MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — Quiz Date: November 5, 2018
Prof. Alan Guth

QUIZ 2

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Please answer all questions in this stapled booklet.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

- (i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.
- (ii) The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.
- (iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.
- (iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

- (i) When the temperature of the universe was about 10^{10} °K (t $\sim$ 1 sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.
- (ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.
- (iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first three minutes, and this is why we call this period the era of nucleosynthesis.
- (iv) After the first three minutes, there were about 7 times more protons than neutrons, and the ratio of protons to neutrons has been almost preserved until today.
- (v) The protons and neutrons became decoupled from the photons after the first three minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.
- (vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s (iii) 1940s (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

— *End of Problem 1.* —

PROBLEM 2: EVOLUTION OF A FRIEDMANN–ROBERTSON–WALKER UNIVERSE *(20 points)*

(a) *(10 points)* The evolution of a homogeneous isotropic model of the universe, known as a Friedmann–Robertson-Walker (FRW) universe, can be described by the following equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \tag{2.1}$$

$$\ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ , \tag{2.2}$$

and

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ . \tag{2.3}$$

These equations are not independent, but any two can in fact be used to derive the third. For example, in Problem Set 6, you were asked to use Eqs. (2.1) and (2.3) to derive Eq. (2.2). Here you are asked to show that Eqs. (2.1) and (2.2) can be used to derive Eq. (2.3).

(b) *(6 points)* If the universe were flat, expanding, and filled with a material for which

$$p = -\rho c^2 \ , \tag{2.4}$$

what would be the form of the scale factor $a(t)$?

(c) *(4 points)* In the universe described in part (b), suppose that, at $t = 0$, my friend Bob emits a photon in my direction. Show that if Bob is more than a certain distance away from me at the time of emission, $t = 0$, then the photon will never reach me. What is this distance?

— End of Problem 2. —

PROBLEM 3: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem was Problem 2 on Problem Set 4.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) \ ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \ .$$

(a) *(5 points)* Find the Hubble expansion rate H as a function of α and θ.

(b) *(5 points)* Find the mass density ρ as a function of α and θ.

(c) *(5 points)* Find the mass density parameter Ω as a function of α and θ.

(d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\mathrm{horizon}}$, as a function of α and θ.

(e) *(7 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

(f) *(7 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

— *End of Problem 3.* —

PROBLEM 4: RADIAL GEODESICS IN A CLOSED UNIVERSE *(20 points)*

As shown in the formula sheets, we can describe a closed universe by choosing $k = 1$, and then using coordinates (t, r, θ, ϕ), with metric

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ , \tag{4.1}$$

or by using coordinates (t, ψ, θ, ϕ), with metric

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ . \tag{4.2}$$

The connection between the two coordinate systems is given by

$$r = \sin\psi\ . \tag{4.3}$$

The general spacetime geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ . \tag{4.4}$$

(a) *(7 points)* Using the coordinates (t, ψ, θ, ϕ) and the metric of Eq. (4.2), compute explicitly the geodesic equation for $\mu = \psi$. By "compute explicitly", I mean that $g_{\mu\nu}$ should be replaced by the relevant expressions from Eq. (4.2), and that the sums over indices should be written out, including only the nonzero terms.

(b) *(7 points)* Using instead the coordinates (t, r, θ, ϕ), compute explicitly the geodesic equation for $\mu = r$.

(c) *(6 points)* Are the results from parts (a) and (b) both valid, or is one valid and the other not? If you believe that they are both valid, use Eq. (4.3) to show that they are equivalent. If you believe that only one is valid, state which one is valid, and explain why the other is not. (4 points will be given for showing the correct understanding of this problem, with 2 points allocated to completing the algebra needed to demonstrate your answer.)

— *End of Problem 4.* —

Problem	Maximum	Score	Initials
1	25		
2	20		
3	35		
4	20		
TOTAL	100		

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QUIZ 2 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \;, \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:

Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right) \;, \quad \vec{p} = \gamma m_0 \vec{v} \;, \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \;,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \;.$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,
where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned} \ell_{p,\text{horizon}}(t) &= a(t) \int_0^t \frac{c}{a(t')} dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases} \end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \quad \ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)} \rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)} \rho_r(t_i) \ \ \text{(radiation).}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) , \quad \Omega \equiv \rho/\rho_c , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$\Omega = 1$.

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)$, $\dfrac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)$,

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\ \frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or: $$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$