

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Physics Department

Physics 8.286: The Early Universe  
Prof. Alan Guth

October 20, 2020

## REVIEW PROBLEMS FOR QUIZ 2

Revised October 27, 2020\*

**QUIZ DATE:** Wednesday, October 28, 2020.

**COVERAGE:** Lecture Notes 4, 5, and through the section on “Dynamics of a Flat Radiation-Dominated Universe” of Lecture Notes 6; Problem Sets 4, 5, and 6; Weinberg, *The First Three Minutes*, Chapters 4 – 7; In Ryden’s *Introduction to Cosmology*, we have read Chapters 4 and Chapter 5 through Section 5.4.1 during this period [First edition: Chapters 4, 5, and Section 6.1]. These chapters, however, parallel what we have done or will be doing in lecture, so you should take them as an aid to learning the lecture material; there will be no questions explicitly based on these sections from Ryden. But we have also read Chapters 7 (*Dark Matter*) and Chapter 9 (*Nucleosynthesis and the Early Universe*) in Ryden [First edition: Chapters 8 and 10], and these are relevant material for the quiz, except for Sec. 9.3 (*Deuterium Synthesis*). I intend to post some notes to serve as background to this section, but since I haven’t done that yet, this section will not be included in Quiz 2. You can also ignore Ryden’s Eqs. (9.11), (9.12), and (9.13) for now, but do not ignore Eq. (9.14) [First edition: same numbers, but in Chapter 10]. Chapters 4 and 5 of Weinberg’s book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. We will not take off for the spelling of names, as long as they are vaguely recognizable. For dates before 1900, it will be sufficient for you to know when things happened to within 100 years. For dates after 1900, it will be sufficient if you can place events within 10 years. You should expect one 25-point problem based on the readings, and several calculational problems. **One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the problem sets listed above (extra credit problems included), or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 7, 8, 12, 14, 19, and 22.

**QUIZ LOGISTICS:** The logistics will be identical to Quiz 1, except of course for the dates. The quiz will be closed book, no calculators, no internet, and 85 minutes long. I assume that most of you will take it during our regular class time on October 28, but you will have the option of starting it any time during a 24-hour window from 11:05 am EDT on October 28 to 11:05 am EDT on Thursday, October 29. If you want to start later than 11:05 am 10/28/20, you should email me your choice of starting time by 11:59 pm on 10/27/2020 (earlier is appreciated). The quiz will

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\* In the original 10/20/20 version, some of the problem solutions were misnumbered and mislinked.

be contained in a PDF file, which I am planning to distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on the uploading, because the time needed for scanning, photographing, or whatever kind of processing you are doing can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the October 28 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times. If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.

**PURPOSE OF THE REVIEW PROBLEMS:** These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

**QUIZZES FROM PREVIOUS YEARS:** In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, 2011, 2013, 2016, and 2018. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, mainly to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match exactly the coverage from all previous years, but I believe that all these review problems would be fair problems for the upcoming quiz. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here. In 2016 we finished Weinberg's book by the time of Quiz 2, but otherwise the coverage has been the same since 2016.

**REVIEW SESSION:** To help you study for the quiz, Bruno Scheihing will hold a review session, at a time and place to be announced.

**FUTURE QUIZ:** Quiz 3 will be given on Wednesday, December 2, 2020.

**INFORMATION TO BE GIVEN ON QUIZ:**

Each quiz in this course will have a section of “useful information” for your reference. For the second quiz, this useful information will be the following:

**DOPPLER SHIFT (Definition:)**

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}},$$

where  $\Delta t_{\text{observer}}$  and  $\Delta t_{\text{source}}$  are the period of the wave as measured by the observer and by the source, respectively, and  $\lambda_{\text{observer}}$  and  $\lambda_{\text{source}}$  are the wavelength of the wave, as measured by the observer and by the source, respectively.

**DOPPLER SHIFT (For motion along a line):**

Nonrelativistic,  $u =$  wave speed, source moving at speed  $v$  away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed  $v$  away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity),  $\beta \equiv v/c$ , where  $c$  is the speed of light and  $v$  is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1 + \beta}{1 - \beta}} - 1$$

**COSMOLOGICAL REDSHIFT:**

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

**SPECIAL RELATIVITY:**

Time Dilation. A clock that is moving at speed  $v$  relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor  $\gamma$ :

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance  $\ell_0$  in their own frame, are moving together, in the direction of the line separating them, at speed  $v$  relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

Energy-Momentum Four-Vector:

$$p^\mu = \left( \frac{E}{c}, \vec{p} \right), \quad \vec{p} = \gamma m_0 \vec{v}, \quad E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2},$$

$$p^2 \equiv |\vec{p}|^2 - (p^0)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -(m_0 c)^2.$$

## KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law:  $v = Hr$ ,

where  $v$  = recession velocity of a distant object,  $H$  = Hubble expansion rate, and  $r$  = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor:  $\ell_p(t) = a(t)\ell_c$ ,

where  $\ell_p(t)$  is the physical distance between any two objects,  $a(t)$  is the scale factor, and  $\ell_c$  is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate:  $H(t) = \frac{1}{a(t)} \frac{da(t)}{dt}$ .

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed  $c$  relative to any observer. In Cartesian coordinates, coordinate speed  $\frac{dx}{dt} = \frac{c}{a(t)}$ . In general,  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$ .

Horizon Distance:

$$\begin{aligned} \ell_{p,\text{horizon}}(t) &= a(t) \int_0^t \frac{c}{a(t')} dt' \\ &= \begin{cases} 3ct & (\text{flat, matter-dominated}), \\ 2ct & (\text{flat, radiation-dominated}). \end{cases} \end{aligned}$$

### COSMOLOGICAL EVOLUTION:

$$\begin{aligned} H^2 &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2}, \quad \ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a, \\ \rho_m(t) &= \frac{a^3(t_i)}{a^3(t)} \rho_m(t_i) \quad (\text{matter}), \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)} \rho_r(t_i) \quad (\text{radiation}). \\ \dot{\rho} &= -3\frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2}\right), \quad \Omega \equiv \rho/\rho_c, \quad \text{where } \rho_c = \frac{3H^2}{8\pi G}. \end{aligned}$$

### EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ( $k = 0$ ):  $a(t) \propto t^{2/3}$   
 $\Omega = 1$ .

Closed ( $k > 0$ ):  $ct = \alpha(\theta - \sin\theta)$ ,  $\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$ ,  
 $\Omega = \frac{2}{1 + \cos\theta} > 1$ ,  
 where  $\alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{k}}\right)^3$ .

Open ( $k < 0$ ):  $ct = \alpha(\sinh\theta - \theta)$ ,  $\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)$ ,  
 $\Omega = \frac{2}{1 + \cosh\theta} < 1$ ,  
 where  $\alpha \equiv \frac{4\pi}{3} \frac{G\rho}{c^2} \left(\frac{a}{\sqrt{\kappa}}\right)^3$ ,  
 $\kappa \equiv -k > 0$ .

**MINKOWSKI METRIC (Special Relativity):**

$$ds^2 \equiv -c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 .$$

**ROBERTSON-WALKER METRIC:**

$$ds^2 \equiv -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

Alternatively, for  $k > 0$ , we can define  $r = \frac{\sin \psi}{\sqrt{k}}$ , and then

$$ds^2 \equiv -c^2 d\tau^2 \equiv -c^2 dt^2 + \tilde{a}^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} ,$$

where  $\tilde{a}(t) = a(t)/\sqrt{k}$ . For  $k < 0$  we can define  $r = \frac{\sinh \psi}{\sqrt{-k}}$ , and then

$$ds^2 \equiv -c^2 d\tau^2 = -c^2 dt^2 + \tilde{a}^2(t) \{ d\psi^2 + \sinh^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} ,$$

where  $\tilde{a}(t) = a(t)/\sqrt{-k}$ . Note that  $\tilde{a}$  can be called  $a$  if there is no need to relate it to the  $a(t)$  that appears in the first equation above.

**SCHWARZSCHILD METRIC:**

$$ds^2 \equiv -c^2 d\tau^2 = - \left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{rc^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,$$

**GEODESIC EQUATION:**

$$\frac{d}{ds} \left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} (\partial_i g_{kl}) \frac{dx^k}{ds} \frac{dx^l}{ds}$$

or:

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

**PROBLEM LIST**

1. Did You Do the Reading? (2018) . . . . . 8 (Sol: 35)
2. Did You Do the Reading? (2016) . . . . . 9 (Sol: 38)
3. Did You Do the Reading? (2013) . . . . . 11 (Sol: 41)
4. Did You Do the Reading? (2011) . . . . . 12 (Sol: 42)
5. Did You Do the Reading? (2007) . . . . . 12 (Sol: 44)
6. Did You Do the Reading? (2000, 2002) . . . . . 16 (Sol: 48)
- \*7. Radial Geodesics in a Closed Universe . . . . . 17 (Sol: 48)
- \*8. Evolution of an Open Universe . . . . . 18 (Sol: 52)
9. Tracing Light Rays in a Closed, Matter-Dominated Universe . . . 18 (Sol: 52)
10. Lengths and Areas in a Two-Dimensional Metric . . . . . 20 (Sol: 55)
11. Geometry in a Closed Universe . . . . . 20 (Sol: 56)
- \*12. The General Spherically Symmetric Metric . . . . . 21 (Sol: 57)
13. Volumes in a Robertson-Walker Universe . . . . . 22 (Sol: 59)
- \*14. The Schwarzschild Metric . . . . . 22 (Sol: 60)
15. Geodesics . . . . . 23 (Sol: 63)
16. An Exercise in Two-Dimensional Metrics . . . . . 23 (Sol: 65)
17. Geodesics in a Closed Universe . . . . . 24 (Sol: 68)
18. A Two-Dimensional Curved Space . . . . . 25 (Sol: 71)
- \*19. Rotating Frames of Reference . . . . . 27 (Sol: 74)
20. The Stability of Schwarzschild Orbits . . . . . 28 (Sol: 77)
21. Gravitational Bending of Light . . . . . 30 (Sol: 81)
- \*22. Pressure and Energy Density of Mysterious Stuff . . . . . 32 (Sol: 84)
23. The Effect of Pressure on Cosmological Evolution . . . . . 33 (Sol: 85)

**PROBLEM 1: DID YOU DO THE READING? (2018)** (25 points)

For clarity, part (b) has been slightly reworded from the original quiz.

- (a) (4 points) Which of the following statements about deuterium is **NOT** true? Choose one.
- (i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.
  - (ii) The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.
  - (iii) The Lyman- $\alpha$  transition in deuterium corresponds to a slightly different wavelength than the Lyman- $\alpha$  transition in hydrogen.
  - (iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or  $^3\text{He}$ .
- (b) (6 points) In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)
- (i) When the temperature of the universe was about  $10^{10}$  °K ( $t \sim 1$  sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.
  - (ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.
  - (iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first  $3\frac{3}{4}$  minutes, and this is why we call this period the era of nucleosynthesis.
  - (iv) After the first  $3\frac{3}{4}$  minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.
  - (v) The protons and neutrons became decoupled from the photons after the first  $3\frac{3}{4}$  minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.
  - (vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.



- (c) (4 points) The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the
- (i) 1920s            (ii) 1930s            (iii) 1940s            (iv) 1950s            (v) 1960s
- (d) (5 points) A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.
- (e) (6 points) In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

**PROBLEM 2: DID YOU DO THE READING? (2016) (25 points)**

- (a) (5 points) In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to  $\Omega$  from clusters of galaxies as
- (i) 0.01            (ii) 0.05            (iii) 0.20            (iv) 0.60            (v) 1.00
- (b) (4 points) One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring
- (i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.
- (ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.
- (iii) the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.
- (iv) the velocity dispersion along the line of sight and also the redshift of the cluster.
- (v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

- (c) (*4 points*) Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.
- (i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.
  - (ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.
  - (iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.
  - (iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.
  - (v) By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.
- (d) (*6 points*) In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a  $3^\circ$  K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)
- (i) The earliest calculations erroneously predicted a cosmic background temperature of only about  $0.1^\circ$  K, and such a background would be too weak to detect.
  - (ii) There was a breakdown in communication between theorists and experimentalists.
  - (iii) It was not technologically possible to detect a signal as weak as a  $3^\circ$  K microwave background until about 1965.
  - (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
  - (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.
  - (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

- (e) (6 points) In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about  $10^9$ . Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)
- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
  - (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
  - (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
  - (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
  - (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about  $10^3$ , not  $10^9$  as Alpher and Herman concluded.

**PROBLEM 3: DID YOU DO THE READING? (2013) (25 points)**

*The following problem comes from Quiz 2, 2013.*

- (a) (6 points) The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity  $v$  as a function of radius  $R$ . If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of  $v(R)$  at large radii?
- (b) (5 points) What is actually found for the behavior of  $v(R)$ ?
- (c) (7 points) An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?
- (d) (7 points) At the end of Chapter 10, Ryden writes “Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and antiquarks in the early universe.” Explain in one or a few sentences how a tiny asymmetry between quarks and antiquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

**PROBLEM 4: DID YOU DO THE READING? (2011)** (20 points)

The following problem comes from Quiz 2, 2011.

- (a) (8 points) During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.
- (i) In *The First Three Minutes*, Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium,  $\text{He}^4$ . Describe at least one of these two chains.
  - (ii) Explain briefly what is the *deuterium bottleneck*, and what is its role during nucleosynthesis.
- (b) (12 points) In Chapter 4 of *The First Three Minutes*, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

*The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures.*

In this part of the problem you will explore more quantitatively this statement.

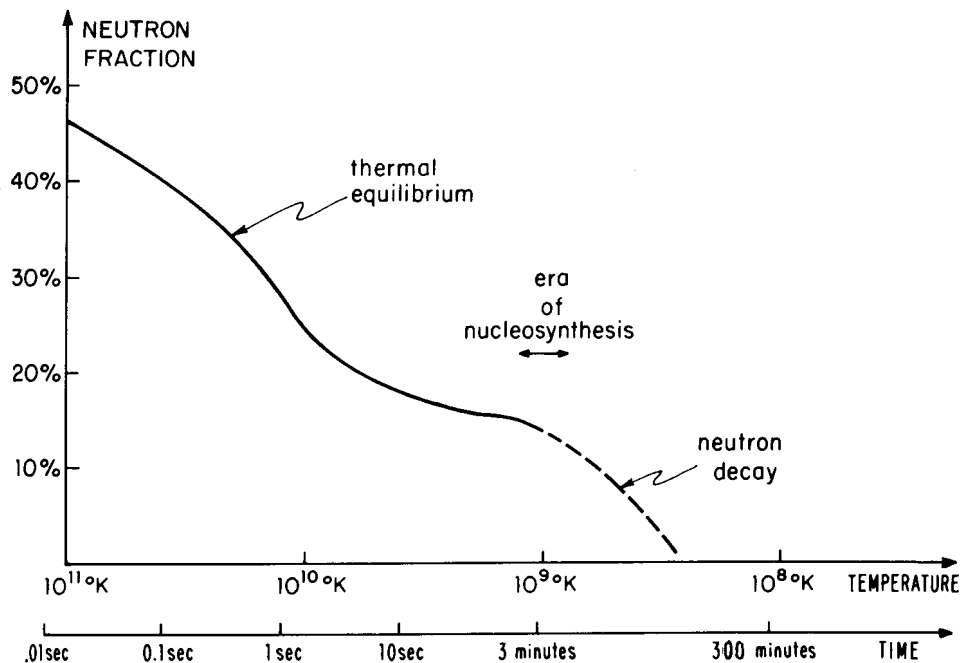
- (i) For a radiation-dominated universe the scale-factor  $a(t) \propto t^{1/2}$ . Find the cosmic time  $t$  as a function of the Hubble expansion rate  $H$ .
- (ii) The mass density stored in radiation  $\rho_r$  is proportional to the temperature  $T$  to the fourth power: i.e.,  $\rho_r \simeq \alpha T^4$ , for some constant  $\alpha$ . For a wide range of temperatures we can take  $\alpha \simeq 4.52 \times 10^{-32} \text{ kg} \cdot \text{m}^{-3} \cdot \text{K}^{-4}$ . If the temperature is measured in degrees Kelvin (K), then  $\rho_r$  has the standard SI units,  $[\rho_r] = \text{kg} \cdot \text{m}^{-3}$ . Use the Friedmann equation for a flat universe ( $k = 0$ ) with  $\rho = \rho_r$  to express the Hubble expansion rate  $H$  in terms of the temperature  $T$ . You will need the SI value of the gravitational constant  $G \simeq 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$ . What is the Hubble expansion rate, in inverse seconds, at the start of nucleosynthesis, when  $T = T_{\text{nucl}} \simeq 0.9 \times 10^9 \text{ K}$ ?
- (iii) Using the results in (i) and (ii), express the cosmic time  $t$  as a function of the temperature. Your result should agree with Weinberg's claim above. What is the cosmic time, in seconds, when  $T = T_{\text{nucl}}$ ?

**PROBLEM 5: DID YOU DO THE READING? (2007)** (24 points)

The following problem was Problem 1 of Quiz 2 in 2007.

- (a) (6 points) In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about  $10^9$ . Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)
- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
  - (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
  - (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
  - (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
  - (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about  $10^3$ , not  $10^9$  as Alpher and Herman concluded.
- (b) (6 points) In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:
- |                  |                     |  |                      |
|------------------|---------------------|--|----------------------|
| Electric Charge: | (i) $\sim 10^9$     | (ii) $\sim 1000$   | (iii) $\sim 1$       |
|                  | (iv) $\sim 10^{-6}$ | (v) either zero or negligible                                      |                      |
| Baryon Number:   | (i) $\sim 10^{-20}$ | (ii) $\sim 10^{-9}$  | (iii) $\sim 10^{-6}$ |
|                  | (iv) $\sim 1$       | (v) anywhere from $10^{-5}$ to 1                                   |                      |
| Lepton Number:   | (i) $\sim 10^9$     | (ii) $\sim 1000$   | (iii) $\sim 1$       |
|                  | (iv) $\sim 10^{-6}$ | (v) could be as high as $\sim 1$ , but is assumed to be very small |                      |

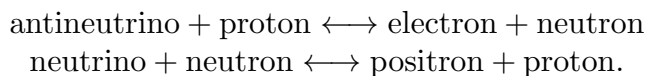
- (c) (12 points) The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance*.



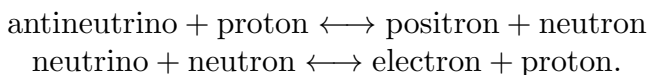
- (i) (3 points) During the period labeled “thermal equilibrium,” the neutron fraction is changing because (choose one):
- (A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.
- (B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.
- (C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.
- (D) Neutrons and protons can be converted from one into the other through reactions such as
- $$\begin{aligned} \text{antineutrino} + \text{proton} &\longleftrightarrow \text{electron} + \text{neutron} \\ \text{neutrino} + \text{neutron} &\longleftrightarrow \text{positron} + \text{proton}. \end{aligned}$$
- (E) Neutrons and protons can be converted from one into the other through reactions such as
- $$\begin{aligned} \text{antineutrino} + \text{proton} &\longleftrightarrow \text{positron} + \text{neutron} \\ \text{neutrino} + \text{neutron} &\longleftrightarrow \text{electron} + \text{proton}. \end{aligned}$$
- (F) Neutrons and protons can be created and destroyed by reactions such as
- $$\begin{aligned} \text{proton} + \text{neutrino} &\longleftrightarrow \text{positron} + \text{antineutrino} \\ \text{neutron} + \text{antineutrino} &\longleftrightarrow \text{electron} + \text{positron}. \end{aligned}$$

(ii) (3 points) During the period labeled “neutron decay,” the neutron fraction is changing because (choose one):

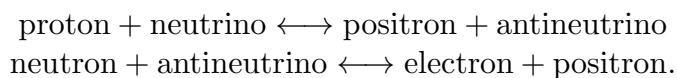
- (A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.
- (B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.
- (C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.
- (D) Neutrons and protons can be converted from one into the other through reactions such as



- (E) Neutrons and protons can be converted from one into the other through reactions such as



- (F) Neutrons and protons can be created and destroyed by reactions such as



(iii) (3 points) The masses of the neutron and proton are not exactly equal, but instead

- (A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV =  $10^9$  eV).
- (B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV =  $10^6$  eV).
- (C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV =  $10^3$  eV).
- (D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.
- (E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

- (F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.
- (iv) (3 points) During the period labeled “era of nucleosynthesis,” (choose one:)
- (A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.
- (B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.
- (C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
- (D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
- (E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.
- (F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

**PROBLEM 6: DID YOU DO THE READING? (2000, 2002)**

*Parts (a)-(c) of this problem come from Quiz 4, 2000, and parts (d) and (e) come from Quiz 3, 2002.*

- (a) (5 points) By what factor does the lepton number per comoving volume of the universe change between temperatures of  $kT = 10$  MeV and  $kT = 0.1$  MeV? You should assume the existence of the normal three species of neutrinos for your answer.
- (b) (5 points) Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do?
- (i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.
- (ii) The deuterium abundance in the Earth’s oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth’s surface.
- (iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.
- (iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.
- (v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.



- (c) (5 points) Give three examples of hadrons.
- (d) (6 points) In Chapter 6 of *The First Three Minutes*, Steven Weinberg posed the question, “Why was there no systematic search for this [cosmic background] radiation, years before 1965?” In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these two elementary particles. (If you name them both correctly, you will get 3 points extra credit. However, one right and one wrong will get you 4 points for the question, compared to 6 points for just naming one particle and getting it right.)

Answer: \_\_\_\_\_  
 2nd Answer (optional): \_\_\_\_\_

- (e) (6 points) In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)
- (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.
  - (ii) There was a breakdown in communication between theorists and experimentalists.
  - (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.
  - (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
  - (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.
  - (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

**\* PROBLEM 7: RADIAL GEODESICS IN A CLOSED UNIVERSE** (20 points)

The following problem was Problem 4, Quiz 2, 2018. (For clarity, part (c) has been reworded slightly.)

As shown in the formula sheets, we can describe a closed universe by choosing  $k = 1$ , and then using coordinates  $(t, r, \theta, \phi)$ , with metric

$$ds^2 \equiv -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \quad (7.1)$$

or by using coordinates  $(t, \psi, \theta, \phi)$ , with metric

$$ds^2 \equiv -c^2 d\tau^2 \equiv -c^2 dt^2 + a^2(t) \{d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)\} . \quad (7.2)$$

The connection between the two coordinate systems is given by

$$r = \sin \psi . \quad (7.3)$$

The general spacetime geodesic equation can be written as

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} . \quad (7.4)$$

- (a) (7 points) Using the coordinates  $(t, \psi, \theta, \phi)$  and the metric of Eq. (7.2), compute explicitly the geodesic equation for  $\mu = \psi$ . By “compute explicitly”, I mean that  $g_{\mu\nu}$  should be replaced by the relevant expressions from Eq. (7.2), and that the sums over indices should be written out, including only the nonzero terms.
- (b) (7 points) Using instead the coordinates  $(t, r, \theta, \phi)$ , compute explicitly the geodesic equation for  $\mu = r$ .
- (c) (6 points) Are the equations that you are asked to derive in parts (a) and (b) both valid, or is one valid and the other not? If you believe that they are both valid, use Eq. (7.3) to show that they are equivalent. If you believe that only one is valid, state which one is valid, and explain why the other is not. (4 points will be given for showing the correct understanding of this problem, with 2 points allocated to completing the algebra needed to demonstrate your answer.)

### \* PROBLEM 8: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time  $t$  at which  $a/\sqrt{\kappa} = 2\alpha$ .

### PROBLEM 9: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE (30 points)

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} ,$$

where I have taken  $k = 1$ . To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate  $\psi$ , related to  $r$  by

$$r = \sin \psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} .$$

- (a) (7 points) A light pulse travels on a null trajectory, which means that  $d\tau = 0$  for each segment of the trajectory. Consider a light pulse that moves along a radial line, so  $\theta = \phi = \text{constant}$ . Find an expression for  $d\psi/dt$  in terms of quantities that appear in the metric.
- (b) (8 points) Write an expression for the physical horizon distance  $\ell_{\text{phys}}$  at time  $t$ . You should leave your answer in the form of a definite integral.

The form of  $a(t)$  depends on the content of the universe. If the universe is matter-dominated (*i.e.*, dominated by nonrelativistic matter), then  $a(t)$  is described by the parametric equations

$$\begin{aligned} ct &= \alpha(\theta - \sin \theta) , \\ a &= \alpha(1 - \cos \theta) , \end{aligned}$$

where

$$\alpha \equiv \frac{4\pi G\rho a^3}{3c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen  $k = 1$ .

- (c) (10 points) Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for  $d\psi/d\theta$ , where  $\theta$  is the parameter used to describe the evolution.
- (d) (5 points) Suppose that a photon leaves the origin of the coordinate system ( $\psi = 0$ ) at  $t = 0$ . How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

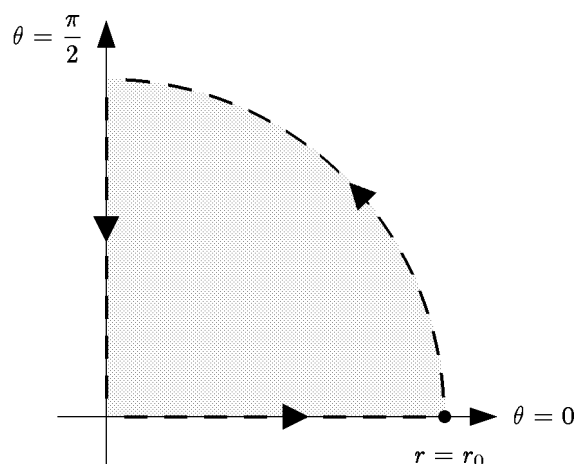
**PROBLEM 10: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC** (25 points)

The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates  $(r, \theta)$ , has a metric given by

$$ds^2 = (1 + ar)^2 dr^2 + r^2(1 + br)^2 d\theta^2 ,$$

where  $a$  and  $b$  are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the  $\theta = 0$  line to  $r = r_0$ , then moving at fixed  $r$  to  $\theta = \pi/2$ , and then moving back to the origin at fixed  $\theta$ . The path is shown below:



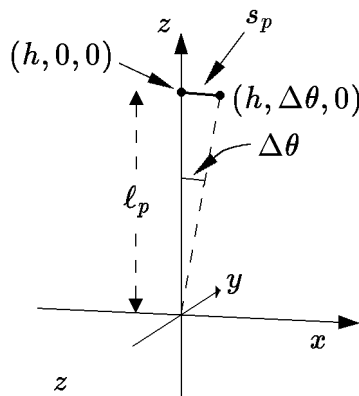
- (10 points) Find the total length of this path.
- (15 points) Find the area enclosed by this path.

**PROBLEM 11: GEOMETRY IN A CLOSED UNIVERSE** (25 points)

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with  $k = 1$ . The questions below all pertain to some fixed time  $t$ , so the scale factor can be written simply as  $a$ , dropping its explicit  $t$ -dependence.

A small rod has one end at the point  $(r = h, \theta = 0, \phi = 0)$  and the other end at the point  $(r = h, \theta = \Delta\theta, \phi = 0)$ . Assume that  $\Delta\theta \ll 1$ .



- (a) Find the physical distance  $\ell_p$  from the origin ( $r = 0$ ) to the first end  $(h, 0, 0)$  of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1-r^2} = \frac{1}{2} \ln \left( \frac{1+r}{1-r} \right) .$$

- (b) Find the physical length  $s_p$  of the rod. Express your answer in terms of the scale factor  $a$ , and the coordinates  $h$  and  $\Delta\theta$ .
- (c) Note that  $\Delta\theta$  is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance  $\ell_p$ , the physical length  $s_p$ , and the scale factor  $a$ .

**\* PROBLEM 12: THE GENERAL SPHERICALLY SYMMETRIC METRIC** (20 points)

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r) [d\theta^2 + \sin^2 \theta d\phi^2]$$

for some function  $\rho(r)$ . The coordinates  $\theta$  and  $\phi$  have their usual ranges:  $\theta$  varies between 0 and  $\pi$ , and  $\phi$  varies from 0 to  $2\pi$ , where  $\phi = 0$  and  $\phi = 2\pi$  are identified. Given this metric, consider the sphere whose outer boundary is defined by  $r = r_0$ .

- Find the physical radius  $a$  of the sphere. (By “radius”, I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)
- Find the physical area of the surface of the sphere.
- Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.
- Suppose a new radial coordinate  $\sigma$  is introduced, where  $\sigma$  is related to  $r$  by

$$\sigma = r^2 .$$

Express the metric in terms of this new variable.

**PROBLEM 13: VOLUMES IN A ROBERTSON-WALKER UNIVERSE** (20 points)

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

Calculate the volume  $V(r_{\max})$  of the sphere described by

$$r \leq r_{\max} .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

**\* PROBLEM 14: THE SCHWARZSCHILD METRIC** (25 points)

The follow problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass  $M$  is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated  $A$  and  $B$ , are located along the same radial line, with values of the coordinate  $r$  given by  $r_A$  and  $r_B$ , respectively, with  $r_A < r_B$ . You should assume that both observers lie outside the Schwarzschild horizon.

- (5 points) Write down the expression for the Schwarzschild horizon radius  $R_S$ , expressed in terms of  $M$  and fundamental constants.
- (5 points) What is the proper distance between  $A$  and  $B$ ? It is okay to leave the answer to this part in the form of an integral that you do not evaluate— but be sure to clearly indicate the limits of integration.

- c) (5 points) Observer  $A$  has a clock that emits an evenly spaced sequence of ticks, with proper time separation  $\Delta\tau_A$ . What will be the coordinate time separation  $\Delta t_A$  between these ticks?
- d) (5 points) At each tick of  $A$ 's clock, a light pulse is transmitted. Observer  $B$  receives these pulses, and measures the time separation on his own clock. What is the time interval  $\Delta\tau_B$  measured by  $B$ .
- e) (5 points) Suppose that the object creating the gravitational field is a static black hole, so the Schwarzschild metric is valid for all  $r$ . Now suppose that one considers the case in which observer  $A$  lies on the Schwarzschild horizon, so  $r_A \equiv R_S$ . Is the proper distance between  $A$  and  $B$  finite for this case? Does the time interval of the pulses received by  $B$ ,  $\Delta\tau_B$ , diverge in this case?

**PROBLEM 15: GEODESICS** (20 points)

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2 d\theta^2 .$$

- (a) Suppose that  $r(\lambda)$  and  $\theta(\lambda)$  describe a geodesic in this space, where the parameter  $\lambda$  is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which  $r(\lambda)$  and  $\theta(\lambda)$  must obey.
- (b) Now introduce the usual Cartesian coordinates, defined by

$$\begin{aligned} x &= r \cos \theta , \\ y &= r \sin \theta . \end{aligned}$$

Use your answer to (a) to show that the line  $y = 1$  is a geodesic curve.

**PROBLEM 16: AN EXERCISE IN TWO-DIMENSIONAL METRICS** (30 points)

- (a) (8 points) Consider first a two-dimensional space with coordinates  $r$  and  $\theta$ . The metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2 .$$

Consider the curve described by

$$r(\theta) = (1 + \epsilon \cos^2 \theta) r_0 ,$$

where  $\epsilon$  and  $r_0$  are constants, and  $\theta$  runs from  $\theta_1$  to  $\theta_2$ . Write an expression, in the form of a definite integral, for the length  $S$  of this curve.

- (b) (5 points) Now consider a two-dimensional space with the same two coordinates  $r$  and  $\theta$ , but this time the metric will be

$$ds^2 = \left(1 + \frac{r}{a}\right) dr^2 + r^2 d\theta^2 ,$$

where  $a$  is a constant.  $\theta$  is a periodic (angular) variable, with a range of 0 to  $2\pi$ , with  $2\pi$  identified with 0. What is the length  $R$  of the path from the origin ( $r = 0$ ) to the point  $r = r_0, \theta = 0$ , along the path for which  $\theta = 0$  everywhere along the path? You can leave your answer in the form of a definite integral. (Be sure, however, to specify the limits of integration.)

- (c) (7 points) For the space described in part (b), what is the total area contained within the region  $r < r_0$ . Again you can leave your answer in the form of a definite integral, making sure to specify the limits of integration.
- (d) (10 points) Again for the space described in part (b), consider a geodesic described by the usual geodesic equation,

$$\frac{d}{ds} \left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} (\partial_i g_{k\ell}) \frac{dx^k}{ds} \frac{dx^\ell}{ds} .$$

The geodesic is described by functions  $r(s)$  and  $\theta(s)$ , where  $s$  is the arc length along the curve. Write explicitly both (i.e., for  $i=1=r$  and  $i=2=\theta$ ) geodesic equations.

### PROBLEM 17: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking  $k = 1$ , the spacetime metric can be written in the form

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

We will assume that this metric is given, and that  $a(t)$  has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction ( $r$ -direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

- (a) (7 points) Express  $d\tau/dt$  in terms of  $dr/dt$ .
- (b) (3 points) Express  $dt/d\tau$  in terms of  $dr/dt$ .



- (c) (10 points) If the object travels on a trajectory given by the function  $r_p(t)$  between some time  $t_1$  and some later time  $t_2$ , write an integral which gives the total amount of time that a clock attached to the object would record for this journey.
- (d) (10 points) During a time interval  $dt$ , the object will move a coordinate distance

$$dr = \frac{dr}{dt} dt .$$

Let  $d\ell$  denote the physical distance that the object moves during this time. By “physical distance,” I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity  $d\ell/dt$  can be regarded as the physical speed  $v_{\text{phys}}$  of the object, since it is the speed that would be measured by a comoving observer. Write an expression for  $v_{\text{phys}}$  as a function of  $dr/dt$  and  $r$ .

- (e) (10 points) Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate  $r$  of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau} \left[ A \frac{dr}{d\tau} \right] = B \left( \frac{dt}{d\tau} \right)^2 + C \left( \frac{dr}{d\tau} \right)^2 + D \left( \frac{d\theta}{d\tau} \right)^2 + E \left( \frac{d\phi}{d\tau} \right)^2 ,$$

where  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  are functions of the coordinates, some of which might be zero.

- (f) (5 points EXTRA CREDIT) On Problem 1 of Problem Set 6 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

$$p = \frac{mv_{\text{phys}}}{\sqrt{1 - \frac{v_{\text{phys}}^2}{c^2}}} ,$$

falls off as  $1/a(t)$ . Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

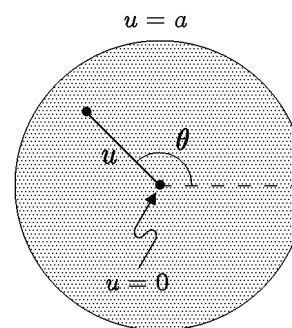
### PROBLEM 18: A TWO-DIMENSIONAL CURVED SPACE (40 points)

The following problem was Problem 3, Quiz 2, 2002.

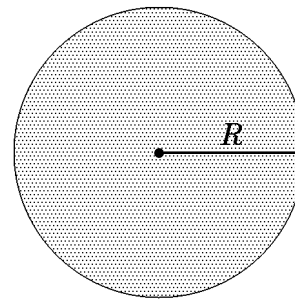
Consider a two-dimensional curved space described by polar coordinates  $u$  and  $\theta$ , where  $0 \leq u \leq a$  and  $0 \leq \theta \leq 2\pi$ , and  $\theta = 2\pi$  is as usual identified with  $\theta = 0$ . The metric is given by

$$ds^2 = \frac{a du^2}{4u(a-u)} + u d\theta^2 .$$

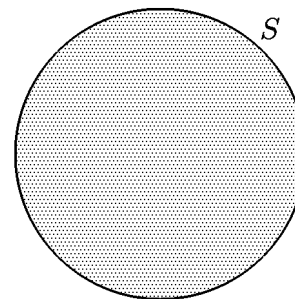
A diagram of the space is shown at the right, but you should of course keep in mind that the diagram does not accurately reflect the distances defined by the metric.



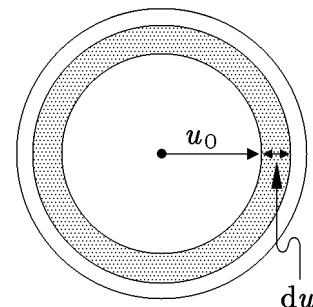
- (a) (6 points) Find the radius  $R$  of the space, defined as the length of a radial (i.e.,  $\theta = \text{constant}$ ) line. You may express your answer as a definite integral, which you need not evaluate. Be sure, however, to specify the limits of integration.



- (b) (6 points) Find the circumference  $S$  of the space, defined as the length of the boundary of the space at  $u = a$ .



- (c) (7 points) Consider an annular region as shown, consisting of all points with a  $u$ -coordinate in the range  $u_0 \leq u \leq u_0 + du$ . Find the physical area  $dA$  of this region, to first order in  $du$ .



- (d) (3 points) Using your answer to part (c), write an expression for the total area of the space.
- (e) (10 points) Consider a geodesic curve in this space, described by the functions  $u(s)$  and  $\theta(s)$ , where the parameter  $s$  is chosen to be the arc length along the curve. Find the geodesic equation for  $u(s)$ , which should have the form

$$\frac{d}{ds} \left[ F(u, \theta) \frac{du}{ds} \right] = \dots ,$$

where  $F(u, \theta)$  is a function that you will find. (Note that by writing  $F$  as a function of  $u$  and  $\theta$ , we are saying that it *could* depend on either or both of them, but we are not saying that it *necessarily* depends on them.) You need not simplify the left-hand side of the equation.

- (f) (8 points) Similarly, find the geodesic equation for  $\theta(s)$ , which should have the form

$$\frac{d}{ds} \left[ G(u, \theta) \frac{d\theta}{ds} \right] = \dots ,$$

where  $G(u, \theta)$  is a function that you will find. Again, you need not simplify the left-hand side of the equation.

**\* PROBLEM 19: ROTATING FRAMES OF REFERENCE** (35 points)

The following problem was Problem 3, Quiz 2, 2004.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + \left[ dr^2 + r^2 (d\phi + \omega dt)^2 + dz^2 \right] , \quad (\text{P19.1})$$

which corresponds to a coordinate system rotating about the  $z$ -axis, where  $\phi$  is the azimuthal angle around the  $z$ -axis. The coordinates have the usual range for cylindrical coordinates:  $-\infty < t < \infty$ ,  $0 \leq r < \infty$ ,  $-\infty < z < \infty$ , and  $0 \leq \phi < 2\pi$ , where  $\phi = 2\pi$  is identified with  $\phi = 0$ .

*EXTRA INFORMATION*

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (P19.1) was obtained by starting with a Minkowski metric in cylindrical coordinates  $\bar{t}$ ,  $\bar{r}$ ,  $\bar{\phi}$ , and  $\bar{z}$ ,

$$c^2 d\tau^2 = c^2 d\bar{t}^2 - [d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2 + d\bar{z}^2] ,$$

and then introducing new coordinates  $t$ ,  $r$ ,  $\phi$ , and  $z$  that are related by

$$\bar{t} = t, \quad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z ,$$

so  $d\bar{t} = dt$ ,  $d\bar{r} = dr$ ,  $d\bar{\phi} = d\phi + \omega dt$ , and  $d\bar{z} = dz$ .

- (a) (8 points) The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2 d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu ,$$

where  $x^0 \equiv t$ ,  $x^1 \equiv r$ ,  $x^2 \equiv \phi$ , and  $x^3 \equiv z$ . Then, for example,  $g_{11}$  (which can also be called  $g_{rr}$ ) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix  $g_{\mu\nu}$ :

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \quad (\text{P19.2})$$

If you cannot answer part (a), you can introduce unspecified functions  $f_1(r)$ ,  $f_2(r)$ ,  $f_3(r)$ , and  $f_4(r)$ , with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) , \end{aligned} \tag{P19.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) (10 points) Using the geodesic equations from the front of the quiz,

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} ,$$

explicitly write the equation that results when the free index  $\mu$  is equal to 1, corresponding to the coordinate  $r$ .

- (c) (7 points) Explicitly write the equation that results when the free index  $\mu$  is equal to 2, corresponding to the coordinate  $\phi$ .
- (d) (10 points) Use the metric to find an expression for  $dt/d\tau$  in terms of  $dr/dt$ ,  $d\phi/dt$ , and  $dz/dt$ . The expression may also depend on the constants  $c$  and  $\omega$ . Be sure to note that your answer should depend on the derivatives of  $t$ ,  $\phi$ , and  $z$  with respect to  $t$ , not  $\tau$ . (*Hint: first find an expression for  $d\tau/dt$ , in terms of the quantities indicated, and then ask yourself how this result can be used to find  $dt/d\tau$ .*)

**PROBLEM 20: THE STABILITY OF SCHWARZSCHILD ORBITS** (30 points)

*This problem was Problem 4, Quiz 2 in 2007. I have modified the reference to the homework problem to correspond to the current (2016) context, where it is Problem 3 of Problem Set 6. In 2007 it had also been a homework problem prior to the quiz.*

This problem is an elaboration of the Problem 3 of Problem Set 6, for which both the statement and the solution are reproduced at the end of this quiz. This material is reproduced for your reference, but you should be aware that the solution to the present problem has important differences. You can copy from this material, but to allow the grader to assess your understanding, you are expected to present a logical, self-contained answer to this question.

In the solution to that homework problem, it was stated that further analysis of the orbits in a Schwarzschild geometry shows that the smallest *stable* circular orbit occurs

for  $r = 3R_S$ . Circular orbits are possible for  $\frac{3}{2}R_S < r < 3R_S$ , but they are not stable. In this problem we will explore the calculations behind this statement.

We will consider a body which undergoes small oscillations about a circular orbit at  $r(t) = r_0$ ,  $\theta = \pi/2$ , where  $r_0$  is a constant. The coordinate  $\theta$  will therefore be fixed, but all the other coordinates will vary as the body follows its orbit.

- (a) (12 points) The first step, since  $r(\tau)$  will not be a constant in this solution, will be to derive the equation of motion for  $r(\tau)$ . That is, for the Schwarzschild metric

$$ds^2 = -c^2 d\tau^2 = -h(r)c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (\text{P20.1})$$

where

$$h(r) \equiv 1 - \frac{R_S}{r},$$

work out the explicit form of the geodesic equation

$$\frac{d}{d\tau} \left[ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}, \quad (\text{P20.2})$$

for the case  $\mu = r$ . You should use this result to find an explicit expression for

$$\frac{d^2 r}{d\tau^2}.$$

You may allow your answer to contain  $h(r)$ , its derivative  $h'(r)$  with respect to  $r$ , and the derivative with respect to  $\tau$  of any coordinate, including  $dt/d\tau$ .

- (b) (6 points) It is useful to consider  $r$  and  $\phi$  to be the independent variables, while treating  $t$  as a dependent variable. Find an expression for

$$\left( \frac{dt}{d\tau} \right)^2$$

in terms of  $r$ ,  $dr/d\tau$ ,  $d\phi/d\tau$ ,  $h(r)$ , and  $c$ . Use this equation to simplify the expression for  $d^2 r/d\tau^2$  obtained in part (a). The goal is to obtain an expression of the form

$$\frac{d^2 r}{d\tau^2} = f_0(r) + f_1(r) \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{P20.3})$$

where the functions  $f_0(r)$  and  $f_1(r)$  might depend on  $R_S$  or  $c$ , and might be positive, negative, or zero. Note that the intermediate steps in the calculation involve a term proportional to  $(dr/d\tau)^2$ , but the net coefficient for this term vanishes.

- (c) (7 points) To understand the orbit we will also need the equation of motion for  $\phi$ . Evaluate the geodesic equation (P20.2) for  $\mu = \phi$ , and write the result in terms of the quantity  $L$ , defined by

$$L \equiv r^2 \frac{d\phi}{d\tau} . \quad (\text{P20.4})$$

- (d) (5 points) Finally, we come to the question of stability. Substituting Eq. (P20.4) into Eq. (P20.3), the equation of motion for  $r$  can be written as

$$\frac{d^2 r}{d\tau^2} = f_0(r) + f_1(r) \frac{L^2}{r^4} .$$

Now consider a small perturbation about the circular orbit at  $r = r_0$ , and write an equation that determines the stability of the orbit. (That is, if some external force gives the orbiting body a small kick in the radial direction, how can you determine whether the perturbation will lead to stable oscillations, or whether it will start to grow?) You should express the stability requirement in terms of the unspecified functions  $f_0(r)$  and  $f_1(r)$ . You are NOT asked to carry out the algebra of inserting the explicit forms that you have found for these functions.

### PROBLEM 21: GRAVITATIONAL BENDING OF LIGHT (30 points)

When a light ray passes by a massive object, general relativity predicts that it will be bent. Since most celestial objects are nearly spherical, we can use the Schwarzschild metric to calculate the bending. Furthermore, since we are usually interested in objects that are not black holes or anywhere nearly as dense, we can obtain an accurate answer by carrying out the calculation in a weak-field approximation. For a photon that grazes the Sun, for example, the value of  $R_{\text{Sch}}/R_{\odot}$ , the Schwarzschild radius over the radius of the Sun, is about  $4 \times 10^{-6}$ .

Starting with the Schwarzschild metric,

$$ds^2 = - \left( 1 - \frac{R_{\text{Sch}}}{r} \right) c^2 dt^2 + \left( 1 - \frac{R_{\text{Sch}}}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (\text{P21.1})$$

where  $R_{\text{Sch}} = 2GM/c^2$ , we can expand in powers of  $R_{\text{Sch}}/r$  and keep only the first order terms:

$$ds^2 = - \left( 1 - \frac{R_{\text{Sch}}}{r} \right) c^2 dt^2 + \left( 1 + \frac{R_{\text{Sch}}}{r} \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (\text{P21.2})$$

For this problem it is useful to switch to Cartesian-like coordinates, defined in terms of  $r$ ,  $\theta$ , and  $\phi$  by the usual Cartesian formulas,

$$\begin{aligned} x &= r \sin \theta \cos \phi , \\ y &= r \sin \theta \sin \phi , \\ z &= r \cos \theta . \end{aligned} \quad (\text{P21.3})$$

General relativity allows us to make any coordinate redefinitions that we might want, as long as we calculate the metric in terms of the new coordinates. It is useful to continue to use the quantity  $r$ , but now it will be thought of as a function of the coordinates  $x$ ,  $y$ , and  $z$ :

$$r = (x^2 + y^2 + z^2)^{1/2} . \quad (\text{P21.4})$$

The metric can then be rewritten as the Minkowski metric of special relativity, plus small corrections:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 + \frac{R_{\text{Sch}}}{r} c^2 dt^2 + \frac{R_{\text{Sch}}}{r} (dr)^2 , \quad (\text{P21.5})$$

where from Eq. (4) one can see that

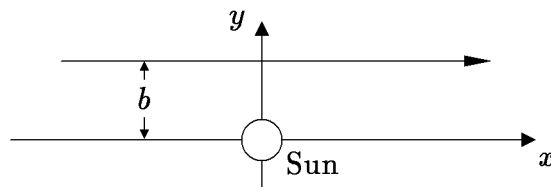
$$dr = \frac{1}{r} (x dx + y dy + z dz) . \quad (\text{P21.6})$$

- (a) (6 points) For the metric as approximated by Eqs. (P21.5) and (P21.6), write the expressions for  $g_{tt}$ ,  $g_{xx}$ , and  $g_{xy}$ .

The trajectory of the photon is lightlike, so we cannot use  $\tau$  to parameterize the trajectory, because proper time intervals along a lightlike trajectory are zero. Nonetheless, it can be shown that one can use an “affine parameter”  $\lambda$ , for which the geodesic equation has the usual form:

$$\frac{d}{d\lambda} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\lambda} \right\} = \frac{1}{2} [\partial_\mu g_{\sigma\tau}] \frac{dx^\sigma}{d\lambda} \frac{dx^\tau}{d\lambda} . \quad (\text{P21.7})$$

To obtain an answer that is accurate to first order in  $G$ , we begin by considering the *unperturbed* photon trajectory — the trajectory it would have if  $G$  were taken as zero, so  $R_{\text{Sch}} = 2GM/c^2 = 0$ . This would be a straight line in the  $(x, y, z)$  coordinates, as shown in the diagram below:



Here  $b$  is called the *impact parameter*. We can parameterize this path by

$$x(\lambda) = \lambda , \quad y(\lambda) = b , \quad z(\lambda) = 0 , \quad t(\lambda) = \lambda/c . \quad (\text{P21.8})$$

We will calculate the deflection (to first order in  $G$ ) by assuming that the photon path is accurately described by Eq. (P21.8), and we will calculate the  $y$ -velocity that the photon acquires due to the gravitational attraction of the Sun.

- (b) (9 points) With the goal of calculating  $d^2y/d\lambda^2$ , we evaluate the geodesic equation for  $\mu = y$ . Start here by evaluating the left-hand side of Eq. (P21.7) for  $\mu = y$ , to first order in  $G$ . Expand the derivative with respect to  $\lambda$  using the product rule, working out explicitly the derivatives of the relevant  $g_{\mu\nu}$  with respect to  $\lambda$ . In parts (b) and (c), you may assume that  $x(\lambda)$ ,  $y(\lambda)$ ,  $z(\lambda)$ , and  $t(\lambda)$ , as well as  $dx/d\lambda$ ,  $dy/d\lambda$ ,  $dz/d\lambda$ , and  $dt/d\lambda$ , are all given to sufficient accuracy by Eq. (P21.8) and its derivatives with respect to  $\lambda$ . (Be careful: it is likely that there are more terms than you will at first notice.)
- (c) (9 points) Evaluate the right-hand side of Eq. (P21.7) for  $\mu = y$ , to first order in  $G$ . Carry out all derivatives explicitly. (It always pays to be careful.)
- (d) (2 points) Use your answers to parts (c) and (d) to find an equation for  $d^2y/d\lambda^2$ .
- (e) (4 points) If the photon starts out on the unperturbed trajectory, its initial value of  $dy/d\lambda$  will be zero. The final value of  $dy/d\lambda$  will then be

$$\left. \frac{dy}{d\lambda} \right|_{\text{final}} = \int_{-\infty}^{\infty} \frac{d^2y}{d\lambda^2} d\lambda. \quad (\text{P21.9})$$

Use this fact to express the deflection angle  $\alpha$ , to first order in  $G$ , as an explicit integral. You need not carry out the integral, but you may wish to use the table of integrals given below to carry it out so that you can check your answer. The correct final answer is

$$\alpha = \frac{4GM}{c^2 b}. \quad (\text{P21.10})$$

#### TABLE OF INTEGRALS:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + b^2)} dx = \frac{\pi}{b} \quad \int_{-\infty}^{\infty} \frac{1}{(x^2 + b^2)^{3/2}} dx = \frac{2}{b^2} \quad \int_{-\infty}^{\infty} \frac{1}{(x^2 + b^2)^2} dx = \frac{\pi}{2b^3}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + b^2)^2} dx = \frac{\pi}{2b} \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + b^2)^{5/2}} dx = \frac{2}{3b^2} \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + b^2)^3} dx = \frac{\pi}{8b^3}$$

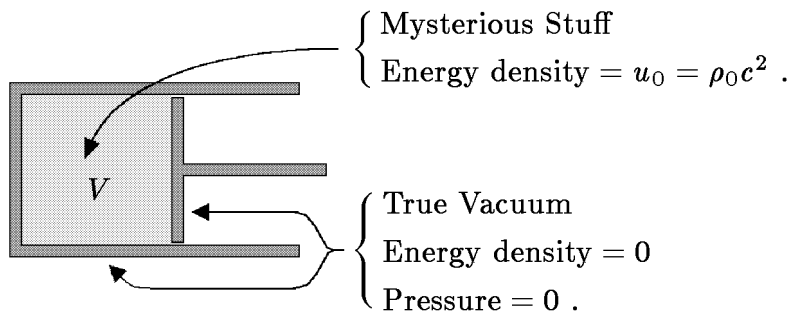
#### \* PROBLEM 22: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF (25 points)

The following problem was Problem 3, Quiz 3, 2002.

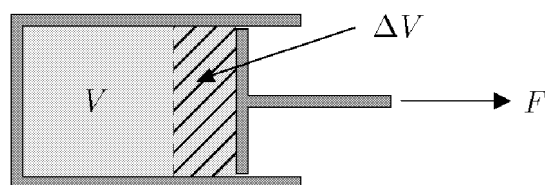
In Lecture Notes 6, with further calculations in Problem 4 of Problem Set 6, a thought experiment involving a piston was used to show that  $p = \frac{1}{3}\rho c^2$  for radiation. In this problem you will apply the same technique to calculate the pressure of **mysterious stuff**, which has the property that the energy density falls off in proportion to  $1/\sqrt{V}$  as the volume  $V$  is increased.



If the initial energy density of the mysterious stuff is  $u_0 = \rho_0 c^2$ , then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume  $V$  is increased to  $V + \Delta V$ . You may consider  $\Delta V$  to be infinitesimal, so  $\Delta V^2$  can be neglected.



- (15 points) Using the fact that the energy density of mysterious stuff falls off as  $1/\sqrt{V}$ , find the amount  $\Delta U$  by which the energy inside the piston changes when the volume is enlarged by  $\Delta V$ . Define  $\Delta U$  to be positive if the energy increases.
- (5 points) If the (unknown) pressure of the mysterious stuff is called  $p$ , how much work  $\Delta W$  is done by the agent that pulls out the piston?
- (5 points) Use your results from (a) and (b) to express the pressure  $p$  of the mysterious stuff in terms of its energy density  $u$ . (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

### PROBLEM 23: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION (25 points)

The following problem was Problem 2 of Quiz 3, 2016. It was also Problem 2 of Problem Set 7 (2016), except that some numerical constants have been changed, so the answers will not be identical.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

- (a) (8 points) For the first fictitious form of matter, the mass density  $\rho$  decreases as the scale factor  $a(t)$  grows, with the relation

$$\rho(t) \propto \frac{1}{a^8(t)} .$$

What is the pressure of this form of matter? [*Hint: the answer is proportional to the mass density.*]

- (b) (9 points) Find the behavior of the scale factor  $a(t)$  for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function  $a(t)$  up to a constant factor.
- (c) (8 points) Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{2}{3}\rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} .$$

Find the power  $n$ .

## SOLUTIONS

### PROBLEM 1: DID YOU DO THE READING? (2018) (25 points)

- (a) (4 points) Which of the following statements about deuterium is **NOT** true? Choose one.
- (i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.
  - (ii) The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.
  - (iii) The Lyman- $\alpha$  transition in deuterium corresponds to a slightly different wavelength than the Lyman- $\alpha$  transition in hydrogen.
  - (iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or  $^3\text{He}$ .

*[Comment: The most promising way to find the primordial value of the deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium in **intergalactic gas clouds** that lie between the quasars and us.]*

- (b) (6 points) In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

- (i) When the temperature of the universe was about  $10^{10}$  °K ( $t \sim 1$  sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.
- (ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.
- (iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first  $3\frac{3}{4}$  minutes, and this is why we call this period the era of nucleosynthesis.
- (iv) After the first  $3\frac{3}{4}$  minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.

- (v) The protons and neutrons became decoupled from the photons after the first  $3\frac{3}{4}$  minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.
- (vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

*[Comment: The statement (i) is described in Chapter 5 of Weinberg's book, and it has also been discussed in class. The correctness of statement (iv) can also be seen from Weinberg's Chapter 5, which says that the fraction of neutrons after the first three minutes is around 14%, and it goes down to around 13% "a little later", which he explains is probably about  $3\frac{3}{4}$  minutes, depending on the ratio of photons to nuclear particles. Weinberg also explains that most of the neutrons present at this time immediately combined with protons to form helium, which causes the ratio to be nearly constant until today. If you did not remember Weinberg's numbers, the statement that the fraction by weight of primordial helium is 26% should allow you to determine the fraction of neutrons, provided that you remember that the helium nucleus consists of 2 protons and two neutrons, that the mass of the proton and neutron are about equal, and that the remaining 74% of the matter is essentially hydrogen, with no neutrons. Thus helium is very nearly half protons and half neutrons by weight, so the neutrons must be about 13% of the matter in the universe. (Note that we are talking about fractions of the total "baryonic" matter, which does not include the dark matter.) (ii) is clearly false, because the temperature of neutrinos becomes lower than that of photons. (iii) is clearly false, because most atoms heavier than helium were made much later in the history of the universe, in the interiors of stars. (v) is false because protons did not decouple from photons until about 350,000 years, and it happened because the plasma of protons and electrons combined to form neutral hydrogen. (vi) is false because most of the helium in the universe today is primordial. Ryden points out, for example, that the abundance of helium in the Sun's atmosphere is only about 28%. Weinberg states, near the end of Chapter 5, that "the 20-30 percent helium abundance could not have been created recently without liberating enormous amounts of radiation that we do not observe."]*

- (c) (4 points) The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s

(ii) 1930s

(iii) 1940s

(iv) 1950s

(v) 1960s

- (d) (5 points) A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

*Answer:*

The neutron decays through the reaction



The quantum numbers of these particles can be described by the following table:

Particle	Charge	Baryon Number	Lepton Number
Neutron ( $n$ )	0	1	0
Proton ( $p$ )	$+e$	1	0
Electron ( $e^-$ )	$-e$	0	1
Anti-electron-neutrino ( $\bar{\nu}_e$ )	0	0	-1

Thus, the total charge of the final state is zero, the total baryon number is 1, and the total lepton number is zero, in all cases matching the initial value of these quantities.

- (e) (6 points) In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

*Answer:* You should have given two of the following three items.

- 1) *Virial theorem:* The virial theorem relates the total kinetic energy of a steady-state cluster to its gravitational potential energy. Since the kinetic energy is proportional to the mass  $M$  of the cluster, while the potential energy is proportional to  $M^2$ , the relation will hold for only one value of  $M$ . By measuring the velocity dispersion (root mean square of the radial galaxy velocities relative to the mean radial velocity) and the size of the cluster, the mass can be inferred.
- 2) *Hot, x-ray emitting gases:* By measuring the x-rays emitted by the cluster, it is possible to model the density, temperature, and composition of the hot gas within the cluster (intracluster gas). By assuming that the gas is in hydrostatic equilibrium — i.e., by assuming that the pressure gradients balance the gravitational forces — one can infer the gravitational field, and hence the total mass.
- 3) *Gravitational lensing:* If one can find a galaxy behind the cluster, so that the image of the galaxy is gravitationally lensed, then the mass of the galaxy can be inferred by the degree to which the galaxy is lensed.

**PROBLEM 2: DID YOU DO THE READING? (2016)** (25 points)

- (a) (5 points) In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to  $\Omega$  from clusters of galaxies as
- (i) 0.01            (ii) 0.05             (iii) 0.20            (iv) 0.60            (v) 1.00
- (b) (4 points) One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring
- (i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.
- (ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.
- (iii) the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.
- (iv) the velocity dispersion along the line of sight and also the redshift of the cluster.
- (v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

*Explanation:* The virial theorem relates the kinetic energy to the potential energy. The key relationship is

$$\frac{1}{2}M\langle v^2 \rangle = \frac{\alpha}{2} \frac{GM^2}{r_h},$$

where  $M$  is the mass of the cluster,  $\langle v^2 \rangle$  is the average squared velocity of its galaxies, and  $r_h$  is the radius containing half the total mass, which is estimated by the radius containing half the luminosity.  $\alpha$  is a numerical factor depending on the structure of the cluster, estimated at 0.4 based on observed clusters. Velocities along the line of sight are measured by the spread in Doppler shifts, while velocities perpendicular to the line of sight are essentially impossible to measure, eliminating answers (ii) and (v). Since  $r_h$  is needed, neither (i) nor (iv) include enough information. (iii) is exactly right.

- (c) (4 points) Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.
- (i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.
- (ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.

- (iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.
- (iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.
- (v) By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.

*Explanation:* The dominant component of the mass is apparently dark matter, so the hot intracluster gas is only a small fraction, and we have no direct way of knowing what fraction. But the gas settles into a state of hydrostatic equilibrium which is determined by pressures and gravitational forces. The gas can be mapped by measuring its x-rays, which allows astronomers to estimate the gravitational forces, and hence the mass.

- (d) (6 points) In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)
  - (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.
  - (ii) There was a breakdown in communication between theorists and experimentalists.
  - (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.
  - (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
  - (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.
  - (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

*Answer:* The correct answers were  (ii),  (v) and  (vi). The others were incorrect for the following reasons:

- (i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

- (iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed “long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s.” To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.
- (iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: “It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe”.
- (e) (6 points) In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about  $10^9$ . Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)
- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
- (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
- (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
- (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
- (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about  $10^3$ , not  $10^9$  as Alpher and Herman concluded.



**PROBLEM 3: DID YOU DO THE READING? (2013)** (25 points)

- (a) (6 points) The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity  $v$  as a function of radius  $R$ . If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of  $v(R)$  at large radii?

*Answer:* If stars contributed most of the mass, then at large radii the mass would appear to be concentrated as a spherical lump at the center, and the orbits of the stars would be “Keplerian,” i.e., orbits in a  $1/r^2$  gravitational field. Then  $\vec{F} = m\vec{a}$  implies that

$$\frac{1}{R^2} \propto \frac{v^2}{R} \implies v \propto \frac{1}{\sqrt{R}} .$$

- (b) (5 points) What is actually found for the behavior of  $v(R)$ ?

*Answer:*  $v(R)$  looks nearly flat at large radii.

- (c) (7 points) An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

*Answer:* For a gravitationally bound system in equilibrium,

$$\text{Kinetic energy} = -\frac{1}{2} (\text{Gravitational potential energy}) .$$

(The equality holds whenever  $\ddot{I} \approx 0$ , where  $I$  is the moment of inertia.)

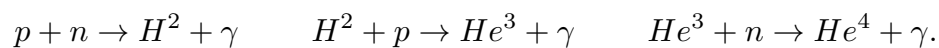
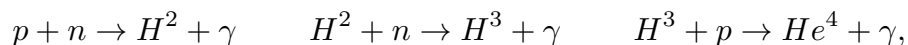
- (d) (7 points) At the end of Chapter 10, Ryden writes “Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and antiquarks in the early universe.” Explain in one or a few sentences how a tiny asymmetry between quarks and antiquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

*Answer:* When  $kT$  was large compared to 150 MeV, the excess of quarks over antiquarks was tiny: only about 3 extra quarks for every  $10^9$  antiquarks. But there was massive quark-antiquark annihilation as  $kT$  fell below 150 MeV, so that today we see the excess quarks, bound into baryons, and almost no sign of antiquarks.

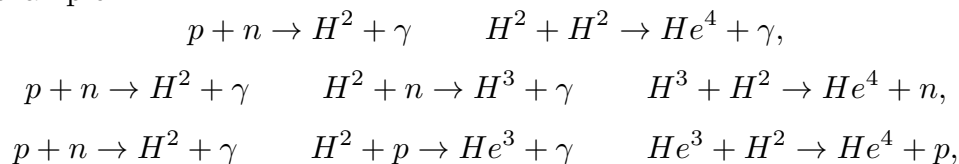
**PROBLEM 4: DID YOU DO THE READING? (2011)** (20 points)<sup>†</sup>

(a) (8 points)

- (i) (4 points) We will use the notation  $X^A$  to indicate a nucleus,\* where  $X$  is the symbol for the element which indicates the number of protons, while  $A$  is the mass number, namely the total number of protons and neutrons. With this notation  $H^1$ ,  $H^2$ ,  $H^3$ ,  $He^3$  and  $He^4$  stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:



These are the two examples given by Weinberg. However, different chains of two particle reactions can take place (in general with different probabilities). For example:



...

Students who described chains different from those of Weinberg, but that can still take place, got full credit for this part. Also, notice that photons in the reactions above carry the additional energy released. However, since the main point was to describe the nuclear reactions, students who didn't include the photons still received full credit.

- (ii) (4 points) The *deuterium bottleneck* is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium ( $H^2$ ) plays a crucial role in nucleosynthesis, since it is the starting point for all the chains. However, the deuterium nucleus is extremely loosely bound compared to  $H^3$ ,  $He^3$ , or especially  $He^4$ . So, there will be a

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\* Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.

range of temperatures which are low enough for  $H^3$ ,  $He^3$ , and  $He^4$  nuclei to be bound, but too high to allow the deuterium nucleus to be stable. This is the temperature range where the *deuterium bottleneck* is in action: even if  $H^3$ ,  $He^3$ , and  $He^4$  nuclei could in principle be stable at those temperatures, they do not form because deuterium, which is the starting point for their formation, cannot be formed yet. Nucleosynthesis cannot proceed at a significant rate until the temperature is low enough so that deuterium nuclei are stable; at this point the deuterium bottleneck has been passed.

(b) (12 points)

(i) (3 points) If we take  $a(t) = bt^{1/2}$ , for some constant  $b$ , we get for the Hubble expansion rate:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \implies \boxed{t = \frac{1}{2H}}.$$

(ii) (6 points) By using the Friedmann equation with  $k = 0$  and  $\rho = \rho_r = \alpha T^4$ , we find:

$$H^2 = \frac{8\pi}{3}G\rho_r = \frac{8\pi}{3}G\alpha T^4 \implies \boxed{H = T^2 \sqrt{\frac{8\pi}{3}G\alpha}}.$$

If we substitute the given numerical values  $G \simeq 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$  and  $\alpha \simeq 4.52 \times 10^{-32} \text{ kg} \cdot \text{m}^{-3} \cdot \text{K}^{-4}$  we get:

$$\boxed{H \simeq T^2 \times 5.03 \times 10^{-21} \text{ s}^{-1} \cdot \text{K}^{-2}}.$$

Notice that the units correctly combine to give  $H$  in units of  $\text{s}^{-1}$  if the temperature is expressed in degrees Kelvin (K). In detail, we see:

$$[G\alpha]^{1/2} = (\text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \cdot \text{kg} \cdot \text{m}^{-3} \cdot \text{K}^{-4})^{1/2} = \text{s}^{-1} \cdot \text{K}^{-2},$$

where we used the fact that  $1 \text{ N} = 1 \text{ kg} \cdot \text{m} \cdot \text{s}^{-2}$ . At  $T = T_{\text{nucl}} \simeq 0.9 \times 10^9 \text{ K}$  we get:

$$\boxed{H \simeq 4.07 \times 10^{-3} \text{ s}^{-1}}.$$

(iii) (3 points) Using the results in parts (i) and (ii), we get

$$\boxed{t = \frac{1}{2H} \simeq \left( \frac{9.95 \times 10^{19}}{T^2} \right) \text{ s} \cdot \text{K}^2}.$$

To good accuracy, the numerator in the expression above can be rounded to  $10^{20}$ . The above equation agrees with Weinberg's claim that, for a radiation dominated universe, time is proportional to the inverse square of the temperature. In particular for  $T = T_{\text{nucl}}$  we get:

$$t_{\text{nucl}} \simeq 123 \text{ s} \approx 2 \text{ min.}$$

†Solution written by Daniele Bertolini.

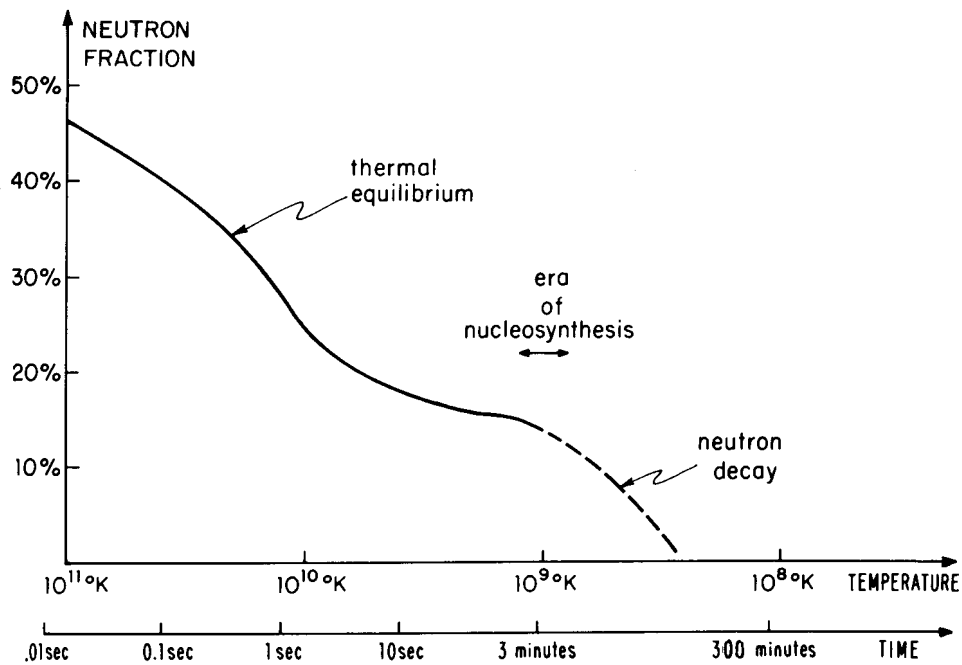
**PROBLEM 5: DID YOU DO THE READING? (2007) (24 points)**

- (a) (6 points) In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about  $10^9$ . Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)
- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
  - (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
  - (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
  - (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
  - (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about  $10^3$ , not  $10^9$  as Alpher and Herman concluded.
- (b) (6 points) In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge,

baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge:	(i) $\sim 10^9$	(ii) $\sim 1000$	(iii) $\sim 1$
	(iv) $\sim 10^{-6}$	(v) either zero or negligible	
Baryon Number:	(i) $\sim 10^{-20}$	(ii) $\sim 10^{-9}$	(iii) $\sim 10^{-6}$
	(iv) $\sim 1$	(v) anywhere from $10^{-5}$ to 1	
Lepton Number:	(i) $\sim 10^9$	(ii) $\sim 1000$	(iii) $\sim 1$
	(iv) $\sim 10^{-6}$	(v) could be as high as $\sim 1$ , but is assumed to be very small	

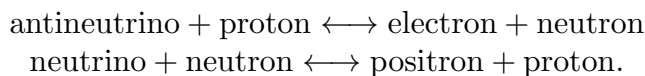
(c) (12 points) The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance*.



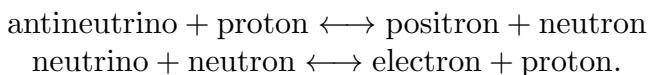
(i) (3 points) During the period labeled “thermal equilibrium,” the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

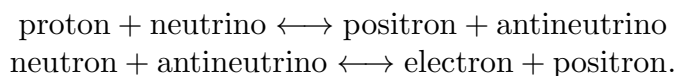
- (B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.
- (C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.
- (D) Neutrons and protons can be converted from one into through reactions such as



- (E) Neutrons and protons can be converted from one into the other through reactions such as



- (F) Neutrons and protons can be created and destroyed by reactions such as

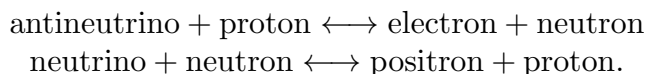


- (ii) (3 points) During the period labeled “neutron decay,” the neutron fraction is changing because (choose one):

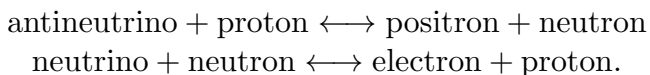
- (A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.
- (B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

- (C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

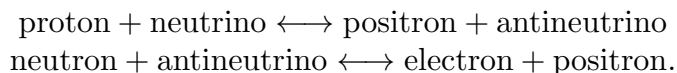
- (D) Neutrons and protons can be converted from one into the other through reactions such as



- (E) Neutrons and protons can be converted from one into the other through reactions such as



- (F) Neutrons and protons can be created and destroyed by reactions such as



- (iii) (3 points) The masses of the neutron and proton are not exactly equal, but instead
- (A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV =  $10^9$  eV).
  - (B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV =  $10^6$  eV).
  - (C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV =  $10^3$  eV).
  - (D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.
  - (E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.
  - (F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.
- (iv) (3 points) During the period labeled “era of nucleosynthesis,” (choose one:)
- (A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.
  - (B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.
  - (C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
  - (D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.
  - (E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.
  - (F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

**PROBLEM 6: DID YOU DO THE READING? (2000, 2002)**

- (a) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.
- (b) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.
- (c) The most obvious answers would be proton, neutron, and pi meson. However, there are many other possibilities, including many that were not mentioned by Weinberg. See Weinberg chapter 7, pages 136-8.
- (d) The correct answers were the neutrino and the antiproton. The neutrino was first hypothesized by Wolfgang Pauli in 1932 (in order to explain the kinematics of beta decay), and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.
- (e) The correct answers were (ii), (v) and (vi). The others were incorrect for the following reasons:
- (i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.
  - (iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed “long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s.” To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.
  - (iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: “It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe”.

**PROBLEM 7: RADIAL GEODESICS IN A CLOSED UNIVERSE** (20 points)

As shown in the formula sheets, we can describe a closed universe by choosing  $k = 1$ , and then using coordinates  $(t, r, \theta, \phi)$ , with metric

$$ds^2 \equiv -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \quad (7.1)$$



or by using coordinates  $(t, \psi, \theta, \phi)$ , with metric

$$ds^2 \equiv -c^2 d\tau^2 \equiv -c^2 dt^2 + a^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} . \quad (7.2)$$

The connection between the two coordinate systems is given by

$$r = \sin \psi . \quad (7.3)$$

The general spacetime geodesic equation can be written as

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} . \quad (7.4)$$

- (a) (7 points) Using the coordinates  $(t, \psi, \theta, \phi)$  and the metric of Eq. (7.2), compute explicitly the geodesic equation for  $\mu = \psi$ . By “compute explicitly”, I mean that  $g_{\mu\nu}$  should be replaced by the relevant expressions from Eq. (7.2), and that the sums over indices should be written out, including only the nonzero terms.

*Answer:*

Since the metric is diagonal, only  $\nu = \psi$  contributes to the sum over  $\nu$ . Similarly  $\lambda$  must equal  $\sigma$ , and the only nonzero values of  $\partial_\psi g_{\lambda\sigma}$  are when  $\lambda = \sigma = \theta$  and  $\lambda = \sigma = \phi$ . So Eq. (7.4) becomes

$$\frac{d}{d\tau} \left\{ g_{\psi\psi} \frac{d\psi}{d\tau} \right\} = \frac{1}{2} \left[ \frac{\partial g_{\theta\theta}}{\partial \psi} \left( \frac{d\theta}{d\tau} \right)^2 + \frac{\partial g_{\phi\phi}}{\partial \psi} \left( \frac{d\phi}{d\tau} \right)^2 \right] .$$

Using  $g_{\psi\psi} = a^2(t)$ ,  $g_{\theta\theta} = a^2(t) \sin^2 \psi$ , and  $g_{\phi\phi} = a^2(t) \sin^2 \psi \sin^2 \theta$ , the equation becomes

$$\frac{d}{d\tau} \left\{ a^2(t) \frac{d\psi}{d\tau} \right\} = a^2(t) \sin \psi \cos \psi \left[ \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right] . \quad (7.5)$$

You were not asked to expand the left-hand-side, but some of you did. If you do expand the left-hand side, it is important to remember that  $a(t)$  depends on  $t$  and  $t$  depends on  $\tau$ , so the equation becomes

$$a^2(t) \frac{d^2 \psi}{d\tau^2} + 2a\dot{a} \frac{dt}{d\tau} \frac{d\psi}{d\tau} = a^2(t) \sin \psi \cos \psi \left[ \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right] ,$$

or

$$\frac{d^2 \psi}{d\tau^2} + 2 \left( \frac{\dot{a}}{a} \right) \frac{dt}{d\tau} \frac{d\psi}{d\tau} = \sin \psi \cos \psi \left[ \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right] .$$

- (b) (7 points) Using instead the coordinates  $(t, r, \theta, \phi)$ , compute explicitly the geodesic equation for  $\mu = r$ .

*Answer:*

Again the equation simplifies significantly, since  $g_{\mu\nu}$  is diagonal. On the right-hand side, only 3 of the 4 possible values of  $\lambda = \sigma$  contribute, as  $\partial_r g_{tt} = 0$ . So,

$$\frac{d}{d\tau} \left\{ g_{rr} \frac{dr}{d\tau} \right\} = \frac{1}{2} \left[ \frac{\partial g_{rr}}{\partial r} \left( \frac{dr}{d\tau} \right)^2 + \frac{\partial g_{\theta\theta}}{\partial r} \left( \frac{d\theta}{d\tau} \right)^2 + \frac{\partial g_{\phi\phi}}{\partial r} \left( \frac{d\phi}{d\tau} \right)^2 \right].$$

Now we use

$$g_{rr} = \frac{a^2(t)}{1-r^2}, \quad g_{\theta\theta} = a^2(t)r^2, \quad g_{\phi\phi} = a^2(t)r^2 \sin^2 \theta,$$

which allows us to rewrite the equation as

$$\frac{d}{d\tau} \left\{ \frac{a^2(t)}{1-r^2} \frac{dr}{d\tau} \right\} = \frac{1}{2} \left[ \frac{2ra^2(t)}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2 + 2ra^2(t) \left( \frac{d\theta}{d\tau} \right)^2 + 2ra^2(t) \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right],$$

or

$$\frac{d}{d\tau} \left\{ \frac{a^2(t)}{1-r^2} \frac{dr}{d\tau} \right\} = ra^2(t) \left[ \frac{1}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2 + \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right].$$

(7.6)

Again you were not asked to expand the left-hand side, but if the left-hand side is expanded, one must remember that  $a(t)$  and  $r$  both depend on  $\tau$ . So

$$\begin{aligned} & \frac{a^2(t)}{1-r^2} \frac{d^2r}{d\tau^2} + \frac{2ra^2(t)}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2 + \frac{2a\dot{a}}{1-r^2} \frac{dt}{d\tau} \frac{dr}{d\tau} \\ &= ra^2(t) \left[ \frac{1}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2 + \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right]. \end{aligned}$$

Rearranging terms, the equation can be simplified to

$$\begin{aligned} \frac{d^2 r}{d\tau^2} + 2 \left( \frac{\dot{a}}{a} \right) \frac{dt}{d\tau} \frac{dr}{d\tau} \\ = r \left\{ -\frac{1}{(1-r^2)} \left( \frac{dr}{d\tau} \right)^2 + (1-r^2) \left[ \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right] \right\}. \end{aligned}$$

- (c) (6 points) Are the equations that you are asked to derive in parts (a) and (b) both valid, or is one valid and the other not? If you believe that they are both valid, use Eq. (7.3) to show that they are equivalent. If you believe that only one is valid, state which one is valid, and explain why the other is not. (4 points will be given for showing the correct understanding of this problem, with 2 points allocated to completing the algebra needed to demonstrate your answer.)

*Answer:*

Both answers are valid, since they are both correct forms of the geodesic equation, in different coordinate systems. To see that they are equivalent, we can start with the equation for  $r$ , Eq. (7.6), and substitute

$$r = \sin \psi \quad \Longrightarrow \quad \frac{dr}{d\tau} = \cos \psi \frac{d\psi}{d\tau} \quad \Longrightarrow \quad \frac{1}{1-r^2} \frac{dr}{d\tau} = \frac{1}{\cos \psi} \frac{d\psi}{d\tau}.$$

So Eq. (7.6) becomes

$$\frac{d}{d\tau} \left\{ \frac{a^2(t)}{\cos \psi} \frac{d\psi}{d\tau} \right\} = a^2 \sin \psi \left[ \frac{1}{\cos^2 \psi} \left( \frac{d\psi}{d\tau} \right)^2 + \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right].$$

Partially expanding the left-hand side,

$$\begin{aligned} \frac{1}{\cos \psi} \frac{d}{d\tau} \left\{ a^2(t) \frac{d\psi}{d\tau} \right\} + \frac{a^2}{\cos^2 \psi} \sin \psi \left( \frac{d\psi}{d\tau} \right)^2 \\ = a^2 \sin \psi \left[ \frac{1}{\cos^2 \psi} \left( \frac{d\psi}{d\tau} \right)^2 + \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right]. \end{aligned}$$

The terms proportional to  $(d\psi/d\tau)^2$  can be seen to cancel, and then multiplication of the equation by  $\cos \psi$  gives

$$\frac{d}{d\tau} \left\{ a^2(t) \frac{d\psi}{d\tau} \right\} = a^2(t) \sin \psi \cos \psi \left[ \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right],$$

which is identical to Eq. (7.5).

**PROBLEM 8: EVOLUTION OF AN OPEN UNIVERSE**

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh \theta - \theta)$$

$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh \theta - 1) .$$

Evaluating the second of these equations at  $a/\sqrt{\kappa} = 2\alpha$  yields a solution for  $\theta$ :

$$2\alpha = \alpha(\cosh \theta - 1) \implies \cosh \theta = 3 \implies \theta = \cosh^{-1}(3) .$$

We can use these results in the first equation to solve for  $t$ . Noting that

$$\sinh \theta = \sqrt{\cosh^2 \theta - 1} = \sqrt{8} = 2\sqrt{2} ,$$

we have

$$t = \frac{\alpha}{c} \left[ 2\sqrt{2} - \cosh^{-1}(3) \right] .$$

Numerically,  $t \approx 1.06567 \alpha/c$ .

**PROBLEM 9: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE**

- (a) Since  $\theta = \phi = \text{constant}$ ,  $d\theta = d\phi = 0$ , and for light rays one always has  $d\tau = 0$ . The line element therefore reduces to

$$0 = -c^2 dt^2 + a^2(t) d\psi^2 .$$

Rearranging gives

$$\left( \frac{d\psi}{dt} \right)^2 = \frac{c^2}{a^2(t)} ,$$

which implies that

$$\frac{d\psi}{dt} = \pm \frac{c}{a(t)} .$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

- (b) The maximum value of the  $\psi$  coordinate that can be reached by time  $t$  is found by integrating its rate of change:

$$\psi_{\text{hor}} = \int_0^t \frac{c}{a(t')} dt' .$$

The physical horizon distance is the proper length of the shortest line drawn at the time  $t$  from the origin to  $\psi = \psi_{\text{hor}}$ , which according to the metric is given by

$$\ell_{\text{phys}}(t) = \int_{\psi=0}^{\psi=\psi_{\text{hor}}} ds = \int_0^{\psi_{\text{hor}}} a(t) d\psi = \boxed{a(t) \int_0^t \frac{c}{a(t')} dt' .}$$

- (c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)} .$$

By differentiating the equation  $ct = \alpha(\theta - \sin\theta)$  stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt} \frac{dt}{d\theta} = \frac{\alpha(1 - \cos\theta)}{a(t)} .$$

Then using  $a = \alpha(1 - \cos\theta)$ , as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 .}$$

- (d) This part is very simple if one knows that  $\psi$  must change by  $2\pi$  before the photon returns to its starting point. Since  $d\psi/d\theta = 1$ , this means that  $\theta$  must also change by  $2\pi$ . From  $a = \alpha(1 - \cos\theta)$ , one can see that  $a$  returns to zero at  $\theta = 2\pi$ , so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 .}$$

If it is not clear why  $\psi$  must change by  $2\pi$  for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture

Notes 5. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates  $(x, y, z, w)$ :

$$x^2 + y^2 + z^2 + w^2 = a^2 ,$$

where  $a$  is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point  $(0, 0, 0, 1)$  as the center of the coordinate system. If we define the  $w$ -direction as “north,” then the point  $(0, 0, 0, 1)$  can be called the north pole. Each point  $(x, y, z, w)$  on the surface of the sphere is assigned a coordinate  $\psi$ , defined to be the angle between the positive  $w$  axis and the vector  $(x, y, z, w)$ . Thus  $\psi = 0$  at the north pole, and  $\psi = \pi$  for the antipodal point,  $(0, 0, 0, -1)$ , which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of  $2\pi$ .

*Discussion:* Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only “matter” present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at  $t = 0$  just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at  $t = t_{\text{Crunch}}$ , the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from  $t = \epsilon$  to  $t = t_{\text{Crunch}} - \epsilon$ , where  $\epsilon$  is arbitrarily small, but we will not try to describe what happens exactly at  $t = 0$  or  $t = t_{\text{Crunch}}$ . Thus, we now consider a photon that starts its journey at  $t = \epsilon$ , and we follow it until  $t = t_{\text{Crunch}} - \epsilon$ . For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as  $\epsilon \rightarrow 0$ . By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as  $\epsilon \rightarrow 0$ . Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

**PROBLEM 10: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC**

- a) Along the first segment  $d\theta = 0$ , so  $ds^2 = (1+ar)^2 dr^2$ , or  $ds = (1+ar) dr$ . Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1+ar) dr = r_0 + \frac{1}{2}ar_0^2 .$$

Along the second segment  $dr = 0$ , so  $ds = r(1+br) d\theta$ , where  $r = r_0$ . So the length of the second segment is

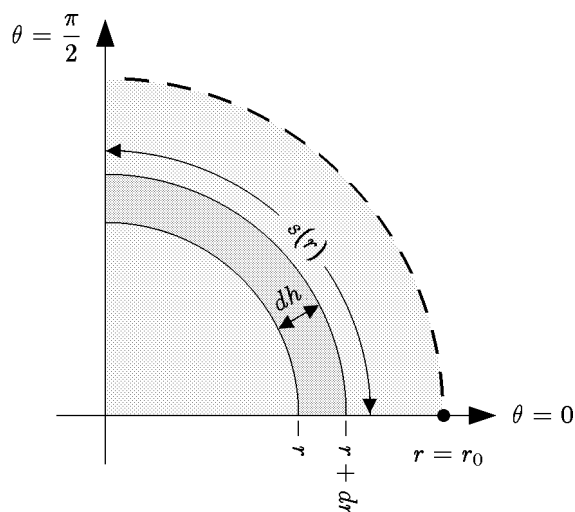
$$S_2 = \int_0^{\pi/2} r_0(1+br_0) d\theta = \frac{\pi}{2}r_0(1+br_0) .$$

Finally, the third segment is identical to the first, so  $S_3 = S_1$ . The total length is then

$$S = 2S_1 + S_2 = 2 \left( r_0 + \frac{1}{2}ar_0^2 \right) + \frac{\pi}{2}r_0(1+br_0)$$

$$= \left( 2 + \frac{\pi}{2} \right) r_0 + \frac{1}{2}(2a + \pi b)r_0^2 .$$

- b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of  $dr$ , but the distance across the width of the strip is determined by the metric to be

$$dh = (1+ar) dr .$$

The length of the strip is calculated the same way as  $S_2$  in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) .$$

The area is then

$$dA = s(r) dh ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r) dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar) dr \\ &= \frac{\pi}{2} \int_0^{r_0} [r + (a + b)r^2 + abr^3] dr \\ &= \boxed{\frac{\pi}{2} \left[ \frac{1}{2} r_0^2 + \frac{1}{3} (a + b) r_0^3 + \frac{1}{4} a b r_0^4 \right]} \end{aligned}$$

### PROBLEM 11: GEOMETRY IN A CLOSED UNIVERSE

- (a) As one moves along a line from the origin to  $(h, 0, 0)$ , there is no variation in  $\theta$  or  $\phi$ . So  $d\theta = d\phi = 0$ , and

$$ds = \frac{a dr}{\sqrt{1 - r^2}} .$$

So

$$\boxed{\ell_p = \int_0^h \frac{a dr}{\sqrt{1 - r^2}} = a \sin^{-1} h .}$$

- (b) In this case it is only  $\theta$  that varies, so  $dr = d\phi = 0$ . So

$$ds = ar d\theta ,$$

so

$$\boxed{s_p = ah \Delta\theta .}$$

- (c) From part (a), one has

$$h = \sin(\ell_p/a) .$$



Inserting this expression into the answer to (b), and then solving for  $\Delta\theta$ , one has

$$\Delta\theta = \frac{s_p}{a \sin(\ell_p/a)} .$$

Note that as  $a \rightarrow \infty$ , this approaches the Euclidean result,  $\Delta\theta = s_p/\ell_p$ .

### PROBLEM 12: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r) [d\theta^2 + \sin^2 \theta d\phi^2] .$$

The radius  $a$  is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of  $ds$ , so

$$a = \int_{\text{radial path from origin to } r_0} ds .$$

The radial path is at a constant value of  $\theta$  and  $\phi$ , so  $d\theta = d\phi = 0$ , and then  $ds = dr$ . So

$$a = \int_0^{r_0} dr = r_0 .$$

(b) On the surface  $r = r_0$ , so  $dr \equiv 0$ . Then

$$ds^2 = \rho^2(r_0) [d\theta^2 + \sin^2 \theta d\phi^2] .$$

To find the area element, consider first a path obtained by varying only  $\theta$ . Then  $ds = \rho(r_0) d\theta$ . Similarly, a path obtained by varying only  $\phi$  has length  $ds = \rho(r_0) \sin \theta d\phi$ . Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a  $dr d\theta$  term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0) \sin \theta d\theta d\phi .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0) \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \\ &= \rho^2(r_0) (2\pi) \left( -\cos \theta \Big|_0^\pi \right) \end{aligned}$$

$$\implies A = 4\pi \rho^2(r_0) .$$

As a check, notice that if  $\rho(r) = r$ , then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere,  $4\pi r^2$ .

- (c) As in Problem 2 of Problem Set 5, we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from  $r$  to  $r + dr$ . By the previous calculation, the area of such a shell is  $A(r) = 4\pi\rho^2(r)$ . (In the previous part we considered only the case  $r = r_0$ , but the same argument applies for any value of  $r$ .) The thickness of the shell is just the path length  $ds$  of a radial path corresponding to the coordinate interval  $dr$ . For radial paths the metric reduces to  $ds^2 = dr^2$ , so the thickness of the shell is  $ds = dr$ . The volume of the shell is then

$$dV = 4\pi\rho^2(r) dr .$$

The total volume is then obtained by integration:

$$V = 4\pi \int_0^{r_0} \rho^2(r) dr .$$

Checking the answer for the Euclidean case,  $\rho(r) = r$ , one sees that it gives  $V = (4\pi/3)r_0^3$ , as expected.

- (d) If  $r$  is replaced by a new coordinate  $\sigma \equiv r^2$ , then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma} ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma} .$$

The function  $\rho(r)$  can then be written as  $\rho(\sqrt{\sigma})$ , so

$$ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}) [d\theta^2 + \sin^2 \theta d\phi^2] .$$

**PROBLEM 13: VOLUMES IN A ROBERTSON-WALKER UNIVERSE**

The product of differential length elements corresponding to infinitesimal changes in the coordinates  $r, \theta$  and  $\phi$  equals the differential volume element  $dV$ . Therefore

$$dV = a(t) \frac{dr}{\sqrt{1 - kr^2}} \times a(t)r d\theta \times a(t)r \sin \theta d\phi$$

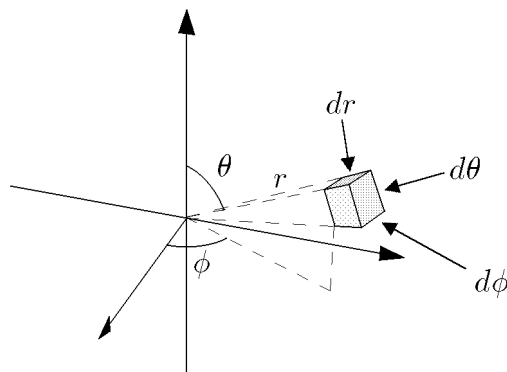
The total volume is then

$$V = \int dV = a^3(t) \int_0^{r_{\max}} dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin \theta}{\sqrt{1 - kr^2}}$$

We can do the angular integrations immediately:

$$V = 4\pi a^3(t) \int_0^{r_{\max}} \frac{r^2 dr}{\sqrt{1 - kr^2}} .$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between  $r$  and  $r + dr$ , between  $\theta$  and  $\theta + d\theta$ , and between  $\phi$  and  $\phi + d\phi$ . In the limit as  $dr$ ,  $d\theta$ , and  $d\phi$  all approach zero, the cell approaches a rectangular solid with sides of length:

$$ds_1 = a(t) \frac{dr}{\sqrt{1 - kr^2}}$$

$$ds_2 = a(t)r d\theta$$

$$ds_3 = a(t)r \sin \theta d\phi .$$

Here each  $ds$  is calculated by using the metric to find  $ds^2$ , in each case allowing only one of the quantities  $dr$ ,  $d\theta$ , or  $d\phi$  to be nonzero. The infinitesimal volume element is then  $dV = ds_1 ds_2 ds_3$ , resulting in the answer above. The derivation relies on the orthogonality of the  $dr$ ,  $d\theta$ , and  $d\phi$  directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as  $dr d\theta$ .]

[Extension: The integral can in fact be carried out, using the substitution

$$\begin{aligned}\sqrt{k} r &= \sin \psi \quad (\text{if } k > 0) \\ \sqrt{-k} r &= \sinh \psi \quad (\text{if } k < 0).\end{aligned}$$

The answer is

$$V = \begin{cases} 2\pi a^3(t) \left[ \frac{\sin^{-1}(\sqrt{k} r_{\max})}{k^{3/2}} - \frac{\sqrt{1 - kr_{\max}^2}}{k} \right] & (\text{if } k > 0) \\ 2\pi a^3(t) \left[ \frac{\sqrt{1 - kr_{\max}^2}}{(-k)} - \frac{\sinh^{-1}(\sqrt{-k} r_{\max})}{(-k)^{3/2}} \right] & (\text{if } k < 0) \end{cases} .]$$

#### PROBLEM 14: THE SCHWARZSCHILD METRIC

- a) The Schwarzschild horizon is the value of  $r$  for which the metric becomes singular. Since the metric contains the factor

$$\left( 1 - \frac{2GM}{rc^2} \right),$$

it becomes singular at

$$R_S = \frac{2GM}{c^2} .$$

- b) The separation between  $A$  and  $B$  is purely in the radial direction, so the proper length of a segment along the path joining them is given by

$$ds^2 = \left( 1 - \frac{2GM}{rc^2} \right)^{-1} dr^2 ,$$

so

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .$$

The proper distance from  $A$  to  $B$  is obtained by adding the proper lengths of all the segments along the path, so

$$s_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .$$

*EXTENSION:* The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for  $s_{AB}$  as

$$s_{AB} = \int_{r_A}^{r_B} \frac{\sqrt{r} dr}{\sqrt{r - R_S}} .$$

Then introduce the hyperbolic trigonometric substitution

$$r = R_S \cosh^2 u .$$

One then has

$$\begin{aligned} \sqrt{r - R_S} &= \sqrt{R_S} \sinh u \\ dr &= 2R_S \cosh u \sinh u du , \end{aligned}$$

and the indefinite integral becomes

$$\begin{aligned} \int \frac{\sqrt{r} dr}{\sqrt{r - R_S}} &= 2R_S \int \cosh^2 u du \\ &= R_S \int (1 + \cosh 2u) du \\ &= R_S \left( u + \frac{1}{2} \sinh 2u \right) \\ &= R_S (u + \sinh u \cosh u) \\ &= R_S \sinh^{-1} \left( \sqrt{\frac{r}{R_S} - 1} \right) + \sqrt{r(r - R_S)} . \end{aligned}$$

Thus,

$$\begin{aligned} s_{AB} &= R_S \left[ \sinh^{-1} \left( \sqrt{\frac{r_B}{R_S} - 1} \right) - \sinh^{-1} \left( \sqrt{\frac{r_A}{R_S} - 1} \right) \right] \\ &\quad + \sqrt{r_B(r_B - R_S)} - \sqrt{r_A(r_A - R_S)} . \end{aligned}$$

- c) A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = - \left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2 ,$$

so

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}} dt .$$

The reading on the observer's clock corresponds to the proper time interval  $d\tau$ , so the corresponding interval of the coordinate  $t$  is given by

$$\Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} .$$

- d) Since the Schwarzschild metric does not change with time, each pulse leaving  $A$  will take the same length of time to reach  $B$ . Thus, the pulses emitted by  $A$  will arrive at  $B$  with a time coordinate spacing

$$\Delta t_B = \Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} .$$

The clock at  $B$ , however, will read the proper time and not the coordinate time. Thus,

$$\begin{aligned} \Delta\tau_B &= \sqrt{1 - \frac{2GM}{r_B c^2}} \Delta t_B \\ &= \sqrt{\frac{1 - \frac{2GM}{r_B c^2}}{1 - \frac{2GM}{r_A c^2}}} \Delta\tau_A . \end{aligned}$$

- e) From parts (a) and (b), the proper distance between  $A$  and  $B$  can be rewritten as

$$s_{AB} = \int_{R_S}^{r_B} \frac{\sqrt{r} dr}{\sqrt{r - R_S}} .$$

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of  $r = R_S$ , say  $R_S < r < R_S + \epsilon$ . For this range the quantity  $\sqrt{r}$  in the numerator can be approximated by  $\sqrt{R_S}$ , so the contribution has the form

$$\sqrt{R_S} \int_{R_S}^{R_S + \epsilon} \frac{dr}{\sqrt{r - R_S}} .$$

Changing the integration variable to  $u \equiv r - R_S$ , the contribution can be easily evaluated:

$$\sqrt{R_S} \int_{R_S}^{R_S+\epsilon} \frac{dr}{\sqrt{r - R_S}} = \sqrt{R_S} \int_0^\epsilon \frac{du}{\sqrt{u}} = 2\sqrt{R_S}\epsilon < \infty .$$

So, although the integrand is infinite at  $r = R_S$ , the integral is still finite.

The proper distance between  $A$  and  $B$  does not diverge.

Looking at the answer to part (d), however, one can see that when  $r_A = R_S$ ,

The time interval  $\Delta\tau_B$  diverges.

### PROBLEM 15: GEODESICS

The geodesic equation for a curve  $x^i(\lambda)$ , where the parameter  $\lambda$  is the arc length along the curve, can be written as

$$\frac{d}{d\lambda} \left\{ g_{ij} \frac{dx^j}{d\lambda} \right\} = \frac{1}{2} (\partial_i g_{k\ell}) \frac{dx^k}{d\lambda} \frac{dx^\ell}{d\lambda} .$$

Here the indices  $j$ ,  $k$ , and  $\ell$  are summed from 1 to the dimension of the space, so there is one equation for each value of  $i$ .

(a) The metric is given by

$$ds^2 = g_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2 ,$$

so

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{r\theta} = g_{\theta r} = 0 .$$

First taking  $i = r$ , the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda} \left\{ g_{rr} \frac{dr}{d\lambda} \right\} = \frac{1}{2} (\partial_r g_{\theta\theta}) \frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda} ,$$

which can be written explicitly as

$$\frac{d}{d\lambda} \left\{ \frac{dr}{d\lambda} \right\} = \frac{1}{2} (\partial_r r^2) \left( \frac{d\theta}{d\lambda} \right)^2 ,$$

or

$$\frac{d^2 r}{d\lambda^2} = r \left( \frac{d\theta}{d\lambda} \right)^2 .$$

For  $i = \theta$ , one has the simplification that  $g_{ij}$  is independent of  $\theta$  for all  $(i, j)$ . So

$$\frac{d}{d\lambda} \left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 .$$

- (b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus  $y = 1$  a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve  $y = 1$  can be parameterized as

$$x(\lambda) = \lambda , \quad y(\lambda) = 1 .$$

(The parameterization is not unique, because one can choose  $\lambda = 0$  to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} ,$$

$$\theta(\lambda) = \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) .$$



Calculating the needed derivatives,\*

$$\begin{aligned}\frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2 + 1}} \\ \frac{d^2r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2 + 1}} - \frac{\lambda^2}{(\lambda^2 + 1)^{3/2}} = \frac{1}{(\lambda^2 + 1)^{3/2}} = \frac{1}{r^3} \\ \frac{d\theta}{d\lambda} &= -\frac{1}{1 + (\frac{1}{\lambda})^2} \frac{1}{\lambda^2} = -\frac{1}{r^2} .\end{aligned}$$

Then, substituting into the geodesic equation for  $i = r$ ,

$$\frac{d^2r}{d\lambda^2} = r \left( \frac{d\theta}{d\lambda} \right)^2 \iff \frac{1}{r^3} = r \left( -\frac{1}{r^2} \right)^2 ,$$

which checks. Substituting into the geodesic equation for  $i = \theta$ ,

$$\frac{d}{d\lambda} \left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 \iff \frac{d}{d\lambda} \left\{ r^2 \left( -\frac{1}{r^2} \right) \right\} = 0 ,$$

which also checks.

**PROBLEM 16: AN EXERCISE IN TWO-DIMENSIONAL METRICS** (30 points)

(a) Since

$$r(\theta) = (1 + \epsilon \cos^2 \theta) r_0 ,$$

as the angular coordinate  $\theta$  changes by  $d\theta$ ,  $r$  changes by

$$dr = \frac{dr}{d\theta} d\theta = -2\epsilon r_0 \cos \theta \sin \theta d\theta .$$

---

\* If you do not remember how to differentiate  $\phi = \tan^{-1}(z)$ , then you should know how to derive it. Write  $z = \tan \phi = \sin \phi / \cos \phi$ , so

$$dz = \left( \frac{\cos \phi}{\cos \phi} + \frac{\sin^2 \phi}{\cos^2 \phi} \right) d\phi = (1 + \tan^2 \phi) d\phi .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1 + \tan^2 \phi} = \frac{1}{1 + z^2} .$$

$ds^2$  is then given by

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 \\ &= 4\epsilon^2 r_0^2 \cos^2 \theta \sin^2 \theta d\theta^2 + (1 + \epsilon \cos^2 \theta)^2 r_0^2 d\theta^2 \\ &= [4\epsilon^2 \cos^2 \theta \sin^2 \theta + (1 + \epsilon \cos^2 \theta)^2] r_0^2 d\theta^2 , \end{aligned}$$

so

$$ds = r_0 \sqrt{4\epsilon^2 \cos^2 \theta \sin^2 \theta + (1 + \epsilon \cos^2 \theta)^2} d\theta .$$

Since  $\theta$  runs from  $\theta_1$  to  $\theta_2$  as the curve is swept out,

$$S = r_0 \int_{\theta_1}^{\theta_2} \sqrt{4\epsilon^2 \cos^2 \theta \sin^2 \theta + (1 + \epsilon \cos^2 \theta)^2} d\theta .$$

(b) Since  $\theta$  does not vary along this path,

$$ds = \sqrt{1 + \frac{r}{a}} dr ,$$

and so

$$R = \int_0^{r_0} \sqrt{1 + \frac{r}{a}} dr .$$

(c) Since the metric does not contain a term in  $dr d\theta$ , the  $r$  and  $\theta$  directions are orthogonal. Thus, if one considers a small region in which  $r$  is in the interval  $r'$  to  $r' + dr'$ , and  $\theta$  is in the interval  $\theta'$  to  $\theta' + d\theta'$ , then the region can be treated as a rectangle. The side along which  $r$  varies has length  $ds_r = \sqrt{1 + (r'/a)} dr'$ , while the side along which  $\theta$  varies has length  $ds_\theta = r' d\theta'$ . The area is then

$$dA = ds_r ds_\theta = r' \sqrt{1 + (r'/a)} dr' d\theta' .$$

To cover the area for which  $r < r_0$ ,  $r'$  must be integrated from 0 to  $r_0$ , and  $\theta'$  must be integrated from 0 to  $2\pi$ :

$$A = \int_0^{r_0} dr' \int_0^{2\pi} d\theta' r' \sqrt{1 + (r'/a)} .$$

But

$$\int_0^{2\pi} d\theta' = 2\pi ,$$

so

$$A = 2\pi \int_0^{r_0} dr' r' \sqrt{1 + (r'/a)} .$$

You were not asked to carry out the integration, but it can be done by using the substitution  $u = 1 + (r'/a)$ , so  $du = (1/a) dr'$ , and  $r' = a(u - 1)$ . The result is

$$A = \frac{4\pi a^2}{15} \left[ 2 + \left( \frac{3r_0^2}{a^2} + \frac{r_0}{a} - 2 \right) \sqrt{1 + \frac{r_0}{a}} \right] .$$

(d) The nonzero metric coefficients are given by

$$g_{rr} = 1 + \frac{r}{a} , \quad g_{\theta\theta} = r^2 ,$$

so the metric is diagonal. For  $i = 1 = r$ , the geodesic equation becomes

$$\frac{d}{ds} \left\{ g_{rr} \frac{dr}{ds} \right\} = \frac{1}{2} \frac{\partial g_{rr}}{\partial r} \frac{dr}{ds} \frac{dr}{ds} + \frac{1}{2} \frac{\partial g_{\theta\theta}}{\partial r} \frac{d\theta}{ds} \frac{d\theta}{ds} ,$$

so if we substitute the values from above, we have

$$\frac{d}{ds} \left\{ \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right\} = \frac{1}{2} \frac{\partial}{\partial r} \left( 1 + \frac{r}{a} \right) \left( \frac{dr}{ds} \right)^2 + \frac{1}{2} \frac{\partial r^2}{\partial r} \left( \frac{d\theta}{ds} \right)^2 .$$

Simplifying slightly,

$$\frac{d}{ds} \left\{ \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right\} = \frac{1}{2a} \left( \frac{dr}{ds} \right)^2 + r \left( \frac{d\theta}{ds} \right)^2 .$$

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

$$\frac{d}{ds} \left\{ \left( 1 + \frac{r}{a} \right) \frac{dr}{ds} \right\} = \frac{1}{a} \left( \frac{dr}{ds} \right)^2 + \left( 1 + \frac{r}{a} \right) \frac{d^2r}{ds^2} .$$

Inserting this expansion into the boxed equation above, the first term can be brought to the right-hand side, giving

$$\left( 1 + \frac{r}{a} \right) \frac{d^2r}{ds^2} = -\frac{1}{2a} \left( \frac{dr}{ds} \right)^2 + r \left( \frac{d\theta}{ds} \right)^2 .$$

The  $i = 2 = \theta$  equation is simpler, because none of the  $g_{ij}$  coefficients depend on  $\theta$ , so the right-hand side of the geodesic equation vanishes. One has simply

$$\frac{d}{ds} \left\{ r^2 \frac{d\theta}{ds} \right\} = 0 .$$

For most purposes this is the best way to write the equation, since it leads immediately to  $r^2(d\theta/ds) = \text{const.}$  However, it is possible to expand the derivative, giving the alternative form

$$r^2 \frac{d^2\theta}{ds^2} + 2r \frac{dr}{ds} \frac{d\theta}{ds} = 0 .$$

### PROBLEM 17: GEODESICS IN A CLOSED UNIVERSE

(a) (7 points) For purely radial motion,  $d\theta = d\phi = 0$ , so the line element reduces to

$$-c^2 d\tau^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} \right\} .$$

Dividing by  $dt^2$ ,

$$-c^2 \left( \frac{d\tau}{dt} \right)^2 = -c^2 + \frac{a^2(t)}{1-r^2} \left( \frac{dr}{dt} \right)^2 .$$

Rearranging,

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)} \left( \frac{dr}{dt} \right)^2} .$$

(b) (3 points)

$$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}} = \frac{1}{\sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)} \left( \frac{dr}{dt} \right)^2}} .$$

(c) (10 points) During any interval of clock time  $dt$ , the proper time that would be measured by a clock moving with the object is given by  $d\tau$ , as given by the metric. Using the answer from part (a),

$$d\tau = \frac{d\tau}{dt} dt = \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)} \left( \frac{dr_p}{dt} \right)^2} dt .$$

Integrating to find the total proper time,

$$\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)} \left( \frac{dr_p}{dt} \right)^2} dt .$$

- (d) (10 points) The physical distance  $d\ell$  that the object moves during a given time interval is related to the coordinate distance  $dr$  by the spatial part of the metric:

$$d\ell^2 = ds^2 = a^2(t) \left\{ \frac{dr^2}{1-r^2} \right\} \implies d\ell = \frac{a(t)}{\sqrt{1-r^2}} dr .$$

Thus

$$v_{\text{phys}} = \frac{d\ell}{dt} = \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt} .$$

*Discussion:* A common mistake was to include  $-c^2 dt^2$  in the expression for  $d\ell^2$ . To understand why this is not correct, we should think about how an observer would measure  $d\ell$ , the distance to be used in calculating the velocity of a passing object. The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval  $dt_{\text{meas}}$ . Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the **same** time  $t$ . Thus, when we compute the distance between the two marks, we set  $dt = 0$ . To compute the speed she would then divide the distance by  $dt_{\text{meas}}$ , which is nonzero.

- (e) (10 points) We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} .$$

This formula is true for each possible value of  $\mu$ , while the Einstein summation convention implies that the indices  $\nu$ ,  $\lambda$ , and  $\sigma$  are summed. We are trying to derive the equation for  $r$ , so we set  $\mu = r$ . Since the metric is diagonal, the only contribution on the left-hand side will be  $\nu = r$ . On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when  $\lambda = \sigma$ . The term will vanish unless  $dx^\lambda/d\tau$  is nonzero, so  $\lambda$  must be either  $r$  or  $t$  (i.e., there is no motion in the  $\theta$  or  $\phi$  directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r} .$$

Since  $g_{tt} = -c^2$ , the derivative with respect to  $r$  will vanish. Thus, the only nonzero contribution on the right-hand side arises from  $\lambda = \sigma = r$ . Using

$$g_{rr} = \frac{a^2(t)}{1-r^2},$$

the geodesic equation becomes

$$\frac{d}{d\tau} \left\{ g_{rr} \frac{dr}{d\tau} \right\} = \frac{1}{2} (\partial_r g_{rr}) \frac{dr}{d\tau} \frac{dr}{d\tau},$$

or

$$\frac{d}{d\tau} \left\{ \frac{a^2}{1-r^2} \frac{dr}{d\tau} \right\} = \frac{1}{2} \left[ \partial_r \left( \frac{a^2}{1-r^2} \right) \right] \frac{dr}{d\tau} \frac{dr}{d\tau},$$

or finally

$$\frac{d}{d\tau} \left\{ \frac{a^2}{1-r^2} \frac{dr}{d\tau} \right\} = a^2 \frac{r}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2.$$

This matches the form shown in the question, with

$$A = \frac{a^2}{1-r^2}, \text{ and } C = a^2 \frac{r}{(1-r^2)^2},$$

with  $B = D = E = 0$ .

- (f) (5 points EXTRA CREDIT) The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for  $p$ . Using the answer from (d),

$$p = \frac{mv_{\text{phys}}}{\sqrt{1 - \frac{v_{\text{phys}}^2}{c^2}}} = \frac{m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt}}{\sqrt{1 - \frac{a^2}{c^2(1-r^2)} \left( \frac{dr}{dt} \right)^2}}.$$

Using the answer from (b), this simplifies to

$$p = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt} \frac{dt}{d\tau} = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{d\tau}.$$

Multiply the geodesic equation by  $m$ , and then use the above result to rewrite it as

$$\frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} = ma^2 \frac{r}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2.$$

Expanding the left-hand side,

$$\begin{aligned} LHS &= \frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} = \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ap \frac{r}{(1-r^2)^{3/2}} \frac{dr}{d\tau} \\ &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ma^2 \frac{r}{(1-r^2)^2} \left( \frac{dr}{d\tau} \right)^2 . \end{aligned}$$

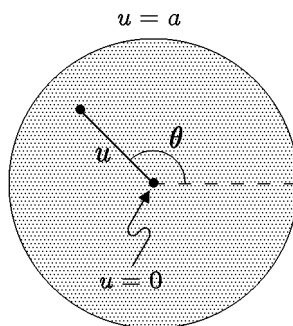
Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} = 0 .$$

Multiplying by  $\sqrt{1-r^2}$ , one has the desired result:

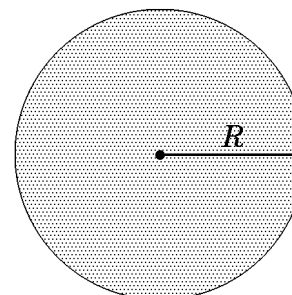
$$\frac{d}{d\tau} \{ap\} = 0 \quad \Longrightarrow \quad \boxed{p \propto \frac{1}{a(t)}} .$$

**PROBLEM 18: A TWO-DIMENSIONAL CURVED SPACE** (40 points)



(a) For  $\theta = \text{constant}$ , the expression for the metric reduces to

$$\begin{aligned} ds^2 &= \frac{a du^2}{4u(a-u)} \quad \Longrightarrow \\ ds &= \frac{1}{2} \sqrt{\frac{a}{u(a-u)}} du . \end{aligned}$$



To find the length of the radial line shown, one must integrate this expression from the value

of  $u$  at the center, which is 0, to the value of  $u$  at the outer edge, which is  $a$ . So

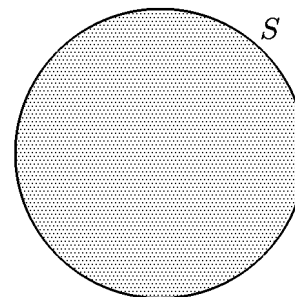
$$R = \frac{1}{2} \int_0^a \sqrt{\frac{a}{u(a-u)}} du .$$

You were not expected to do it, but the integral can be carried out, giving  $R = (\pi/2)\sqrt{a}$ .

(b) For  $u = \text{constant}$ , the expression for the metric reduces to

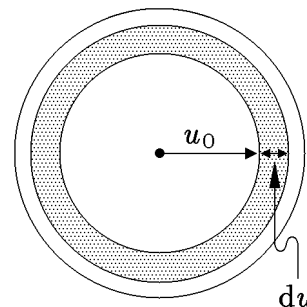
$$ds^2 = u d\theta^2 \quad \implies \quad ds = \sqrt{u} d\theta .$$

Since  $\theta$  runs from 0 to  $2\pi$ , and  $u = a$  for the circumference of the space,



$$S = \int_0^{2\pi} \sqrt{a} d\theta = 2\pi\sqrt{a} .$$

(c) To evaluate the answer to first order in  $du$  means to neglect any terms that would be proportional to  $du^2$  or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case we can imagine bending it into a rectangle without changing its area. The area is then equal to the circumference times the width. Both the circumference and the width must be calculated by using the metric:



$dA = \text{circumference} \times \text{width}$

$$= [2\pi\sqrt{u_0}] \times \left[ \frac{1}{2} \sqrt{\frac{a}{u_0(a-u_0)}} du \right]$$

$$= \pi \sqrt{\frac{a}{(a-u_0)}} du .$$



- (d) We can find the total area by imagining that it is broken up into annuluses, where a single annulus starts at radial coordinate  $u$  and extends to  $u + du$ . As in part (a), this expression must be integrated from the value of  $u$  at the center, which is 0, to the value of  $u$  at the outer edge, which is  $a$ .

$$A = \pi \int_0^a \sqrt{\frac{a}{a-u}} du .$$

You did not need to carry out this integration, but the answer would be  $A = 2\pi a$ .

- (e) From the list at the front of the exam, the general formula for a geodesic is written as

$$\frac{d}{ds} \left[ g_{ij} \frac{dx^j}{ds} \right] = \frac{1}{2} \frac{\partial g_{k\ell}}{\partial x^i} \frac{dx^k}{ds} \frac{dx^\ell}{ds} .$$

The metric components  $g_{ij}$  are related to  $ds^2$  by

$$ds^2 = g_{ij} dx^i dx^j ,$$

where the Einstein summation convention (sum over repeated indices) is assumed. In this case

$$g_{11} \equiv g_{uu} = \frac{a}{4u(a-u)}$$

$$g_{22} \equiv g_{\theta\theta} = u$$

$$g_{12} = g_{21} = 0 ,$$

where I have chosen  $x^1 = u$  and  $x^2 = \theta$ . The equation with  $du/ds$  on the left-hand side is found by looking at the geodesic equations for  $i = 1$ . Of course  $j$ ,  $k$ , and  $\ell$  must all be summed, but the only nonzero contributions arise when  $j = 1$ , and  $k$  and  $\ell$  are either both equal to 1 or both equal to 2:

$$\frac{d}{ds} \left[ g_{uu} \frac{du}{ds} \right] = \frac{1}{2} \frac{\partial g_{uu}}{\partial u} \left( \frac{du}{ds} \right)^2 + \frac{1}{2} \frac{\partial g_{\theta\theta}}{\partial u} \left( \frac{d\theta}{ds} \right)^2 .$$

$$\begin{aligned} \frac{d}{ds} \left[ \frac{a}{4u(a-u)} \frac{du}{ds} \right] &= \frac{1}{2} \left[ \frac{d}{du} \left( \frac{a}{4u(a-u)} \right) \right] \left( \frac{du}{ds} \right)^2 + \frac{1}{2} \left[ \frac{d}{du}(u) \right] \left( \frac{d\theta}{ds} \right)^2 \\ &= \frac{1}{2} \left[ \frac{a}{4u(a-u)^2} - \frac{a}{4u^2(a-u)} \right] \left( \frac{du}{ds} \right)^2 + \frac{1}{2} \left( \frac{d\theta}{ds} \right)^2 \end{aligned}$$

$$= \frac{1}{8} \frac{a(2u-a)}{u^2(a-u)^2} \left( \frac{du}{ds} \right)^2 + \frac{1}{2} \left( \frac{d\theta}{ds} \right)^2 .$$

- (f) This part is solved by the same method, but it is simpler. Here we consider the geodesic equation with  $i = 2$ . The only term that contributes on the left-hand side is  $j = 2$ . On the right-hand side one finds nontrivial expressions when  $k$  and  $\ell$  are either both equal to 1 or both equal to 2. However, the terms on the right-hand side both involve the derivative of the metric with respect to  $x^2 = \theta$ , and these derivatives all vanish. So

$$\frac{d}{ds} \left[ g_{\theta\theta} \frac{d\theta}{ds} \right] = \frac{1}{2} \frac{\partial g_{uu}}{\partial \theta} \left( \frac{du}{ds} \right)^2 + \frac{1}{2} \frac{\partial g_{\theta\theta}}{\partial \theta} \left( \frac{d\theta}{ds} \right)^2 ,$$

which reduces to

$$\boxed{\frac{d}{ds} \left[ u \frac{d\theta}{ds} \right] = 0 .}$$

### PROBLEM 19: ROTATING FRAMES OF REFERENCE (35 points)

- (a) The metric was given as

$$-c^2 d\tau^2 = -c^2 dt^2 + \left[ dr^2 + r^2 (d\phi + \omega dt)^2 + dz^2 \right] ,$$

and the metric coefficients are then just read off from this expression:

$$g_{11} \equiv g_{rr} = 1$$

$$g_{00} \equiv g_{tt} = \text{coefficient of } dt^2 = -c^2 + r^2\omega^2$$

$$g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } d\phi dt = r^2\omega$$

$$g_{22} \equiv g_{\phi\phi} = \text{coefficient of } d\phi^2 = r^2$$

$$g_{33} \equiv g_{zz} = \text{coefficient of } dz^2 = 1 .$$

Note that the off-diagonal term  $g_{\phi t}$  must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu$$

includes the two equal terms  $g_{20} d\phi dt + g_{02} dt d\phi$ , where  $g_{20} \equiv g_{02}$ .

- (b) Starting with the general expression

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} ,$$

we set  $\mu = r$ :

$$\frac{d}{d\tau} \left\{ g_{r\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_r g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} .$$

When we sum over  $\nu$  on the left-hand side, the only value for which  $g_{r\nu} \neq 0$  is  $\nu = 1 \equiv r$ . Thus, the left-hand side is simply

$$\text{LHS} = \frac{d}{d\tau} \left( g_{rr} \frac{dx^1}{d\tau} \right) = \frac{d}{d\tau} \left( \frac{dr}{d\tau} \right) = \frac{d^2 r}{d\tau^2} .$$

The RHS includes every combination of  $\lambda$  and  $\sigma$  for which  $g_{\lambda\sigma}$  depends on  $r$ , so that  $\partial_r g_{\lambda\sigma} \neq 0$ . This means  $g_{tt}$ ,  $g_{\phi\phi}$ , and  $g_{\phi t}$ . So,

$$\begin{aligned} \text{RHS} &= \frac{1}{2} \partial_r (-c^2 + r^2 \omega^2) \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{2} \partial_r (r^2) \left( \frac{d\phi}{d\tau} \right)^2 + \partial_r (r^2 \omega) \frac{d\phi}{d\tau} \frac{dt}{d\tau} \\ &= r\omega^2 \left( \frac{dt}{d\tau} \right)^2 + r \left( \frac{d\phi}{d\tau} \right)^2 + 2r\omega \frac{d\phi}{d\tau} \frac{dt}{d\tau} \\ &= r \left( \frac{d\phi}{d\tau} + \omega \frac{dt}{d\tau} \right)^2 . \end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from  $g_{\phi t}$  and  $g_{t\phi}$ , where the two terms were combined to cancel the factor of  $1/2$  in the general expression. Finally,

$$\boxed{\frac{d^2 r}{d\tau^2} = r \left( \frac{d\phi}{d\tau} + \omega \frac{dt}{d\tau} \right)^2 .}$$

If one expands the RHS as

$$\frac{d^2 r}{d\tau^2} = r \left( \frac{d\phi}{d\tau} \right)^2 + r\omega^2 \left( \frac{dt}{d\tau} \right)^2 + 2r\omega \frac{d\phi}{d\tau} \frac{dt}{d\tau} ,$$

then one can identify the term proportional to  $\omega^2$  as the centrifugal force, and the term proportional to  $\omega$  as the Coriolis force.

(c) Substituting  $\mu = \phi$ ,

$$\frac{d}{d\tau} \left\{ g_{\phi\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\phi g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} .$$

But none of the metric coefficients depend on  $\phi$ , so the right-hand side is zero. The left-hand side receives contributions from  $\nu = \phi$  and  $\nu = t$ :

$$\frac{d}{d\tau} \left( g_{\phi\phi} \frac{d\phi}{d\tau} + g_{\phi t} \frac{dt}{d\tau} \right) = \frac{d}{d\tau} \left( r^2 \frac{d\phi}{d\tau} + r^2 \omega \frac{dt}{d\tau} \right) = 0 ,$$

so

$$\frac{d}{d\tau} \left( r^2 \frac{d\phi}{d\tau} + r^2 \omega \frac{dt}{d\tau} \right) = 0 .$$

Note that one cannot “factor out”  $r^2$ , since  $r$  can depend on  $\tau$ . If this equation is expanded to give an equation for  $d^2\phi/d\tau^2$ , the term proportional to  $\omega$  would be identified as the Coriolis force. There is no term proportional to  $\omega^2$ , since the centrifugal force has no component in the  $\phi$  direction.

(d) If Eq. (P19.1) of the problem is divided by  $c^2 dt^2$ , one obtains

$$\left( \frac{d\tau}{dt} \right)^2 = 1 - \frac{1}{c^2} \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} + \omega \right)^2 + \left( \frac{dz}{dt} \right)^2 \right] .$$

Then using

$$\frac{dt}{d\tau} = \frac{1}{\left( \frac{d\tau}{dt} \right)} ,$$

one has

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2} \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} + \omega \right)^2 + \left( \frac{dz}{dt} \right)^2 \right]}} .$$

Note that this equation is really just

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} ,$$

adapted to the rotating cylindrical coordinate system.

**PROBLEM 20: THE STABILITY OF SCHWARZSCHILD ORBITS\*** (30 points)

From the metric:

$$ds^2 = -c^2 d\tau^2 = -h(r) c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (\text{S20.1})$$

and the convention  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$  we read the nonvanishing metric components:

$$g_{tt} = -h(r)c^2, \quad g_{rr} = \frac{1}{h(r)}, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta. \quad (\text{S20.2})$$

We are told that the orbit has  $\theta = \pi/2$ , so on the orbit  $d\theta = 0$  and the relevant metric and metric components are:

$$ds^2 = -c^2 d\tau^2 = -h(r) c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\phi^2, \quad (\text{S20.3})$$

$$g_{tt} = -h(r)c^2, \quad g_{rr} = \frac{1}{h(r)}, \quad g_{\phi\phi} = r^2. \quad (\text{S20.4})$$

We also know that

$$h(r) = 1 - \frac{R_S}{r}. \quad (\text{S20.5})$$

(a) The geodesic equation

$$\frac{d}{d\tau} \left[ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}, \quad (\text{S20.6})$$

for the index value  $\mu = r$  takes the form

$$\frac{d}{d\tau} \left[ g_{rr} \frac{dr}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{\lambda\sigma}}{\partial r} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}.$$

Expanding out

$$\frac{d}{d\tau} \left[ \frac{1}{h} \frac{dr}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{tt}}{\partial r} \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{2} \frac{\partial g_{rr}}{\partial r} \left( \frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial r} \left( \frac{d\phi}{d\tau} \right)^2.$$

Using the values in (S20.4) to evaluate the right-hand side and taking the derivatives on the left-hand side:

$$\underline{-\frac{h'}{h^2} \left( \frac{dr}{d\tau} \right)^2} + \frac{1}{h} \frac{d^2 r}{d\tau^2} = -\frac{1}{2} c^2 h' \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{2} \frac{h'}{h^2} \left( \frac{dr}{d\tau} \right)^2 + r \left( \frac{d\phi}{d\tau} \right)^2.$$

---

\* Solution by Barton Zwiebach.

Here  $h' \equiv \frac{dh}{dr}$  and we have suppressed the arguments of  $h$  and  $h'$  to avoid clutter. Collecting the underlined terms to the right and multiplying by  $h$ , we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2} h' h c^2 \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{2} \frac{h'}{h} \left( \frac{dr}{d\tau} \right)^2 + r h \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{S20.7})$$

(b) Dividing the expression (S20.3) for the metric by  $d\tau^2$  we readily find

$$-c^2 = -h c^2 \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{h} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2,$$

and rearranging,

$$h c^2 \left( \frac{dt}{d\tau} \right)^2 = c^2 + \frac{1}{h} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{S20.8})$$

This is the most useful form of the answer. Of course, we also have

$$\left( \frac{dt}{d\tau} \right)^2 = \frac{1}{h} + \frac{1}{h^2 c^2} \left( \frac{dr}{d\tau} \right)^2 + \frac{r^2}{h c^2} \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{S20.9})$$

We use now (S20.8) to simplify (S20.7):

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2} h' \left( c^2 + \frac{1}{h} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2 \right) + \frac{1}{2} \frac{h'}{h} \left( \frac{dr}{d\tau} \right)^2 + r h \left( \frac{d\phi}{d\tau} \right)^2.$$

Expanding out, the terms with  $\left( \frac{dr}{d\tau} \right)^2$  cancel and we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2} h' c^2 + \left( r h - \frac{1}{2} h' r^2 \right) \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{S20.10})$$

This is an acceptable answer. One can simplify (S20.10) further by noting that  $h' = R_S/r^2$  and  $r h = r - R_S$ :

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2} \frac{R_S c^2}{r^2} + \left( r - \frac{3}{2} R_S \right) \left( \frac{d\phi}{d\tau} \right)^2. \quad (\text{S20.11})$$

In the notation of the problem statement, we have

$$f_0(r) = -\frac{1}{2} \frac{R_S c^2}{r^2}, \quad f_1(r) = r - \frac{3}{2} R_S. \quad (\text{S20.12})$$

(c) The geodesic equation (S20.6) for  $\mu = \phi$  gives

$$\frac{d}{d\tau} \left[ g_{\phi\phi} \frac{d\phi}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{\lambda\sigma}}{\partial \phi} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}.$$

Since no metric component depends on  $\phi$ , the right-hand side vanishes and we get:

$$\frac{d}{d\tau} \left[ r^2 \frac{d\phi}{d\tau} \right] = 0 \quad \rightarrow \quad \frac{d}{d\tau} L = 0, \quad \text{where} \quad L \equiv r^2 \frac{d\phi}{d\tau}. \quad (\text{S20.13})$$

The quantity  $L$  is a constant of the motion, namely, it is a number independent of  $\tau$ .

(d) Using (S20.13) the second-order differential equation (S20.11) for  $r(\tau)$  takes the form stated in the problem:

$$\frac{d^2 r}{d\tau^2} = f_0(r) + \frac{f_1(r)}{r^4} L^2 \equiv H(r), \quad (\text{S20.14})$$

where we have introduced the function  $H(r)$  (recall that  $L$  is a constant!). The differential equation then takes the form

$$\frac{d^2 r}{d\tau^2} = H(r). \quad (\text{S20.15})$$

Since we are told that a circular orbit with radius  $r_0$  exists, the function  $r(\tau) = r_0$  must solve this equation. Being the constant function, the left-hand side vanishes and, consequently, the right-hand side must also vanish:

$$H(r_0) = f_0(r_0) + \frac{f_1(r_0)}{r_0^4} L^2 = 0. \quad (\text{S20.16})$$

To investigate stability we consider a small perturbation  $\delta r(\tau)$  of the orbit:

$$r(\tau) = r_0 + \delta r(\tau), \quad \text{with} \quad \delta r(\tau) \ll r_0 \quad \text{at some initial } \tau.$$

Substituting this into (S20.15) we get, to first nontrivial approximation

$$\frac{d^2 \delta r}{d\tau^2} = H(r_0 + \delta r) \simeq H(r_0) + \delta r H'(r_0) = \delta r H'(r_0),$$

where  $H'(r) = \frac{dH(r)}{dr}$  and we used  $H(r_0) = 0$  from (S20.16). The resulting equation

$$\frac{d^2 \delta r(\tau)}{d\tau^2} = H'(r_0) \delta r(\tau), \quad (\text{S20.17})$$

is familiar because  $H'(r_0)$  is just a number. The condition of stability is that this number is negative:  $H'(r_0) < 0$ . Indeed, in this case (S20.17) is the harmonic oscillator equation

$$\frac{d^2 x}{dt^2} = -\omega^2 x, \quad \text{with replacements} \quad x \leftrightarrow \delta r, \quad t \leftrightarrow \tau, \quad -\omega^2 \leftrightarrow H'(r_0),$$

and the solution describes bounded oscillations. So stability requires:

$$\text{Stability Condition: } H'(r_0) = \frac{d}{dr} \left[ f_0(r) + \frac{f_1(r)}{r^4} L^2 \right]_{r=r_0} < 0. \quad (\text{S20.18})$$

This is the answer to part (d).

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For students interested in getting the famous result that orbits are stable for  $r > 3R_S$  we complete this part of the analysis below. First we evaluate  $H'(r_0)$  in (S20.18) using the values of  $f_0$  and  $f_1$  in (S20.12):

$$H'(r_0) = \frac{d}{dr} \left[ -\frac{1}{2} \frac{R_S c^2}{r^2} + \left( \frac{1}{r^3} - \frac{3R_S}{2r^4} \right) L^2 \right]_{r=r_0} = \frac{R_S c^2}{r_0^3} - \frac{3L^2}{r_0^5} (r_0 - 2R_S).$$

The inequality in (S20.18) then gives us

$$R_S c^2 - \frac{3L^2}{r_0^2} (r_0 - 2R_S) < 0, \quad (\text{S20.19})$$

where we multiplied by  $r_0^3 > 0$ . To complete the calculation we need the value of  $L^2$  for the orbit with radius  $r_0$ . This value is determined by the vanishing of  $H(r_0)$ :

$$-\frac{1}{2} \frac{R_S c^2}{r_0^2} + (r_0 - \frac{3}{2} R_S) \frac{L^2}{r_0^4} = 0 \quad \rightarrow \quad \frac{L^2}{r_0^2} = \frac{1}{2} \frac{R_S c^2}{(r_0 - \frac{3}{2} R_S)}.$$

Note, incidentally, that the equality to the right demands that for a circular orbit  $r_0 > \frac{3}{2} R_S$ . Substituting the above value of  $L^2/r_0^2$  in (S20.19) we get:

$$R_S c^2 - \frac{3}{2} \frac{R_S c^2}{(r_0 - \frac{3}{2} R_S)} (r_0 - 2R_S) < 0.$$

Cancelling the common factors of  $R_S c^2$  we find

$$1 - \frac{3}{2} \frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2} R_S)} < 0,$$

which is equivalent to

$$\frac{3}{2} \frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2} R_S)} > 1.$$

For  $r_0 > \frac{3}{2} R_S$ , we get

$$3(r_0 - 2R_S) > 2(r_0 - \frac{3}{2} R_S) \quad \rightarrow \quad r_0 > 3R_S. \quad (\text{S20.20})$$

This is the desired condition for stable orbits in the Schwarzschild geometry.



**PROBLEM 21: GRAVITATIONAL BENDING OF LIGHT** (30 points)

(a) (6 points) Note that

$$\begin{aligned} dr^2 &= \frac{1}{r^2} (x dx + y dy + z dz)^2 \\ &= \frac{1}{r^2} (x^2 dx^2 + y^2 dy^2 + z^2 dz^2 + 2xy dx dy + 2xz dx dz + 2yz dy dz) . \end{aligned} \tag{S21.1}$$

By using this expression for  $(dr)^2$  in Eq. (P21.5), we have the full expression for  $ds^2$  written out, from which we can read off the components of  $g_{\mu\nu}$ :

$g_{tt} = \text{coefficient of } dt^2 = -c^2 \left( 1 - \frac{R_{\text{Sch}}}{r} \right)$ $g_{xx} = \text{coefficient of } dx^2 = 1 + \frac{R_{\text{Sch}}}{r^3} x^2$ $g_{xy} = \frac{1}{2} \text{ of coefficient of } dx dy = \frac{R_{\text{Sch}}}{r^3} xy .$	(S21.2)
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A number of people missed the factor of 1/2 in the value of  $g_{xy}$ . It arises because the general formula is written as  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ , which when expanded becomes

$$ds^2 = g_{xx} dx^2 + g_{yy} dy^2 + g_{zz} dz^2 + g_{tt} dt^2 + g_{xy} dx dy + g_{yx} dy dx + \dots .$$

Since  $dx dy = dy dx$ , the coefficient of  $dx dy$  is  $g_{xy} + g_{yx} = 2g_{xy}$ .

(b) (9 points) It will be useful to know the derivatives of  $r$ :

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2} \\ &= \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = \frac{x}{r} . \end{aligned} \tag{S21.3}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r} , \tag{S21.4}$$

and

$$\begin{aligned} \frac{dr}{d\lambda} &= \frac{\partial r}{\partial x} \frac{dx}{d\lambda} + \frac{\partial r}{\partial y} \frac{dy}{d\lambda} + \frac{\partial r}{\partial z} \frac{dz}{d\lambda} \\ &= \frac{x}{r} . \end{aligned} \tag{S21.5}$$

In the 2nd line I used the value of  $\partial r/\partial x$  from Eq. (S21.3), and the derivatives

$$\frac{dx}{d\lambda} = 1, \quad \frac{dy}{d\lambda} = \frac{dz}{d\lambda} = 0 \quad (\text{S21.6})$$

that can be found from Eq. (P21.8).

Now, to expand the left-hand side of the geodesic equation:

$$\begin{aligned} \frac{d}{d\lambda} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\lambda} \right\} &= \frac{d}{d\lambda} \left\{ g_{yy} \frac{dy}{d\lambda} + g_{yx} \frac{dx}{d\lambda} \right\} \\ &= \frac{d}{d\lambda} \left\{ \left[ 1 + \frac{R_{\text{Sch}}}{r^3} y^2 \right] \frac{dy}{d\lambda} + \frac{R_{\text{Sch}}}{r^3} xy \frac{dx}{d\lambda} \right\} \\ &= \frac{d^2 y}{d\lambda^2} - 3 \frac{R_{\text{Sch}}}{r^4} \frac{x}{r} xy \frac{dx}{d\lambda} + \frac{R_{\text{Sch}}}{r^3} \frac{dx}{d\lambda} y \frac{dx}{d\lambda} \\ &= \boxed{\frac{d^2 y}{d\lambda^2} - 3 \frac{R_{\text{Sch}} b}{r^5} x^2 + \frac{R_{\text{Sch}} b}{r^3}}. \end{aligned} \quad (\text{S21.7})$$

Note that I dropped a term

$$\frac{R_{\text{Sch}} y^2}{r^3} \frac{d^2 y}{d\lambda^2}$$

and a term

$$\frac{R_{\text{Sch}}}{r^3} xy \frac{d^2 x}{d\lambda^2},$$

which is justified because the acceleration  $\frac{d^2 y}{d\lambda^2}$  will be proportional to  $G$ , and  $R_{\text{Sch}}$  is proportional to  $G$ , so this term is 2nd order in  $G$ . The problem stated that we are to work to first order in  $G$ . No points were taken off, however, from students who retained these or other negligible terms.

Note, however, that  $d^2 y/d\lambda^2$  is not negligible, and appears in the answer. This is because  $dy/d\lambda$  is not actually zero, but is of order  $G$ .  $dy/d\lambda$  is zero for the unperturbed path, but in reality the photon picks up a small velocity in the  $y$ -direction, caused by the gravitational attraction of the Sun and proportional to  $G$ .  $d^2 y/d\lambda^2$  will also be proportional to  $G$ . When  $dy/d\lambda$  multiplies a factor proportional to  $R_{\text{Sch}}$ , the product is of order  $G^2$  and hence negligible. But  $d^2 y/d\lambda^2$  by itself is of order  $G$  and is not negligible.

*Note on propagation of errors:* I normally do not take off points for propagating errors, so for example a student who forgot the factor of 1/2 in determining  $g_{xy}$  would get full credit on part (b), even though the answer would contain terms that are wrong by a factor of 1/2. However, it seems right to me to make an exception to this rule in cases where an error on part (a) causes the consequent answer on a later part to become trivial. For

example, if a student described a metric in part (a) which had no dependence on  $r$ , then many of the terms in parts (b) and (c) would not be present. In such cases I still took off points in parts (b) and (c), because it didn't seem fair to me to give such a student credit for calculating these terms, when the student exhibited no such capability.

(c) (9 points)

$$\begin{aligned}
 \frac{1}{2} \frac{\partial}{\partial y} (g_{\sigma\tau}) \frac{dx^\sigma}{d\lambda} \frac{dx^\tau}{d\lambda} &= \frac{1}{2} \frac{\partial}{\partial y} (g_{xx}) \left( \frac{dx}{d\lambda} \right)^2 + \frac{1}{2} \frac{\partial}{\partial y} (g_{tt}) \left( \frac{dt}{d\lambda} \right)^2 \\
 &= \frac{1}{2} \frac{\partial}{\partial y} \left( 1 + \frac{R_{\text{Sch}}}{r^3} x^2 \right) - \frac{1}{2} c^2 \frac{\partial}{\partial y} \left( 1 - \frac{R_{\text{Sch}}}{r} \right) \left( \frac{1}{c^2} \right) \\
 &= -\frac{1}{2} \left( 3 \frac{R_{\text{Sch}}}{r^4} \frac{y}{r} x^2 \right) - \frac{1}{2} \left( \frac{R_{\text{Sch}}}{r^2} \frac{y}{r} \right) \\
 &= \boxed{-\frac{3}{2} \frac{R_{\text{Sch}} b}{r^5} x^2 - \frac{1}{2} \frac{R_{\text{Sch}} b}{r^3}}.
 \end{aligned} \tag{S21.8}$$

(d) (2 points) Combining Eqs. (S21.7) and (S21.8), we find

$$\begin{aligned}
 \frac{d^2 y}{d\lambda^2} &= -\frac{3}{2} \frac{R_{\text{Sch}} b}{r^5} x^2 - \frac{1}{2} \frac{R_{\text{Sch}} b}{r^3} + 3 \frac{R_{\text{Sch}} b}{r^5} x^2 - \frac{R_{\text{Sch}} b}{r^3} \\
 &= \boxed{\frac{3}{2} R_{\text{Sch}} b \left[ \frac{x^2}{r^5} - \frac{1}{r^3} \right]}.
 \end{aligned} \tag{S21.9}$$

(e) (4 points) The final value of  $dy/d\lambda$  is given by Eq. (P21.9), while the final value of  $dx/d\lambda$  will be equal to 1, at least up to possible corrections proportional to  $G$ . Thus, the final velocity will make an angle  $\alpha$  relative to the horizontal, where

$$\begin{aligned}
 \tan \alpha &= \frac{dy/d\lambda|_{\text{final}}}{dx/d\lambda|_{\text{final}}} \\
 &= \int_{-\infty}^{\infty} \frac{d^2 y}{d\lambda^2} d\lambda.
 \end{aligned}$$

Since  $\tan \alpha$  will be proportional to  $G$ , the small angle approximation  $\tan \alpha = \alpha$  will apply, and

$$\boxed{\alpha \approx \int_{-\infty}^{\infty} \frac{d^2 y}{d\lambda^2} d\lambda}. \tag{S21.10}$$

Then, using Eqs. (S21.9) and combining with Eqs. (P21.4) and (P21.8),

$$\alpha = \frac{3}{2} R_{\text{Sch}} b \int_{-\infty}^{\infty} \left[ \frac{x^2}{r^5} - \frac{1}{r^3} \right] d\lambda$$

$$= \boxed{\frac{3}{2} R_{\text{Sch}} b \int_{-\infty}^{\infty} \left[ \frac{\lambda^2}{(\lambda^2 + b^2)^{5/2}} - \frac{1}{(\lambda^2 + b^2)^{3/2}} \right] d\lambda .} \quad (\text{S21.11})$$

You were not asked to carry out these integrals, but using the table of integrals given with the problem, one finds

$$\alpha = \frac{3}{2} R_{\text{Sch}} b \left[ \frac{2}{3b^2} - \frac{2}{b^2} \right] = -\frac{2R_{\text{Sch}}}{b} = -\frac{4GM}{c^2 b} . \quad (\text{S21.12})$$

The minus sign indicates that the deflection is downward, as one would expect.

## PROBLEM 22: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If  $u \propto 1/\sqrt{V}$ , then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} .$$

(The above expression is proportional to  $1/\sqrt{V + \Delta V}$ , and reduces to  $u = u_0$  when  $\Delta V = 0$ .) Expanding to first order in  $\Delta V$ ,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2} \frac{\Delta V}{V}} = u_0 \left( 1 - \frac{1}{2} \frac{\Delta V}{V} \right) .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left( 1 - \frac{1}{2} \frac{\Delta V}{V} \right) V \left( 1 + \frac{\Delta V}{V} \right) = U_0 \left( 1 + \frac{1}{2} \frac{\Delta V}{V} \right) ,$$

where  $U_0 = u_0 V$ . Then

$$\boxed{\Delta U = \frac{1}{2} \frac{\Delta V}{V} U_0 .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is  $p \Delta V$ . So

$$\boxed{\Delta W = -p \Delta V} .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2} \frac{\Delta V}{V} U_0 .$$

Combining this with the expression for  $\Delta W$  from part (b), one sees immediately that

$$p = -\frac{1}{2} \frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 .}$$

**PROBLEM 23: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION** (25 points)

(a) (8 points) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by  $dU = -p dV$ . Using the fact that the energy density  $u$  is equal to  $\rho c^2$ , the energy conservation relation can be written

$$\frac{dU}{dt} = -p \frac{dV}{dt} \implies \frac{d}{dt} (\rho c^2 a^3) = -p \frac{d}{dt} (a^3) .$$

Setting

$$\rho = \frac{\alpha}{a^8}$$

for some constant  $\alpha$ , the conservation of energy formula becomes

$$\frac{d}{dt} \left( \frac{\alpha c^2}{a^5} \right) = -p \frac{d}{dt} (a^3) ,$$

which implies

$$-5 \frac{\alpha c^2}{a^6} \frac{da}{dt} = -3pa^2 \frac{da}{dt} .$$

Thus

$$p = \frac{5}{3} \frac{\alpha c^2}{a^8} = \boxed{\frac{5}{3} \rho c^2 .}$$

Alternatively, one may start from the equation for the time derivative of  $\rho$ ,

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) .$$

Since  $\rho = \frac{\alpha}{a^8}$ , we take the time derivative to find  $\dot{\rho} = -8(\dot{a}/a)\rho$ , and therefore

$$-8\frac{\dot{a}}{a}\rho = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right),$$

and therefore

$$p = \frac{5}{3}\rho c^2.$$

(b) (9 points) For a flat universe, the Friedmann equation reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho.$$

Using  $\rho \propto 1/a^8$ , this implies that

$$\dot{a} = \frac{\beta}{a^3},$$

for some constant  $\beta$ . Rewriting this as

$$a^3 da = \beta dt,$$

we can integrate the equation to give

$$\frac{1}{4}a^4 = \beta t + \text{const},$$

where the constant of integration has no effect other than to shift the origin of the time variable  $t$ . Using the standard big bang convention that  $a = 0$  when  $t = 0$ , the constant of integration vanishes. Thus,

$$a \propto t^{1/4}.$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that “You should be able to determine the function  $a(t)$  up to a constant factor.” Note that we could have expressed the constant of proportionality in terms of the constant  $\alpha$  that we used in part (a), but there would not really be any point in doing that. The constant  $\alpha$  was not a given variable. If the comoving coordinates are measured in “notches,” then  $a$  is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) (8 points) We start from the conservation of energy equation in the form

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right).$$

Substituting  $\dot{\rho} = -n(\dot{a}/a)\rho$  and  $p = (2/3)\rho c^2$ , we have

$$-nH\rho = -3H\left(\frac{5}{3}\rho\right)$$

and therefore

$$n = 5.$$