MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 20, 2018
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QUIZ 2 SOLUTIONS

Quiz Date: November 5, 2018
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Please answer all questions in this stapled booklet.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

(i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.

[(ii)] The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.

(iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.

(iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

[Comment: The most promising way to find the primordial value of the deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium in ***intergalactic gas clouds*** *that lie between the quasars and us.]*

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

[(i)] When the temperature of the universe was about 10^{10} °K (t $\sim$ 1 sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.

(ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.

(iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first three minutes, and this is why we call this period the era of nucleosynthesis.

(iv) After the first three minutes, there were about 7 times more protons than neutrons, and the ratio of protons to neutrons has been almost preserved until today.

(v) The protons and neutrons became decoupled from the photons after the first three minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.

(vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

[Comment: The statement (i) is described in Chapter 5 of Weinberg's book, and it has also been discussed in class. The correctness of statement (iv) can also be seen from Weinberg's Chapter 5, which says that the fraction of neutrons after the first three minutes is around 14%, and it goes down to around 13% "a little later". Thus there were about 7 times more protons compared to neutrons. Weinberg also explains that most of the neutrons present at this time immdiately combined with protons to form helium, which causes the ratio to be nearly constant until today. If you did not remember Weinberg's numbers, the statement that the fraction by weight of primordial helium is 26% should allow you to determine the neutron to proton ratio, provided that you remember that the helium nucleus consists of 2 protons and two neutrons, that the mass of the proton and neutron are about equal, and that the remaining 74% of the matter is essentially hydrogen, with no neutrons. Thus helium is very nearly half protons and half neutrons by weight, so the neutrons must be about 13% of the matter in the universe. (Note that we are talking about fractions of the total "baryonic" matter, which does not include the dark matter.) (ii) is clearly false, because the temperature of neutrinos becomes lower than that of photons. (iii) is clearly false, because most atoms heavier than helium were made much later in the history of the universe, in the interiors of stars. (v) is false because protons did not decouple from photons until about 350,000 years, and it happened because the plasma of protons and electrons combined to form neutral hydrogen. (vi) is false because most of the helium in the universe today is primordial. Ryden points out, for example, that the abundance of helium in the Sun's atmosphere is only about 28%. Weinberg states, near the end of Chapter 5, that "the 20-30 percent helium abundance could not have been created recently without liberating enormous amounts of radiation that we do not observe."]

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s **(iii) 1940s** (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

Answer:

The neutron decays through the reaction

$$n \to p + e^- + \bar{\nu}_e \ .$$

The quantum numbers of these particles can be described by the following table:

Particle	Charge	Baryon Number	Lepton Number
Neutron (n)	0	1	0
Proton (p)	$+e$	1	0
Electron (e^-)	-e	0	1
Anti-electron-neutrino ($\bar{\nu}_e$)	0	0	-1

Thus, the total charge of the final state is zero, the total baryon number is 1, and the total lepton number is zero, in all cases matching the initial value of these quantities.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

Answer: You should have given two of the following three items.

1) Virial theorem: The virial theorem relates the total kinetic energy of a steady-state cluster to is gravitational potential energy. Since the kinetic energy is proportional to the mass M of the cluster, while the potential energy is proportional to M^2, the relation will hold for only one value of M. By measuring the velocity dispersion (root mean square of the radial galaxy velocities relative to the mean radial velocity) and the size of the cluster, the mass can be inferred.

2) Hot, x-ray emitting gases: By measuring the x-rays emitted by the cluster, it is possible to model the density, temperature, and composition of the hot gas within the cluster (intracluster gas). By assuming that the gas is in hydrostatic equilibrium — i.e., by assuming that the pressure gradients balance the gravitational forces — one can infer the gravitational field, and hence the total mass.

3) Gravitational lensing: If one can find a galaxy behind the cluster, so that the image of the galaxy is gravitationally lensed, then the mass of the galaxy can be inferred by the degree to which the galaxy is lensed.

— End of Problem 1. —

PROBLEM 2: EVOLUTION OF A FRIEDMANN–ROBERTSON–WALKER UNIVERSE *(20 points)*

(a) *(10 points)* The evolution of a homogeneous isotropic model of the universe, known as a Friedmann–Robertson-Walker (FRW) universe, can be described by the following equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \tag{2.1}$$

$$\ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ , \tag{2.2}$$

and

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ . \tag{2.3}$$

These equations are not independent, but any two can in fact be used to derive the third. For example, in Problem Set 6, you were asked to use Eqs. (2.1) and (2.3) to derive Eq. (2.2). Here you are asked to show that Eqs. (2.1) and (2.2) can be used to derive Eq. (2.3).

Answer:

We can rewrite Eq. (2.1) as

$$\dot{a}^2 = \frac{8\pi}{3} G\rho a^2 - kc^2 \ ,$$

which can then be differentiated to give

$$2\dot{a}\ddot{a} = \frac{16\pi}{3} G\rho a\dot{a} + \frac{8\pi}{3} G\dot{\rho}a^2 \ .$$

The above equation can be solved for $\dot{\rho}$, giving

$$\dot{\rho} = -2\,\frac{\dot{a}}{a}\,\rho + \frac{3}{4\pi G}\,\frac{\ddot{a}\dot{a}}{a^2} \ .$$

Then if Eq. (2.2) is used to replace $\ddot{a}$, one finds

$$\begin{aligned}\dot{\rho} &= -2\,\frac{\dot{a}}{a}\,\rho - \frac{\dot{a}}{a}\left(\rho + \frac{3p}{c^2}\right) \\ &= -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ .\end{aligned}$$

(b) *(6 points)* If the universe were flat, expanding, and filled with a material for which

$$p = -\rho c^2 \ , \tag{2.4}$$

what would be the form of the scale factor $a(t)$?

Answer:

If $p = -\rho c^2$, then Eq. (2.3) implies that

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \ ,$$

so ρ is a constant. Then, since $k = 0$ for a flat universe, Eq. (2.1) implies

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho = \text{const} \ ,$$

so

$$\dot{a} = \pm\sqrt{\frac{8\pi}{3}G\rho}\, a \ .$$

Only the positive option describes an expanding universe, so

$$\dot{a} = \sqrt{\frac{8\pi}{3}G\rho}\, a \quad \Longrightarrow \quad \boxed{a(t) = b\, e^{\sqrt{\frac{8\pi}{3}G\rho}\, t} \ ,}$$

where b is an arbitrary constant.

(c) *(4 points)* In the universe described in part (b), suppose that, at $t = 0$, my friend Bob emits a photon in my direction. Show that if Bob is more than a certain distance away from me at the time of emission, $t = 0$, then the photon will never reach me. What is this distance?

Answer:

The metric for this universe is

$$\mathrm{d}s^2 = -c^2\, \mathrm{d}t^2 + b^2 e^{2Ht}\, \mathrm{d}\vec{x}^2 \ ,$$

where

$$H = \sqrt{\frac{8\pi}{3}G\rho} \ .$$

Light pulses travel with $\mathrm{d}s^2 = 0$, so the coordinate speed of light, for a pulse traveling along the x axis, is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{be^{Ht}} \ .$$

So the coordinate distance that the pulse will travel between time 0 and some arbitrary time t is given by

$$\ell_{\text{coord}}(t) = \int_0^t \frac{\mathrm{d}x}{\mathrm{d}t}(t')\,\mathrm{d}t' = \frac{c}{b}\int_0^t e^{-Ht'}\,\mathrm{d}t' = \frac{c}{bH}\left(1 - e^{-Ht}\right) \ .$$

Therefore, even if we let $t \to \infty$, the coordinate distance traveled by the light pulse will always be less than $c/(bH)$. Since I am stationary in the comoving coordinates, if the initial coordinate distance between Bob and me was more than $c/(bH)$, the light pulse will never reach me. Since the initial time was $t = 0$, with $a(0) = b$,

the light pulse will never reach me if the initial physical distance between Bob and me was more than cH^{-1}, which is called the Hubble length.

[Comment: many students tried to use the formulas that we have learned for the horizon distance, but that is not the same thing. The horizon distance is the present proper distance to the most distant objects from which light has had time to reach us, since $t = 0$. This is often called the "particle horizon," and clearly it is determined solely by the history of the universe, up to the present. The current question concerns whether a photon emitted at the present time by Bob will ever reach me. This question is determined solely by the future evolution of the universe, and is a completely different question from the particle horizon issue. It is also called a horizon, however. The distance beyond which light will never reach me is called the "event horizon."]

— *End of Problem 2.* —

PROBLEM 3: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem was Problem 2 on Problem Set 4.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots\ .$$

(a) *(5 points)* Find the Hubble expansion rate H as a function of α and θ.

Answer:

Using the chain rule, the standard formula for the Hubble expansion rate can be rewritten as

$$H(\theta) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\ .$$

The parametric equations for a and t for an open, matter-dominated universe are given by

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ .$$

Recall that the hyperbolic trigonometric functions are differentiated as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sinh\theta = \cosh\theta\ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh\theta = \sinh\theta\ .$$

So, differentiating the parametric equations,

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \alpha\sqrt{k}\sinh\theta\ ,$$

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\alpha}{c}(\cosh\theta - 1) = \frac{1}{\mathrm{d}\theta/\mathrm{d}t}\ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta - 1)}\right]$$

$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta - 1)^2}\ .}$$

(b) *(5 points)* Find the mass density ρ as a function of α and θ.

Answer:

This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) One way to find ρ is to use

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\ .$$

This is usually the safest method to find ρ for a cosmological model, since the above equation is one of the general Friedmann equations. The equation requires that the universe be homogeneous and isotropic, but it is valid for any form of matter. By contrast, the two other methods that will be shown below are valid only for "matter-dominated" universes (i.e., universes that are dominated by nonrelativistic matter, for which the pressure is always negligible). One can rewrite this equation as

$$\frac{8\pi}{3}G\rho = H^2 + \frac{kc^2}{a^2}\ .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3}G\rho = H^2 - \frac{\kappa c^2}{a^2}\ .$$

Replacing H by the answer in part (a) and a by its parametric equation, one finds

$$\frac{8\pi}{3}G\rho = \frac{c^2\sinh^2\theta}{\alpha^2(\cosh\theta - 1)^4} - \frac{\kappa c^2}{\alpha^2\kappa(\cosh\theta - 1)^2}$$

$$= \frac{c^2}{\alpha^2(\cosh\theta - 1)^4}\left[\sinh^2\theta - (\cosh\theta - 1)^2\right]\ .$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\sinh^2\theta - (\cosh\theta - 1)^2 = \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1$$
$$= 2(\cosh\theta - 1) \ ,$$

so

$$\frac{8\pi}{3} G\rho = \frac{2c^2}{\alpha^2(\cosh\theta - 1)^3} \ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one finds

$$\boxed{\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \ .}$$

(ii) Use the definition of α,

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ ,$$

from Eq. (4.17) of Lecture Notes 4, with Eq. (4.39),

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{\kappa}} \ .$$

One can then solve for ρ, finding

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{Ga^3} \ .$$

By substituting for $a(\theta)$ by using the parametric equation, one finds the final result:

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{G}\frac{1}{\alpha^3\kappa^{3/2}(\cosh\theta - 1)^3}$$
$$= \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \ .$$

(iii) ρ can also be found from $\ddot{a} = -(4\pi/3)G\rho a$, as long as we know that the universe is matter-dominated. (Be careful, however, about applying this formula in other situations: if the pressure cannot be neglected, then this equation has to be modified.) To evaluate $\ddot{a}$, again use the chain rule. Starting with $\dot{a}$,

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha\sqrt{\kappa}\sinh\theta \frac{c}{\alpha(\cosh\theta - 1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1} \ .$$

Then

$$\ddot{a} = \frac{\mathrm{d}\dot{a}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)}$$

$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right]$$

$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right]$$

$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} \; .$$

So

$$\ddot{a} = -\frac{4\pi}{3}G\rho a \quad \Longrightarrow \quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1) \; ,$$

and

$$\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \; .$$

(c) *(5 points)* Find the mass density parameter Ω as a function of α and θ.

Answer:

The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \; .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \; .$$

Now replace H by the answer to part (a), and ρ by the answer to part (b):

$$\Omega = \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta - 1)^3}\right]\left[\frac{\alpha^2(\cosh\theta - 1)^4}{c^2\sinh^2\theta}\right]$$

$$= 2\frac{\cosh\theta - 1}{\sinh^2\theta} = 2\frac{\cosh\theta - 1}{\cosh^2\theta - 1}$$

$$= 2\frac{\cosh\theta - 1}{(\cosh\theta + 1)(\cosh\theta - 1)} = \boxed{\frac{2}{\cosh\theta + 1}} \; .$$

The answer can be written even more compactly, if one wishes, by using a further hypertrigonometric identity:

$$\Omega = \frac{2}{\cosh\theta + 1} = \frac{1}{\cosh^2 \frac{1}{2}\theta} = \mathrm{sech}^2 \frac{1}{2}\theta \ .$$

(d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

Answer:

The basic formula that determines the physical value of the horizon distance is given by Eq. (4.7) of the lecture notes:

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' \ .$$

The complication here is that a is given as a function of θ, rather than t. The problem is handled, however, by a simple change of integration variables. One can change the integral over t' to an integral over θ', provided that one replaces

$$\mathrm{d}t' \to \frac{dt'}{\mathrm{d}\theta'}\mathrm{d}\theta' = \frac{\alpha}{c}(\cosh\theta' - 1)\mathrm{d}\theta' \ .$$

One must also re-express the limits of integration in terms of θ. So

$$\begin{aligned}
\ell_{p,\text{horizon}}(t) &= a\big(\theta(t)\big)\int_0^{\theta(t)} \frac{c}{a(\theta')}\frac{dt'}{\mathrm{d}\theta'}\mathrm{d}\theta' \\
&= \alpha\sqrt{\kappa}\big(\cosh\theta(t) - 1\big)\int_0^{\theta(t)} \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta' - 1)}\frac{\alpha}{c}(\cosh\theta' - 1)\,\mathrm{d}\theta' \ . \\
&= \alpha\big(\cosh\theta(t) - 1\big)\int_0^{\theta(t)} \mathrm{d}\theta' = \boxed{\ \alpha\,\theta(t)\,\big(\cosh\theta(t) - 1\big)\ .\ }
\end{aligned}$$

(e) *(7 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

Answer:

The key to this problem is the use of power series expansions. When this problem appeared as a quiz problem in 1992, I was rather surprised to find that many of the students seemed very inexperienced in this technique. It is a very useful method of approximation, so I strongly urge you to learn it if you don't know it already. In general, any sufficiently smooth function $f(x)$ can be expanded about the point x_0 by the series

$$f(x) = f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots ,$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$$

$$\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!} \ldots$$

$$\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots .$$

For this problem, we expand the parametric equations for $a(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$t = \frac{\alpha}{c}(\sinh\theta - \theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!} + \ldots\right)$$

$$a = \alpha\sqrt{\kappa}(\cosh\theta - 1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!} + \ldots\right) .$$

The first expression can be solved for θ, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} ,$$

which can be substituted into the second expression to give

$$\boxed{a \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between θ and t, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c}\ .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for t^*.

(f) *(7 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

Answer:

From part (c), the expression for Ω is given by

$$\Omega = \frac{2}{\cosh\theta + 1}\ .$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\cosh\theta + 1}\ .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}{2 + \frac{\theta^2}{2!} + \ldots}\ .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2\ ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3}\ .}$$

This result shows that the deviation of Ω from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.

A common mistake (very minor) was to keep extra terms, especially in the denominator. Keeping extra terms allows a higher degree of accuracy, so there is nothing wrong with it. However, one should always be sure to keep ***all*** terms of a given order, since keeping only a subset of terms may or may not increase the accuracy. In this case, an extra term in the denominator can be rewritten as a term in the numerator:

$$\begin{aligned}\frac{\frac{\theta^2}{2!}}{2+\frac{\theta^2}{2!}} = \frac{1}{4}\frac{\theta^2}{1+\frac{\theta^2}{4}} &= \frac{1}{4}\theta^2\left(1-\frac{\theta^2}{4}+\dots\right)\\ &= \frac{1}{4}\theta^2 - \frac{1}{16}\theta^4 + \dots\ ,\end{aligned}$$

where I used the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots\ .$$

Thus, the extra term in the denominator is equivalent to a term in the numerator of order θ^4, but other terms proportional to θ^4 have been dropped. So, it is not worthwhile to keep the 2nd term in the expansion of the denominator.

— *End of Problem 3.* —

PROBLEM 4: RADIAL GEODESICS IN A CLOSED UNIVERSE *(20 points)*

As shown in the formula sheets, we can describe a closed universe by choosing $k = 1$, and then using coordinates (t, r, θ, ϕ), with metric

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} , \tag{4.1}$$

or by using coordinates (t, ψ, θ, ϕ), with metric

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} . \tag{4.2}$$

The connection between the two coordinate systems is given by

$$r = \sin\psi . \tag{4.3}$$

The general spacetime geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} . \tag{4.4}$$

(a) *(7 points)* Using the coordinates (t, ψ, θ, ϕ) and the metric of Eq. (4.2), compute explicitly the geodesic equation for $\mu = \psi$. By "compute explicitly", I mean that $g_{\mu\nu}$ should be replaced by the relevant expressions from Eq. (4.2), and that the sums over indices should be written out, including only the nonzero terms.

Answer:

Since the metric is diagonal, only $\nu = \psi$ contributes to the sum over ν. Similarly λ must equal σ, and the only nonzero values of $\partial_\psi g_{\lambda\sigma}$ are when $\lambda = \sigma = \theta$ and $\lambda = \sigma = \phi$. So Eq. (4.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\psi\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left[\frac{\partial g_{\theta\theta}}{\partial\psi}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \frac{\partial g_{\phi\phi}}{\partial\psi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

Using $g_{\psi\psi} = a^2(t)$, $g_{\theta\theta} = a^2(t)\sin^2\psi$, and $g_{\phi\phi} = a^2(t)\sin^2\psi\sin^2\theta$, the equation becomes

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{a^2(t)\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right\} = a^2(t)\,\sin\psi\,\cos\psi\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .} \tag{4.5}$$

You were not asked to expand the left-hand-side, but some of you did. If you do expand the left-hand side, it is important to remember that $a(t)$ depends on t and t depends on τ, so the equation becomes

$$a^2(t)\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 2a\dot{a}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = a^2(t)\,\sin\psi\,\cos\psi\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] ,$$

or

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \sin\psi\,\cos\psi\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

(b) *(7 points)* Using instead the coordinates (t, r, θ, ϕ), compute explicitly the geodesic equation for $\mu = r$.

Answer:

Again the equation simplifies significantly, since $g_{\mu\nu}$ is diagonal. On the right-hand side, only 3 of the 4 possible values of $\lambda = \sigma$ contribute, as $\partial_r g_{tt} = 0$. So,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left[\frac{\partial g_{rr}}{\partial r}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\partial g_{\theta\theta}}{\partial r}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

Now we use

$$g_{rr} = \frac{a^2(t)}{1-r^2} \ , \quad g_{\theta\theta} = a^2(t)r^2 \ , \quad g_{\phi\phi} = a^2(t)r^2\sin^2\theta \ ,$$

which allows us to rewrite the equation as

$$\boxed{\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{a^2(t)}{1-r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} = \frac{1}{2}\Bigg[&\frac{2ra^2(t)}{(1-r^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + 2ra^2(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 \\ &+ 2ra^2(t)\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\Bigg] ,\end{aligned}}$$

or

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{a^2(t)}{1-r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} = ra^2(t)\left[\frac{1}{(1-r^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .} \tag{4.6}$$

Again you were not asked to expand the left-hand side, but if the left-hand side is expanded, one must remember that $a(t)$ and r both depend on τ. So

$$\frac{a^2(t)}{1-r^2}\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{2ra^2(t)}{(1-r^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{2a\dot{a}}{1-r^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}$$
$$= ra^2(t)\left[\frac{1}{(1-r^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

Rearranging terms, the equation can be simplified to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}$$
$$= r\left\{-\frac{1}{(1-r^2)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + (1-r^2)\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\right\} .$$

(c) *(6 points)* Are the results from parts (a) and (b) both valid, or is one valid and the other not? If you believe that they are both valid, use Eq. (4.3) to show that they are equivalent. If you believe that only one is valid, state which one is valid, and explain why the other is not. (4 points will be given for showing the correct understanding of this problem, with 2 points allocated to completing the algebra needed to demonstrate your answer.)

Answer:

Both answers are valid, since they are both correct forms of the geodesic equation, in different coordinate systems. To see that they are equivalent, we can start with the equation for r, Eq. (4.6), and substitute

$$r = \sin\psi \quad \Longrightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \cos\psi\frac{\mathrm{d}\psi}{\mathrm{d}\tau} \quad \Longrightarrow \quad \frac{1}{1-r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{1}{\cos\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\tau} .$$

So Eq. (4.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{a^2(t)}{\cos\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\} = a^2\sin\psi\left[\frac{1}{\cos^2\psi}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

Partially expanding the left-hand side,

$$\frac{1}{\cos\psi}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{a^2(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\} + \frac{a^2}{\cos^2\psi}\sin\psi\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2$$
$$= a^2\sin\psi\left[\frac{1}{\cos^2\psi}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] .$$

The terms proportional to $(\mathrm{d}\psi/\mathrm{d}\tau)^2$ can be seen to cancel, and then multiplication of the equation by $\cos\psi$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{a^2(t)\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right\} = a^2(t)\,\sin\psi\,\cos\psi\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] ,$$

which is identical to Eq. (4.5).

— End of Problem 4. —

Problem	Maximum	Score	Initials
1	25		
2	20		
3	35		
4	20		
TOTAL	100		

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — Quiz Date: November 5, 2018

Prof. Alan Guth

QUIZ 2 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \; , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:

Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right) \; , \quad \vec{p} = \gamma m_0 \vec{v} \; , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \; ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \; .$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,
where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)}\dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned}\ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\rho_r(t_i) \ \ \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$\Omega = 1$.

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$$\Omega = \frac{2}{1 + \cos\theta} > 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 .$$

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)$, $\dfrac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)$,

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 ,$$

$$\kappa \equiv -k > 0 .$$

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$