MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 28, 2020
Prof. Alan Guth

QUIZ 2

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) *(5 points)* What is a necessary condition for two photons in a head-on collision to be able to produce an electron-positron pair?

- (i) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 TeV.
- (ii) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 GeV.
- (iii) The energy of the photons must be below the "rest energy" of the electron-positron pair, which is roughly 1.02 keV.
- (iv) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 MeV.
- (v) Two colliding massless objects cannot produce massive particles as an outcome of their collision.

(b) *(5 points)* In Chapter 4, *Recipe for a Hot Universe,* Weinberg states that there are three conserved quantities that must be specified in the recipe for the early universe. What are they? [Grading: one correct = 2 pts; two correct = 4 pts; three correct = 5 pts.]

- (i) ______________________
- (ii) ______________________
- (iii) ______________________

— *Problem 1 continues on next page.* —

(c) *(5 points)* Why are "complicated" Feynman diagrams needed for calculations of the strong interactions, whereas accurate predictions can be made for electromagnetism with only the "simple" diagrams?

- (i) Only the simplest Feynman diagrams in electromagnetism give a non-zero contribution; the more complicated diagrams cancel each other out.
- (ii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," about 1/137.036. Complicated diagrams therefore give small contributions. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to one, so complicated diagrams are not suppressed.
- (iii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," which is about equal to one, meaning that adding internal lines to a diagram doesn't change its value significantly. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to 137, so complicated diagrams are strongly enhanced.
- (iv) Simple Feynman diagrams are mathematically ill-defined in the theory of strong interactions, whereas in electromagnetism they give a sensible numerical result.
- (v) Calculations for the strong interactions involve six quarks, while there are only three charged leptons. If there were three more electron-like particles (in addition to the muon and the tau lepton), then electromagnetism would be as complicated as the strong interactions.

— *Problem 1 continues on next page.* —

(d) *(5 points)* Consider the following observable phenomena:

- (A) The motion of stars moving in large-radius circular orbits around the centers of spiral galaxies.
- (B) The motion of some galaxies relative to the galaxy clusters they inhabit.
- (C) Gravitational lensing of light emitted by stars in nearby galaxies.

Which one of the following combinations constitutes observational evidence that dark matter exists?

- (i) Only (A).
- (ii) Only (B).
- (iii) Only (C).
- (iv) (A) and (B).
- (v) (A), (B), and (C).

(e) *(5 points)* Consider the following claims about hadrons:

- (A) Electrons and neutrinos make up 90% of the Universe's hadronic matter content.
- (B) They are the class of particles affected by strong interactions (i.e., by the strong nuclear force).
- (C) It is a synonym of "baryons."
- (D) They are theorized to be made up of more fundamental particles, called "quarks."
- (E) They cannot be made up of more fundamental constituents, because if they were, we should be able to break them up into their building blocks and observe the hypothetical "quarks" directly.

Which combination of these claims is true?

- (i) Only (A).
- (ii) Only (B).
- (iii) (B) and (D).
- (iv) (B) and (E).
- (v) (B), (C), and (D).

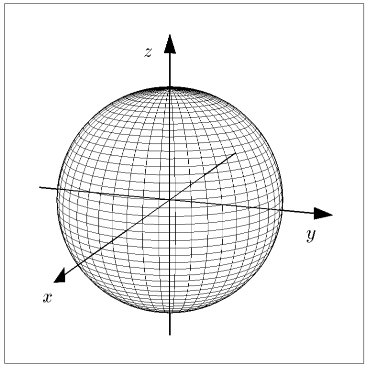
— Problem 1 continues on next page. —

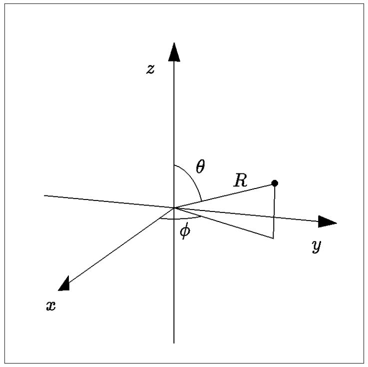
(f) *(5 points)* Which of the following is the range of typical binding energies per nucleon in an atomic nucleus? (This is relevant to nuclear fusion and nuclear fission studies, and to the epoch of Big Bang nucleosynthesis.)

- (i) 1 eV – 10 eV.
- (ii) 100 eV – 1 keV.
- (iii) 10 keV – 100 keV.
- (iv) 1 MeV – 10 MeV.
- (v) 100 MeV – 1 GeV.

— End of Problem 1. —

PROBLEM 2: GEODESICS ON THE SURFACE OF A SPHERE *(25 points)*





In this problem we will explore the geodesic equation for the metric describing the surface of a sphere. We will describe the sphere as in Lecture Notes 5, with metric given by

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ , \tag{2.1}$$

where R is the radius of the sphere. As given on the formula sheet, the geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s} \ . \tag{2.2}$$

(a) *(5 points)* Using the metric of Eq. (2.1), write explicitly the geodesic equation for $i = 1 = \theta$. By "explicitly," we mean that the equation that you give should not contain any instances of g, i, j, or k. All instances of the metric should be replaced using the metric from Eq. (2.1), and all repeated indices should be summed explicitly.

(b) *(5 points)* Again using the metric of Eq. (2.1), write explicitly the geodesic equation for $i = 2 = \phi$.

(c) *(5 points)* Consider a curve that circles the sphere at fixed latitude,

$$\theta(s) = \theta_0 \ , \qquad \phi(s) = \beta s \ , \tag{2.3}$$

where β and θ_0 are constants. Use the geodesic equations you found in parts (a) and (b) to find out if these curves are geodesics (i.e., see if they statisfy the geodesic equations). For what values of θ_0, if any, are these curves geodesics, and for what values of θ_0, if any, are they not?

— *Problem 2 continues on next page.* —

(d) *(5 points)* Consider a curve that moves along a line of fixed longitude:

$$\theta(s) = \gamma s \ , \qquad \phi(s) = \phi_0 \ , \tag{2.4}$$

where γ and ϕ_0 are constants. Use the equations you found in parts (a) and (b) to find the values of ϕ_0, if any, for which these curves are geodesics, and for what values, if any, they are not.

(e) *(2 points)* The geodesic equations imply that there is a conserved quantity of the form

$$L \equiv F(\theta, \phi)\frac{\mathrm{d}\phi}{\mathrm{d}s} \ . \tag{2.5}$$

(By "conserved," we mean that this quantity is guaranteed to be constant along a geodesic.) Find the function $F(\theta, \phi)$ for which this is true. Note (a) that $F(\theta, \phi)$ is defined by the conservation statement only up to a multiplicative constant; and (b) that a function need not actually vary as its argument is varied (i.e., $q(x) = 5$ is a perfectly valid function of x).

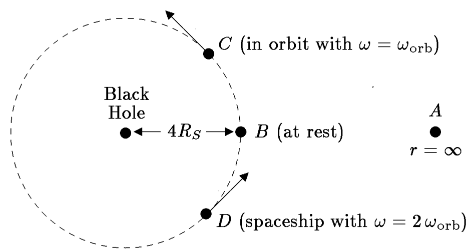
(f) *(3 points)* Use your answers from part (e), (a), and (b) to find an equation for geodesics of the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \text{expression} \ , \tag{2.6}$$

where the "expression" depends only on θ and the conserved quantity L. (This is useful, since differential equations involving just one variable are generally easier to solve than differential equations involving two variables.) You may leave $F(\theta, \phi)$ in your answer, if you have not evaluated it.

— *End of Problem 2.* —

PROBLEM 3: TRAVEL IN THE SCHWARZSCHILD METRIC *(20 points)*



Consider a black hole of mass M, described by the Schwarzschild metric as written on the formula sheets:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{3.1}$$

In this problem we will consider a set of different observers in this spacetime, labeled by A, B, C, and D, as shown in the diagram above and described below.

(a) *(5 points)* Let observer A be stationary at an arbitrarily large value of r (you can take $r_A = \infty$), and let observer B be stationary at

$$r_B = 4R_S\,, \tag{3.2}$$

where R_S is the Schwarzschild radius, $R_S = 2GM/c^2$. (Note that observer B needs to be using rocket propulsion to prevent herself from falling into the black hole.) Suppose that observer B is sending out pulses of electromagnetic radiation, at evenly spaced intervals $\Delta\tau_B$, as measured on B's local clock. What is the time interval $\Delta\tau_A$ that observer A will measure between the pulses that he receives?

(b) *(5 points)* Observer C is in orbit around the black hole, also at radius

$$r_C = 4R_S\,. \tag{3.3}$$

The orbit is in the equatorial plane, with θ fixed at $\theta = \frac{\pi}{2}$, and ϕ changing with time as

$$\phi(t) = \omega_{\rm orb}\,t\,. \tag{3.4}$$

In Problem 3 of Problem Set 6 you showed that the orbital angular velocity $\omega_{\rm orb}$, for a circular orbit, must satisfy

$$r\omega_{\rm orb}^2 = \frac{GM}{r^2}\,. \tag{3.5}$$

As measured on C's local clock, how long does it take for C to complete one orbit (i.e., for ϕ to change by 2π)?

— *Problem 3 continues on next page.* —

(c) *(5 points)* Now consider an observer D on a spaceship traveling in a circle at the same radius,

$$r_D = 4R_S\,, \tag{3.6}$$

also in the equatorial plane, $\theta = \frac{\pi}{2}$. Using its rocket power, the spaceship travels twice as fast as observer C, with

$$\phi(t) = \omega_D\,t = 2\,\omega_{\rm orb}\,t\,. \tag{3.7}$$

As measured on D's local clock, how long does it take for D to make one full circle (i.e., for ϕ to change by 2π).

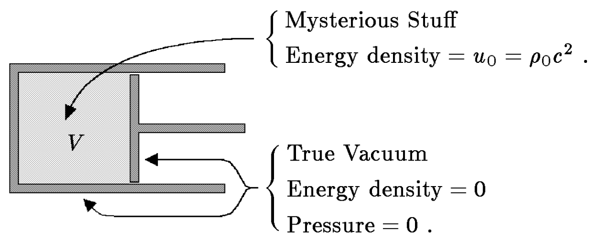
(d) *(5 points)* If a tape measure, at rest with respect to the Schwarzschild coordinates, were stretched around the circle at radius $r = 4R_S$, what circumference would it measure?

PROBLEM 4: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

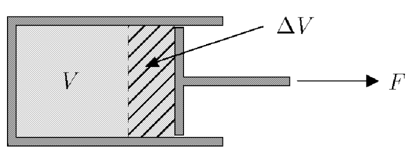
The following problem was Problem 22 of the Review Problems for Quiz 2, and earlier it was Problem 3, Quiz 3, 2002.

In Lecture Notes 6, with further calculations in Problem 4 of Problem Set 6, a thought experiment involving a piston was used to show that $p = \frac{1}{3}\rho c^2$ for radiation. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

Problem	Maximum	Score	Initials
1	30		
2	25		
3	20		
4	25		
TOTAL	100		