecules is generally referred to as baryonic matter, since only the baryons (protons and neutrons) contribute significantly to the mass density.

(v) About three-fourths of the baryonic matter in the universe is currently in the form of helium.

[Comment: about three-fourths of the baryonic matter in the universe is currently in the form of hydrogen.]

(f) *(5 points)* If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?*

(i) 1000 Mpc. (1 Mpc $= 10^6$ pc, 1 pc $= 3.086 \times 10^{16}$ m $= 3.262$ light-year).

(ii) 100 Mpc.

(iii) 1 Mpc.

(iv) 100 kpc (1 kpc $= 1000$ pc).

(v) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

— End of Problem 1. —

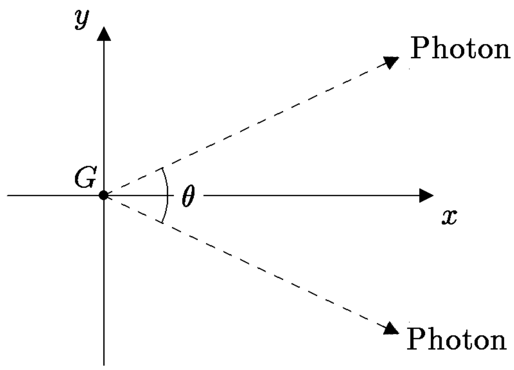
* This question was a replacement, listed on a separate sheet when the quiz was administered.

PROBLEM 2: LIGHT RAYS TRAVELING THROUGH A MATTER-DOMINATED FLAT UNIVERSE *(40 points)*

Consider a flat, matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Now consider a galaxy G in this universe which at time t_1 emits two photons, with an angular separation θ between their paths, as shown in the diagram:



(a) *(10 points)* At cosmic time t (for $t > t_1$), what is the physical distance $\ell_{1,\text{phys}}(t)$ of each of these photons from the galaxy G?

Answer:

The coordinate speed of light is $c/a(t)$, so the coordinate distance traveled is

$$\begin{aligned}\ell_{1,c}(t) &= \int_{t_1}^{t} \frac{c}{a(t')} \mathrm{d}t' \\ &= \left(\frac{3c}{b}\right) t^{1/3}\Big|_{t_1}^{t} \\ &= \left(\frac{3c}{b}\right) t^{1/3} \left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .\end{aligned}$$

The physical distance is then

$$\begin{aligned}\ell_{1,\text{phys}}(t) &= a(t)\ell_{1,c}(t) = bt^{2/3}\ell_{1,c}(t) \\ &= \boxed{3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .}\end{aligned}$$

(b) *(5 points)* If the frequency of the photons was ν_1 when they were emitted, what is their frequency $\nu(t)$ at cosmic time t (for $t > t_1$)? $\nu(t)$ should be the frequency as it would be measured by a comoving observer, i.e. an observer at rest with respect to the matter at the same location.

Answer:

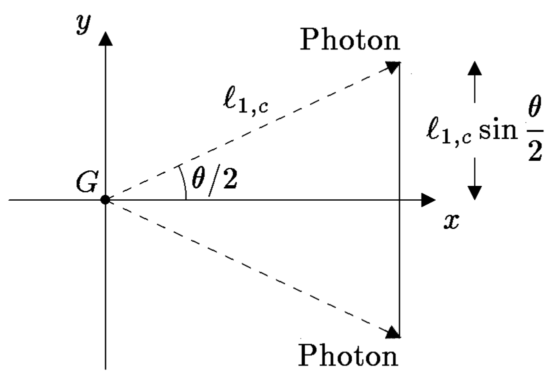
The wavelength of a photon is stretched in proportion to the scale factor, so the frequency is inversely proportional to the scale factor. So

$$\boxed{\nu(t) = \frac{a(t_1)}{a(t)}\nu_1 = \left(\frac{t_1}{t}\right)^{2/3} \nu_1 \ .}$$

(c) *(10 points)* What is the physical distance $\ell_{2,\text{phys}}(t)$ between the two photons at time t (for $t > t_1$)?

Answer:

Since the universe is flat, we can use ordinary Euclidean geometry, as shown in the diagram:



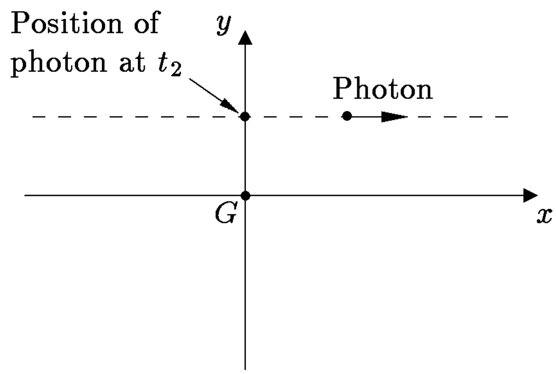
The coordinate distance between the two photons is then given by

$$\ell_{2,c} = 2\ell_{1,c} \sin\frac{\theta}{2} \ .$$

The physical distance is then

$$\begin{aligned}
\ell_{2,\text{phys}} &= 2a(t)\ell_{1,c} \sin\frac{\theta}{2} \\
&= 2bt^{2/3}\left(\frac{3c}{b}\right) t^{1/3} \left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] \sin\frac{\theta}{2} \\
&= \boxed{6ct \left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] \sin\frac{\theta}{2} \ .}
\end{aligned}$$

Now consider a different situation, but in the same universe. This time we consider a photon that travels past the galaxy G, traveling in the x direction, in the x-y plane, as shown in the diagram below. We are told that the photon crosses the y axis at time t_2, and at that time the photon is a physical distance h from the galaxy.



(d) *(10 points)* What is the physical distance $\ell_{3,\text{phys}}(t)$ between the photon and the galaxy G at arbitrary time t, which might be earlier or later than t_2?

Answer:

It is important to recognize here that the coordinates shown are comoving coordinates, or map coordinates, so that physical distances are obtained by multiplying by the scale factor. (If these coordinates represented physical distances from the origin, then the Hubble expansion would be driving all particles outward, and the photon trajectory would not be a straight line.) So, if the physical distance between the photon and the galaxy is equal to h at time t_2, then the y coordinate of the photon is equal to

$$y = \frac{h}{a(t_2)} \ .$$

The x coordinate is determined by the fact that it vanishes at time t_2, and then moves toward positive values at the coordinate speed of light,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} \ .$$

Thus,

$$x(t) = \int_{t_2}^{t} \frac{c}{bt'^{2/3}} \mathrm{d}t' = \frac{3c}{b} t^{1/3} \left[1 - \left(\frac{t_2}{t} \right)^{1/3} \right] \ .$$

The coordinate distance from the origin is then given by the Pythagorean theorem,

$$\ell_{3,c}(t) = \left[x^2(t) + y^2(t) \right]^{1/2} \ ,$$

so

$$\ell_{3,\text{phys}}(t) = bt^{2/3}\left[x^2(t) + y^2(t)\right]^{1/2}$$

$$= \boxed{\left\{9c^2t^2\left[1 - \left(\frac{t_2}{t}\right)^{1/3}\right]^2 + \left(\frac{t}{t_2}\right)^{4/3} h^2\right\}^{1/2}} \ .$$

(e) *(5 points)* At time t_2, what is the recessional speed $d\ell_{3,\text{phys}}(t)/dt$ of the photon from the galaxy. *Hint:* if you are clever, this can be done with very little calculation.

Answer:

Note, first of all, that one cannot blindly assume that the photon obeys Hubble's law, since Hubble's law applies only to the comoving matter in the model universe, which is undergoing uniform expansion. It does not apply to objects, such as photons, that are moving relative to the comoving matter. (Motion relative to the comoving matter is called proper motion.)

The answer can be obtained by simply differentiating the above expression for $\ell_{3,\text{phys}}(t)$ with respect to t, and then setting $t = t_2$, but there is a shorter way. If we go back to

$$\ell_{3,\text{phys}}(t) = a(t)\ell_{3,c}(t) \ ,$$

we note that, unlike the the description of uniform Hubble expansion, in this case the coordinate distance $\ell_{3,c}$ depends on time. The coordinate distance between two pieces of comoving matter (i.e., matter expanding with the universe) does not change with time, but here we have the distance between a galaxy (at fixed coordinates) and a photon (which is traveling). However, we can easily see from the diagram that at time t_2, the coordinate distance $\ell_{3,c}(t)$ is at its minimum, and therefore its time derivative at t_2 must be zero. Therefore,

$$\left.\frac{\mathrm{d}\ell_{3,\text{phys}}}{\mathrm{d}t}\right|_{t_2} = \dot{a}(t_2)\ell_{3,c}(t_2)$$

$$= \left(\frac{\dot{a}}{a}\right)\left[a\ell_{3,c}(t_2)\right] = H(t_2)\ell_{3,\text{phys}}(t_2)$$

$$= \boxed{\left(\frac{2}{3t_2}\right) h \ .}$$

The average grade on this problem was only 2.8/5, or 55%, which was the lowest for any problem on the quiz. Many students assumed that Hubble's law applied to the photon. This assumption leads to the correct answer at time t_2, when the photon proper velocity is perpendicular to the direction from the photon to the galaxy, but not at other times. Students who gave the correct answer, but attributed it to Hubble's law, were given 4 points out of 5. Another common error was to assert that the speed of light is always measured as c, so $\mathrm{d}\ell_{3,\mathrm{phys}}(t)/\mathrm{d}t = c$. The correct description of the invariance of the speed of light is to say that any inertial observer (which includes all comoving observers) will measure the speed of a photon that passes him as being equal to c. But if the photon is at a different location, then one has to take into account the expansion of the universe, which is done by basing all calculations on the principle that the coordinate speed of light is always equal to $c/a(t)$.

PROBLEM 3: THE STEADY-STATE UNIVERSE THEORY *(30 points)*

The following problem was Problem 2, Quiz 1, 2000. It was also Problem 2 of the Quiz 1 Review Problems, 2018.

Until the discovery of the cosmic microwave background, the steady state theory was considered a viable model of the universe. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density does not fall. Like the conventional theory, the steady-state theory describes a homogeneous, isotropic, expanding universe, so the same comoving coordinate formulation can be used.

a) *(15 points)* The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $a(t)$ which is consistent with this hypothesis.

Answer:

The Hubble expansion rate is related to $a(t)$ by

$$H(t) = \frac{1}{a(t)} \frac{da}{dt} ,$$

so in this case

$$\frac{1}{a(t)} \frac{da}{dt} = H_0 ,$$

which can be rewritten as

$$\frac{da}{a} = H_0 \, dt .$$

Integrating,

$$\ln a = H_0\, t + c\ ,$$

where c is a constant of integration. Exponentiating,

$$\boxed{a = b e^{H_0\, t}\ ,}$$

where $b = e^c$ is an arbitrary constant.

b) *(15 points)* Suppose that the mass density of the universe is ρ_0, which of course does not change with time. In terms of the general form for $a(t)$ that you found in part (a), calculate the rate at which new matter must be created for ρ_0 to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. *[If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $a(t)$.]*

Answer:

Consider a cube of side ℓ_c drawn on the comoving coordinate system diagram. The physical length of each side is then $a(t)\,\ell_c$, so the physical volume is

$$V(t) = a^3(t)\,\ell_c^3\ .$$

Since the mass density is fixed at $\rho = \rho_0$, the total mass inside this cube at any given time is given by

$$M(t) = a^3(t)\,\ell_c^3\,\rho_0\ .$$

In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

$$\begin{aligned}
\text{Rate} &= \frac{1}{V(t)}\frac{dM}{dt} \\
&= \frac{1}{a^3(t)\,\ell_c^3}\,3a^2(t)\,\frac{da}{dt}\,\ell_c^3\,\rho_0 \\
&= \frac{3}{a}\,\frac{da}{dt}\,\rho_0 \\
&= \boxed{3H_0\,\rho_0\ .}
\end{aligned}$$

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose we take $H_0 = 70$ km-sec^{-1}-Mpc^{-1}, and take ρ_0 to be the critical density, $\rho_c = 3H_0^2/8\pi G$. Then

$$\begin{aligned}
\text{Rate} &= \frac{9H_0^3}{8\pi G} \\
&= \frac{9 \times (70 \text{ km-s}^{-1}\text{-Mpc}^{-1})^3}{8\pi \times 6.67260 \times 10^{-11} \text{ m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\
&= \frac{9 \times (70 \cancel{\text{km}}\text{-s}^{-1}\text{-}\cancel{\text{Mpc}}^{-1})^3}{8\pi \times 6.67260 \times 10^{-11} \text{ m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\
&\qquad \times \left(\frac{1 \cancel{\text{Mpc}}}{3.086 \times 10^{22} \cancel{\text{m}}}\right)^3 \times \left(\frac{10^3 \cancel{\text{m}}}{\cancel{\text{km}}}\right)^3 \\
&= 6.26 \times 10^{-44} \text{ kg-m}^{-3}\text{-s}^{-1} \;.
\end{aligned}$$

To put this number into more meaningful terms, note that the mass of a hydrogen atom is 1.67×10^{-27} kg, and that 1 year $= 3.156 \times 10^7$ s. The rate of matter production required for the steady-state universe theory can then be expressed as roughly one hydrogen atom per cubic meter per billion years! Needless to say, such a rate of matter production is totally undetectable, so the steady-state theory cannot be ruled out by the failure to detect matter production.

— End of Problem 3. —

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QUIZ 1 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \; , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,

where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with speed $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{c}{a(t)}$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \; , \quad \ddot{a} = -\frac{4\pi}{3} G\rho a \; ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \; , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \; .$$

Flat ($k = 0$): $a(t) \propto t^{2/3}$, $\Omega = 1$