MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 19, 2018
Prof. Alan Guth

QUIZ 3 SOLUTIONS

Quiz Date: December 5, 2018

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Please answer all questions in this stapled booklet.

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(5 points)* Which one of the following statements about CMB is NOT correct?

(i) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(ii) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is <T> = 2.725K.

$\boxed{\text{(iii)}}$ After the dipole distortion of the CMB is subtracted away, the temperature of the CMB varies by 0.3 microKelvin across the sky.

(iv) The photons of the CMB have mostly been traveling on straight lines since they were last scattered at $t \approx 370,000$ yr, at a location called the surface of last scattering.

[Comment: The actual variation is about 30 microKelvin, or maybe a few times that much. Ryden quotes the COBE root mean square fractional variation of the CMB temperature as

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5}$$

as Eq. (8.8) (2nd Edition), which gives a value of about 30 microKelvin, given that $T \approx 3\,K$. In Lecture Notes 2 we quoted a value of 4.14×10^{-5} computed from Planck data. The root mean square fluctuations increase with better angular resolution, because fluctuations with small angular wavelengths are not seen unless the resolution is high.

(b) *(5 points)* The nonuniformities in the cosmic microwave background allow us to measure the ripples in the mass density of the universe at the time when the plasma combined to form neutral atoms, about 300,000 - 400,000 years after the big bang. These ripples are crucial for understanding what happened later, since they are the seeds which led to the complicated tapestry of galaxies, clusters of galaxies, and voids. Which of the following sentences describes how these ripples are created in the context of inflationary models:

(i) Magnetic monopoles can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(ii) Cosmic strings, which are linelike topological defects, can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

$\boxed{\text{(iii)}}$ They are generated by quantum fluctuations during inflation.

(iv) Since the early universe was very hot, there were large thermal fluctuations which ultimately evolved into the ripples in the mass density.

(c) *(5 points)* In Chapter 8 of *The First Three Minutes*, Steven Weinberg describes the future of the universe (assuming, as was thought then to be the case, that the cosmological constant is zero). One possibility that he discusses is that the cosmic matter density could be greater than the critical density. Assuming that we live in such a universe, which of the following statements is NOT true?

(i) The universe is finite and its expansion will eventually cease, giving way to an accelerating contraction.

$\boxed{\text{(ii)}}$ Three minutes after the temperature reaches a thousand million degrees (10^9 K), the laws of physics guarantee that the universe will crunch, and time will stop.

(iii) During at least the early part of the contracting phase, we will be able to observe both redshifts and blueshifts.

(iv) When the universe has recontracted to one-hundredth its present size, the radiation background will begin to dominate the sky, with a temperature of about 300 K.

[Comment: Weinberg is very clear no speculations about the end of the universe are guaranteed to be true: "Does time really have to stop some three minutes after the temperature reaches a thousand million degrees? Obviously, we cannot be sure. All the uncertainties that we met in the preceding chapter, in trying to explore the first hundredth of a second, will return to perplex us as we look into the last hundredth of a second."]

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(iv) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

$\boxed{\text{(v)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(vi) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

[Comment: Ryden discusses the Sachs-Wolfe effect on pp. 161–162 (2nd Edition).

— End of Problem 1. —

PROBLEM 2: TIME EVOLUTION OF A UNIVERSE INCLUDING A HYPOTHETICAL KIND OF MATTER *(30 points)*

Suppose that a flat universe includes nonrelativistic matter, radiation, and also mysticium, where the mass density of mysticium behaves as

$$\rho_{\text{myst}} \propto \frac{1}{a^5(t)}$$

as the universe expands. In this problem we will define

$$x(t) \equiv \frac{a(t)}{a(t_0)} \ ,$$

where t_0 is the present time. For the following questions, you need not evaluate any of the integrals that might arise, but they must be integrals of explicit functions with explicit limits of integration; remember that $a(t)$ is not given. You may express your answers in terms of the present value of the Hubble expansion rate, H_0, and the various contributions to the present value of Ω: $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{myst},0}$.

(a) *(7 points)* Write an expression for the Hubble expansion rate $H(x)$.

(b) *(7 points)* Write an expression for the current age of the universe.

(c) *(3 points)* Write an expression for the time $t(x)$ in terms of the value of x.

(d) *(3 points)* Write an expression for the total mass density $\rho(x)$ as a function of x.

(e) *(10 points)* Write an expression for present value of the physical horizon distance, $\ell_{p,\text{hor}}(t_0)$.

(a) Since the universe is flat, the first Friedmann equation becomes

$$H^2 = \frac{8\pi}{3} G\rho \ ,$$

but then we can write ρ as

$$H^2 = \frac{8\pi}{3} G \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)} \right]^3 + \rho_{\text{rad},0} \left[\frac{a(t_0)}{a(t)} \right]^4 + \rho_{\text{myst},0} \left[\frac{a(t_0)}{a(t)} \right]^5 \right\} \ .$$

Now use

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \ \text{and} \ \ \Omega \equiv \frac{\rho}{\rho_c} \ ,$$

so

$$H^2 = \frac{H_0^2}{\rho_{c,0}} \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)} \right]^3 + \rho_{\rm rad,0} \left[\frac{a(t_0)}{a(t)} \right]^4 + \rho_{\rm myst,0} \left[\frac{a(t_0)}{a(t)} \right]^5 \right\}$$

$$= H_0^2 \left\{ \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\rm rad,0}}{x^4} + \frac{\Omega_{\rm myst,0}}{x^5} \right\} .$$

Finally,

$$\boxed{H(x) = \frac{H_0}{x^2} \sqrt{\Omega_{m,0} x + \Omega_{\rm rad,0} + \frac{\Omega_{\rm myst,0}}{x}} .}$$

(b) To find the current age t_0, we start with

$$H = \frac{\dot a}{a} = \frac{\dot x}{x} \quad \Longrightarrow \quad \frac{{\rm d}x}{{\rm d}t} = xH \quad \Longrightarrow \quad {\rm d}t = \frac{{\rm d}x}{xH} .$$

So t_0 can be found by integrating over the range of x, from 0 to 1:

$$t_0 = \int_0^1 \frac{{\rm d}x}{xH(x)}$$

$$= \boxed{\frac{1}{H_0} \int_0^1 \frac{x\,{\rm d}x}{\sqrt{\Omega_{m,0} x + \Omega_{\rm rad,0} + \frac{\Omega_{\rm myst,0}}{x}}} .}$$

(c) To find the time t corresponding to some value of x other than 1, one simply integrates ${\rm d}t$ from $x' = 0$ to $x' = x$:

$$t(x) = \int_0^x \frac{{\rm d}x'}{x' H(x')}$$

$$= \boxed{\frac{1}{H_0} \int_0^x \frac{x'\,{\rm d}x'}{\sqrt{\Omega_{m,0} x' + \Omega_{\rm rad,0} + \frac{\Omega_{\rm myst,0}}{x'}}} .}$$

(d) From the first Friedmann equation,

$$H^2 = \frac{8\pi}{3} G\rho \quad \Longrightarrow \quad \rho = \frac{3}{8\pi G} H^2(x) .$$

Given the answer in part (a), this becomes

$$\boxed{\rho(x) = \frac{3}{8\pi G}\frac{H_0^2}{x^4}\left[\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}\right] .}$$

(e) The general formula for the physical horizon distance is given on the formula sheet:

$$\ell_{p,\mathrm{hor}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' .$$

Here we are not given the function $a(t)$, but we can change the variable of integration to integrate over x:

$$\mathrm{d}t' = \frac{\mathrm{d}t'}{\mathrm{d}a}\mathrm{d}a = \frac{1}{\dot{a}}\mathrm{d}a = \frac{1}{a}\frac{a}{\dot{a}}\mathrm{d}a = \frac{\mathrm{d}a}{aH(x)} .$$

So

$$\begin{aligned}\ell_{p,\mathrm{hor}}(t_0) &= a(t_0)\int_0^{a(t_0)} \frac{c\,\mathrm{d}a}{a^2 H(a)} \\ &= \int_0^1 \frac{c\,\mathrm{d}x}{x^2 H(x)} \\ &= \boxed{\frac{c}{H_0}\int_0^1 \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}}} .}\end{aligned}$$

PROBLEM 3: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 6 of the Review Problems for Quiz 3.

In answering the following questions, remember that you can refer to the formulas on the formula sheets, circulated separately. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos at temperature T. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

Solution:

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\,\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}\,kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is 7/8 times the number of spin states, and g^* is 3/4 times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)}\,kT\ .}$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again $7/8$ and $3/4$ respectively, so

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8} \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{135\zeta(3)}\, k\ .}$$

Numerically, this gives $S = 4.202\, k$.

PROBLEM 4: THE CONSEQUENCES OF AN ALT-PHOTON *(25 points)*

Suppose that, in addition to the particles that are known to exist, there also existed an alt-photon, which has exactly the properties of a photon: it is massless, has two spin states (or polarization states), and has the same interactions with other particles that photons do. Like photons, it is its own antiparticle.

(a) *(5 points)* In thermal equilibrium at temperature T, what is the total energy density of alt-photons?

(b) *(5 points)* In thermal equilibrium at temperature T, what is the number density of alt-photons?

(c) *(10 points)* In this situation, what would be the temperature ratios T_ν/T_γ and $T_\nu/T_{\text{alt}\gamma}$ today?

(d) *(5 points)* Would the existence of this particle increase or decrease the abundance of helium, or would it have no effect?

Solution:

(a) The energy density will be the same as for photons, since there is no difference. The general formula is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ ,$$

as given on the formula sheets, and $g = 2$ for alt-photons (or photons), since there are two polarization states, and the particles are bosons. So

$$\boxed{u_{\text{alt}\gamma} = \frac{\pi^2}{15}\,\frac{(kT)^4}{(\hbar c)^3}\ .} \tag{4.1}$$

(b) For the number density, the general formula is

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\ ,$$

where $g^* = 2$ since again the alt-photons are bosons with two polarization states. So

$$\boxed{n_{\text{alt}\gamma} = 2\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\ .} \tag{4.2}$$

(c) As in the actual scenario, the event that causes a temperature difference is the disappearance of the electron-positron pairs from the thermal equilibrium mix, which occurs as kT changes from values large compared to $m_e c^2 = 0.511$ MeV to values that are small compared to it. The key point is that this disappearance occurs after the neutrinos have decoupled from the other particles, so all of the entropy from the electron-positron pairs is given to the photons, and none is given to the neutrinos. In this case the entropy is given to both the photons and the alt-photons.

The general formula for entropy density is on the formula sheet, and it can be rewritten as

$$s = AgT^3 \ , \tag{4.3}$$

where

$$A = \frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3} \ . \tag{4.4}$$

The value of A will in fact not be needed for this problem.

Since the neutrinos have decoupled by the time the e^+e^- pairs disappear, the entropy of neutrinos and the entropy of everything else will be separately conserved. Entropy conservation means that the entropy per comoving volume does not change. During the period before e^+e^- freeze-out, g is constant, so the constancy of entropy per comoving volume implies that

$$S = sV_{\rm phys} = gT^3 AV_{\rm phys} = ga^3T^3AV_{\rm coord} \ , \tag{4.5}$$

so $S/V_{\rm coord} = $ const implies that a^3T^3 is constant, and so aT is constant. Here T is the common temperature of photons, alt-photons, electrons and positrons, and neutrinos, all of which were in thermal equilibrium during this period. Since aT is constant during this period, we can give the constant a name,

$$aT = [aT]_{\rm before} \ . \tag{4.6}$$

For the neutrinos, the formula sheet tells us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \ , \tag{4.7}$$

while

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ . \tag{4.8}$$

Thus

$$g_{\text{else}} = g_\gamma + g_{\text{alt}\gamma} + g_{e^+e^-} = 2 + 2 + \frac{7}{2} = \frac{15}{2} \ . \tag{4.9}$$

Thus before the e^+e^- freezeout, the two conserved quantities were

$$\frac{S_\nu}{V_{\text{coord}}} = A g_\nu [aT]^3_{\text{before}} \ , \quad \frac{S_{\text{else}}}{V_{\text{coord}}} = A g_{\text{else}} [aT]^3_{\text{before}} \ . \tag{4.10}$$

After e^+e^- freezeout, the temperature of the neutrinos T_ν will no longer be the same as the temperature T_γ of the photons and alt-photons, and of course e^+e^- pairs will no longer be present. But T_γ and $T_{\text{alt}\gamma}$ will be equal to each other, since they have the same interactions; we know that the interactions of the photons keep them in thermal equilibrium until $t_{\text{decoupling}} \sim 380,000$ years, so both the photons and the alt-photons will remain in thermal equilibrium until long after the era of e^+e^- freezeout, which is of order 1–10 seconds. Thus the two conserved quantities will be

$$\frac{S_\nu}{V_{\text{coord}}} = A g_\nu [aT_\nu]^3_{\text{after}} \ , \quad \frac{S_{\text{else}}}{V_{\text{coord}}} = A (g_\gamma + g_{\text{alt}\gamma}) [aT_\gamma]^3_{\text{after}} \ . \tag{4.11}$$

By equating the values of S_ν / V_{coord} before and after, we see that

$$[aT_\nu]_{\text{after}} = [aT]_{\text{before}} \ , \tag{4.12}$$

and then by equating the values of $S_{\text{else}}/V_{\text{coord}}$ before and after, we see that

$$[aT_\gamma]_{\text{after}} = \left(\frac{g_{\text{else}}}{g_\gamma + g_{\text{alt}\gamma}} \right)^{1/3} [aT]_{\text{before}} = \left(\frac{g_{\text{else}}}{g_\gamma + g_{\text{alt}\gamma}} \right)^{1/3} [aT_\nu]_{\text{after}} \ , \tag{4.13}$$

where we used Eq. (4.12) in the last step. It follows that

$$\left[\frac{T_\nu}{T_\gamma} \right]_{\text{after}} = \left(\frac{g_\gamma + g_{\text{alt}\gamma}}{g_{\text{else}}} \right)^{1/3} = \left(\frac{2+2}{\frac{15}{2}} \right)^{1/3} = \boxed{\left(\frac{8}{15} \right)^{1/3} \ .} \tag{4.14}$$

(d) It would increase the abundance of helium. The main effect of the alt-photon would be to increase the expansion rate of the universe, which in turn would cause the neutrinos to decouple earlier from the thermal equilibrium mix, which in turn would mean that the ratio n_n/n_p, the ratio of neutrons to protons, would become frozen at a larger value. The increased expansion rate would also mean less time available for free neutron decay, which further increases the number of neutrons that remain when the temperature falls low enough for helium formation to complete. Essentially all the neutrons become bound into helium, so more neutrons implies more helium.

Problem	**Maximum**	**Score**	**Initials**
1	20		
2	30		
3	25		
4	25		
TOTAL	100		