MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 5, 2020
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QUIZ 1 SOLUTIONS

Quiz Date: September 30, 2020

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) *(5 points)* In what way do spectral lines help us understand the properties of nearby stars? Consider the following options:

(A) We can determine their chemical composition.

(B) We can determine their speed relative to us.

(C) We can determine their acceleration relative to us.

(D) We can determine their mass.

(E) We can determine their volume.

Which one of the following combinations is correct:

(i) **(A) and (B)**

(ii) (A) and (C)

(iii) (B) and (C)

(iv) (D) and (E)

(v) (A), (B), (C), (D), and (E)

[*Comment:* The redshift of the spectral lines determines the speed of the star relative to us, and the presence and strengths of lines tells us about the composition. Although this information can identify the type of star and thereby give some indirect information about the mass and volume, the word "determine" does not really apply to the mass or volume.]

(b) *(5 points)* What is the cosmological principle?

(i) We live in the center of the universe.

(ii) **The universe is homogeneous and isotropic on large distance scales.**

(iii) The universe is isotropic.

(iv) All galaxies are evenly spaced throughout the universe.

(v) Newtonian gravity applies on large distance scales.

(c) *(5 points)* Roughly, what is the size of the Milky Way? And, at what length scale does the cosmological principle become valid?

(i) 10^7 light years and 100 Mpc

(ii) 10^5 light years and 1 Mpc

(iii) 10^5 light years and 100 Mpc

(iv) 10^3 light years and 1 Mpc

(v) 10^3 light years and 100 Mpc

(d) *(5 points)* Which one of the following statements is a valid description of the history of what is now known as the "Cosmic Microwave Background"?

(i) The CMB radiation initially had a temperature less than 2.7 degrees Kelvin, but it increased gradually to its current value as it absorbed the energy of the light from stars.

(ii) Plasma instabilities in the early universe generated large fluctuations in electromagnetic radiation, which we now see as a "background" to astronomical observations.

(iii) The universe was originally filled with microwave radiation, and it persists until today.

(iv) It is a primordial kind of radiation with a blackbody spectrum of 2.7 degrees Kelvin at the time it was generated.

(v) At some point in the history of our universe, photons, which were once in thermal equilibrium with other particles, stopped interacting because electrons and atomic nuclei combined into atoms. The photons then free-streamed throughout the universe, from then until now.

[*Comment:* Choices (i), (iii), and (iv) are all contradicted by the fact that the temperature of the CMB has been cooling as the universe expands. 2.7 degrees is the current temperature; in the past it was much hotter, and hence not in the microwave band. The origin is not related to plasma instabilities, which are non-equilibrium phenomena which would not be capable of producing the thermal spectrum of the CMB.]

(e) *(5 points)* Why is the night sky not uniformly bright due to the universe being homogeneously populated with stars (or other bright objects)? Consider the following options:

(A) Space is not transparent.

(B) The universe is not infinitely large.

(C) The universe is not infinitely old.

(D) The absolute brightness of stars depends on their distance away from us.

(E) The cosmological redshift makes stars look dimmer and dimmer as they are further away from us.

Which one of the following combinations is correct?

(i) (A) and (C) — accepted

(ii) (B) and (D) — accepted

(iii) **(C) and (E)** — preferred

(iv) (A), (C), and (E) — accepted

(v) (B), (C), and (D) — accepted

[*Comment:* When we (Alan and Bruno) wrote this problem we intended (iii) to be the correct answer, but after the quiz we decided that all the answers are arguably correct, so we decided to give full credit to everybody.

Despite the fact that this question is irrelevant to your grade, we hope you will find the following discussion interesting and informative.

The question is complicated by the fact that we do not fully agree with Ryden's statement that "the primary resolution to Olbers' paradox comes from the fact that the universe has a finite age," which corresponds to choice (C) in the problem statement. She also mentions that "However, in an expanding universe, the surface brightness of distant light sources is decreased relative to what you would see in a static universe," which is a way of describing choice (E). It's hard to know what "primary resolution" means, since these two explanations are both relevant and are somewhat intertwined. It is clear that a finite age is not essential to resolving Olber's paradox. In the steady-state theory (discussed by Ryden on pp. 17 and 18), which was considered very plausible before the discovery of the CMB, the age of the universe was infinite, but Olber's paradox was entirely avoided by the redshift effects.

Ryden was not quantitative when she spoke of the decrease in surface brightness, but in fact you have already done the calculations to show this. You calculated on Problem 6 of Problem Set 2 that the power received from a distant source is reduced by a factor of $(1+z)^2$, where one factor of $(1+z)$ came from the reduction in energy of each photon, and the other factor came from the reduction in the arrival rate of photons. You also found in Problem 5 that the angular size is enhanced by a factor of $(1+z)$, which means that the solid angle is enhanced by a factor of $(1+z)^2$. This means that the surface brightness, the amount of power that we receive per surface area on the sky, is actually reduced by a factor of $(1+z)^4$. So as we look to very large redshifts, the stars contribute very little to the brightness of the night sky.

The effect of the finite age means that we cannot see stars at distances larger than the horizon distance, which (as discussed in Lecture Notes 4) is the present physical distance of the furthest objects for which light has had time to reach us, since the beginning. The light we receive from such objects left them at $t = 0$, when $a(t) = 0$, so the redshift is infinite. Since the brightness of high z stars was suppressed by a factor of $1/(1+z)^4$, the fact that we see nothing beyond $z =$ infinity does not seem as relevant. But the finite age can still be considered relevant in the sense that if we considered a toy model of a universe that was static, so there would be no redshift, Olber's paradox could still be resolved if the universe were not infinitely old.

Option (A): the lack of spatial transparency is not usually mentioned in discussions of Olber's paradox. From observations of the CMB, we know that the universe is highly transparent, out to redshifts of about $z \approx 1000$. From the point of view of Olber's paradox, however, $z \approx 1000$ is very close. As Ryden explains, Olber's paradox involves thinking about distances of order 10^{18} Mpc, which is about 10^{14} times larger than the horizon distance. At $z \approx 1000$ we are looking so far back in time that we see the early plasma phase of the universe, when space was very opaque (non-transparent). This region is called the *surface of last scattering*, since most of the CMB photons were last scattered in this region. The surface of last scattering is fairly close to the horizon distance, so the lack of transparency in this region can be used as a variant to discussing the effects of the horizon.

Ryden dismisses transparency as an explanation by pointing out that if the universe is infinitely old, then eventually any light-absorbing material would be heated to the temperature of the stars, and would glow brightly. But if the universe is not infinitely old (as it does not appear to be), then this argument does not hold.

Option (B): the possibility that the universe has a finite size. Ryden points out that our universe might not be infinite — we don't know. If *our* universe has a finite size, however, it would not have much relevance to Olber's paradox, since the finite size would have to be about equal to or larger than the horizon distance, or else we would probably see evidence of it. In toy models of a universe, however, a finite size can serve as a way to avoid Olber's paradox.

Option (D): The validity of (D) as an explanation depends on the interpretation of the words. At a fixed cosmic time, we believe that the universe is on average homogeneous, so the absolute brightness of stars does not depend on their distance from us. However, if the statement is assumed to refer to the absolute brightness of the stars at the time that we see them, then of course we are seeing distant stars as they appeared long ago, and the absolute brightness can be different. If we look so far back that there are no stars, then the brightness would be zero. By this interpretation (D) can be valid, and is really just another way of discussing the finite age.]

(f) *(5 points)* Which of the following statements best describes Einstein's equivalence principle?

(i) Physical laws do not depend on the place of the universe where we make measurements.

(ii) General relativity is equivalent to special relativity when only massless particles are considered.

(iii) An observer on the surface of the Earth experiences the same external forces as a free-falling observer.

(iv) An observer on the surface of the Earth, without looking through a window, cannot tell whether they are actually on Earth or if they are on a rocket ship accelerating at 9.8 m/s^2 in empty space.

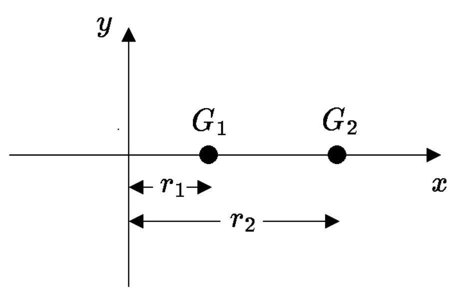
(v) Any calculation done in the framework of General Relativity must be equivalent to another in Newtonian gravity in the appropriate limit.

PROBLEM 2: INTERGALACTIC SIGNALING AND TRAVEL *(30 points)*

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{2/3} ,$$

where b is a constant. Suppose that galaxy G_1 is located at comoving coordinates $(x, y, z) = (r_1, 0, 0)$ and galaxy G_2 is located at $(x, y, z) = (r_2, 0, 0)$, as shown in the following diagram:



(a) *(10 points)* At cosmic time $t = t_1$, a light signal is sent from G_1, in the direction of G_2. At what time t_2 does the light signal reach G_2? Your answer should be expressed in terms of some or all of the given variables t_1, b, r_1, r_2, and the speed of light c.

Answer: The coordinate distance that must be traveled is $r_2 - r_1$, and the coordinate speed of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} .$$

The integral of the coordinate speed over time is the coordinate distance traveled, so

$$\int_{t_1}^{t_2} \frac{c\,\mathrm{d}t'}{b\,t'^{2/3}} = r_2 - r_1 . \tag{2.1}$$

Carrying out the integration,

$$\int_{t_1}^{t_2} \frac{c\,\mathrm{d}t'}{b\,t'^{2/3}} = 3\frac{c}{b}\left[t_2^{1/3} - t_1^{1/3}\right] = r_2 - r_1 \ . \tag{2.2}$$

Solving for t_2, one finds

$$\boxed{t_2 = \left[t_1^{1/3} + \frac{b}{3c}(r_2 - r_1)\right]^3 \ .} \tag{2.3}$$

(b) *(6 points)* When the light signal arrives at G_2, what is its redshift z? Your answer can be expressed in terms of any or all of the variables mentioned in part (a), and also t_2, the answer to part (a).

Answer: This is an example of the cosmological redshift, given by

$$1 + z = \frac{a(t_{\text{observer}})}{a(t_{\text{source}})} = \frac{a(t_2)}{a(t_1)} \ , \tag{2.4}$$

so

$$\boxed{z = \frac{a(t_2)}{a(t_1)} - 1 = \left(\frac{t_2}{t_1}\right)^{2/3} - 1 \ .} \tag{2.5}$$

The equation above is an acceptable answer, but one could also insert Eq. (2.3) for t_2:

$$\boxed{z = \left[1 + \frac{b}{3ct_1^{1/3}}(r_2 - r_1)\right]^2 - 1 \ .} \tag{2.6}$$

(c) *(7 points)* Now suppose that at time t_1, the galaxy G_1 also launches a starship towards G_2. This is a science-fiction starship, which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity, and no time to decelerate at the end of its journey. At what time t_3 does the starship arrive at G_2? Your answer may depend on any of the allowed variables in part (b).

Answer: This part is analogous to part (a), except that the trajectory has a physical speed $\frac{1}{2}c$ instead of c, and hence a coordinate speed

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{2a(t)} \ . \tag{2.7}$$

The answer is then identical to Eq. (2.3), except that c is replaced by $\frac{1}{2}c$:

$$t_3 = \left[t_1^{1/3} + \frac{2b}{3c}(r_2 - r_1) \right]^3 . \tag{2.8}$$

(d) *(7 points)* We have not discussed how the clocks on such a starship would behave, but general relativity implies that, as the starship passes any comoving observer, the comoving observer would measure the clocks on the starship to be time-dilated exactly as they would be in special relativity, for the same relative velocity. Given this fact, what time will the clocks on the starship read when the ship arrives at G_2? Your answer may include any of the allowed variables in part (c), and also t_3, the answer to part (c).

Answer: The time dilation factor for $v = \frac{1}{2}c$ is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2}{\sqrt{3}} . \tag{2.9}$$

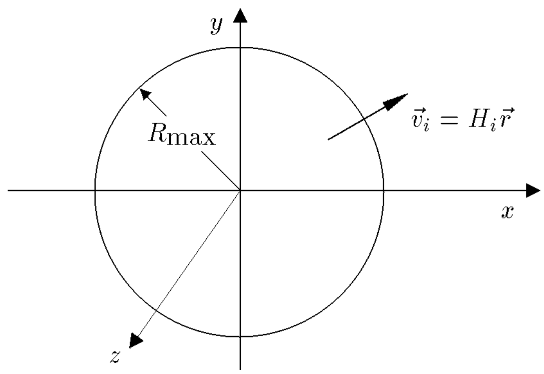
When the starship clocks are passing the clocks of the comoving observers, they will appear to be running slowly by this factor. During the starship voyage the comoving clocks will advance from time t_1 to t_3, so the clocks on the starship will advance from t_1 to

$$t_{\text{final,starship}} = t_1 + \frac{\sqrt{3}}{2}(t_3 - t_1) . \tag{2.10}$$

PROBLEM 3: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(30 points)*

The following problem was Problem 4 on Problem Set 3.

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i\vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} \ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} + \gamma r^n(r_i, t)\,\hat{r} \ ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i, t)} + \gamma r^n(r_i, t) \ .$$

(a) *(7 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u. *(Hint: be sure that u is the only time-dependent quantity in your equation; r, ρ, etc. must be rewritten in terms of u, ρ_i, etc.)*

Answer: Substituting the equation for $M(r_i)$, given on the quiz, into the differential equation for r, also given on the quiz, one finds:

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} + \gamma r^n \ .$$

Dividing both sides of the equation by r_i, one has

$$\frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G r_i^2 \rho_i}{r^2} + \gamma \frac{r^n}{r_i} \ .$$

Substituting $u = r/r_i$, this becomes

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} + \gamma u^n r_i^{n-1} \ .}$$

(b) *(6 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

Answer: The only dependence on r_i occurs in the last term, which is proportional to r_i^{n-1}. This dependence disappears if $\boxed{n = 1}$, since the zeroth power of any positive number is 1.

(c) *(7 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

Answer: This is exactly the same as the case discussed in the lecture notes, since the initial conditions do not depend on the differential equation. At $t = 0$,

$$\begin{aligned} r(r_i, 0) &= r_i && \text{(definition of } r_i\text{)} \\ \dot{r}(r_i, 0) &= H_i r_i && \text{(since } \vec{v}_i = H_i \vec{r}\text{).} \end{aligned}$$

Dividing these equations by r_i one has the intial conditions

$$\boxed{\begin{aligned} u &= 1 \\ \dot{u} &= H_i \ . \end{aligned}}$$

(d) *(10 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$a(t) \equiv u(r_i, t) \ .$$

Show, for this value of n, that the differential equation for a can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on a and $\dot{a}$, but not $\ddot{a}$ or any higher derivatives.

Answer: $a(t)$ should obey the differential equation obtained in part (a) for u, for the value of n that was obtained in part (b): $n = 1$. So,

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a \ .$$

Multiplying the equation by $\dot{a} \equiv da/dt$, one finds

$$\frac{d^2 a}{dt^2}\frac{da}{dt} = \left\{ -\frac{4\pi}{3}\frac{G\rho_i}{a^2} + \gamma a \right\} \frac{da}{dt} \ ,$$

which can be rewritten as

$$\frac{d}{dt}\left\{ \frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 \right\} = 0 \ .$$

Thus the quantity inside the curly brackets must be constant. Following the lecture notes, I will call this constant E:

$$\boxed{\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 = E \ .}$$

Or one can use the conventionally defined quantity

$$k = -\frac{2E}{c^2} \ ,$$

in which case the equation can be written

$$\left\{ \frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} - \frac{1}{2}\gamma a^2 \right\} = -\frac{kc^2}{2} \ .$$

One can then rewrite the equation in the more standard form

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho + \gamma - \frac{kc^2}{a^2} \ ,}$$

where I have used $\rho(t) = \rho_i/a^3(t)$.

Additional Note: Historically, the constant γ corresponds to the "cosmological constant" which was introduced by Albert Einstein in 1917 in an effort to build a static model of the universe. The cosmological constant Λ, as defined by Einstein, is related to γ by

$$\gamma = \frac{1}{3}\Lambda c^2 \ .$$

The differential equation can be rewritten as

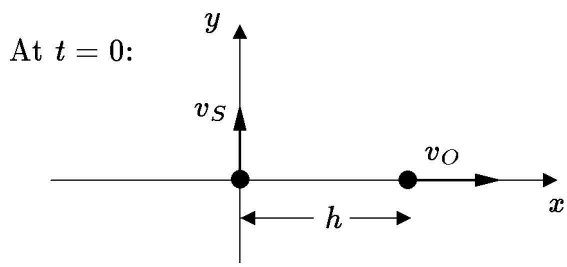
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{a^2} \ ,$$

which shows that the cosmological constant contributes like a constant addition to the mass density. Modern physicists interpret the cosmological constant as a manifestation of the mass density of the vacuum. From the above equation, we can see that the mass density of the vacuum is related to Einstein's cosmological constant Λ by

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G} \ .$$

PROBLEM 4: THE NONRELATIVISTIC TRANSVERSE DOPPLER SHIFT *(10 points)*

At time $t = 0$, a source of sound waves is located at the origin of a coordinate system, and an observer is located at a distance h along the positive x axis, as shown in the diagram below. The source is moving at constant speed v_S along the positive y axis, and the observer is moving at constant speed v_O along the positive x axis. Denote the speed of sound by u.



(a) *(6 points)* At $t = 0$, the sound source emits a wave crest. At what time t_1 does it arrive at the observer?

Answer: The wave crest follows the trajectory

$$x_c = ut \ .$$

The motion of the observer is described by

$$x_O = h + v_O t \ .$$

The intersection of these equations determines when the wave crest intersects the observer:

$$x_c = x_O \quad \Longrightarrow \quad ut_1 = h + v_O t_1 \quad \Longrightarrow$$

$$\boxed{t_1 = \frac{h}{u - v_O} \ .}$$

(b) *(4 points)* Under the usual assumption that the wavelength of the wave is very short compared to any other distance, what is the Doppler shift of the sound wave, as received by the observer?

Answer: Since this is a nonrelativistic problem, there is no time dilation. Thus, the Doppler shift is caused entirely by the rate of change of the path length $\ell(t)$, the length of the path of wave crests emitted at time t. Since the motion of the source is perpendicular to the direction of the observer, it does not contribute to $\mathrm{d}\ell/\mathrm{d}t$. Thus, the Doppler shift is given by the formula for the nonrelativistic Doppler shift when the observer is moving along a line, away from the source, which is given in the formula sheet:

$$\boxed{z = \frac{v_O/u}{1 - v_O/u} \ .}$$

This was derived in Lecture Notes 1, as Eq. (1.8).

Problem	**Maximum**	**Score**	**Initials**
1	30		
2	30		
3	30		
4	10		
TOTAL	100		