MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 22, 2020
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

QUIZ DATE: Wedneday, December 2, 2020, during the normal class time.

COVERAGE: Lecture Notes 6 (pp. 12–end) and Lecture Notes 7. Problem Sets 7 and 8; Steven Weinberg, *The First Three Minutes,* Chapter 8 and the Afterword; Barbara Ryden, *Introduction to Cosmology*, Chapters 8 *(The Cosmic Microwave Background)* and 10 *(Inflation and the Very Early Universe)* [First Edition: Chapters 9 and 11]; Alan Guth, *Inflation and the New Era of High-Precision Cosmology,*

http://web.mit.edu/physics/news/physicsatmit/physicsatmit_02_cosmology.pdf .

One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 6, 7, 11, 15, 17, and 19.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2007, 2009, 2011, 2013, 2016, and 2018. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

QUIZ LOGISTICS: The logistics will be identical to Quizzes 1 and 2, except of course for the dates. The quiz will be closed book, no calculators, no internet, and 85 minutes long. I assume that most of you will take it during our regular class time on December 2, but you will have the option of starting it any time during a 24-hour window from 11:05 am EST on December 2 to 11:05 am EST on Thursday, December 3. If you want to start later than 11:05 am 12/2/20, you should email me your choice of starting time by 11:59 pm on the night before the quiz (earlier is appreciated). The quiz will be contained in a PDF file, which I will distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on the uploading, because the time needed for scanning, photographing, or whatever kind of processing you are doing can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the December 2 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times. If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.

PURPOSE OF THE REVIEW PROBLEMS: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

REVIEW SESSION AND OFFICE HOURS: A review session and special office hours will be held to help you study for the quiz. Details will follow.

QUIZZES FROM PREVIOUS YEARS: In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, 2011, 2013, 2016, and 2018. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, mainly to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match exactly the coverage from all previous years, but I believe that all these review problems would be fair problems for the upcoming quiz. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here. In 2016 we finished Weinberg's book by the time of Quiz 2, but otherwise the coverage has been the same since 2016.

INFORMATION TO BE GIVEN ON QUIZ:

For the third quiz, the following information will be made available to you:

DOPPLER SHIFT (Definition:)

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} ,$$

where $\Delta t_{\text{observer}}$ and Δt_{source} are the period of the wave as measured by the observer and by the source, respectively, and $\lambda_{\text{observer}}$ and λ_{source} are the wavelength of the wave, as measured by the observer and by the source, respectively.

DOPPLER SHIFT (For motion along a line):

Nonrelativistic, u = wave speed, source moving at speed v away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed v away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity), $\beta \equiv v/c$, where c is the speed of light and v is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation. A clock that is moving at speed v relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor γ:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance ℓ_0 in their own frame, are moving together, in the direction of the line separating them, at speed v relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,
where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned}\ell_{p,\mathrm{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\,\rho_r(t_i) \ \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$):

$$a(t) \propto t^{2/3}$$

$$\Omega = 1 \ .$$

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2$$
$$+ r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or: $$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

BLACK-BODY RADIATION:

Whenever $kT \gg mc^2$ for any particle, where k is the Boltzmann constant, T is the temperature, and m is the (rest) mass of the particle, in thermal equilibrium there will be a black-body radiation, in which the particle will make the following contributions to the energy density, mass density, pressure, number density, and energy density:

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3} u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \, , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} \, , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 \, .$$

The values of g and g^* for photons, neutrinos, and electron-positron pairs are as follows:

$$g_\gamma = g^*_\gamma = 2 \, ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e, \nu_\mu, \nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \, ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e, \nu_\mu, \nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2} \, ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \ .$$

Spectrum of Black-Body Radiation:

The energy density for radiation in the frequency interval between ν and $\nu + \mathrm{d}\nu$ is given by

$$\rho_\nu(\nu)d\nu = \frac{8\pi^2\, g\, \hbar\nu^3}{c^3} \frac{1}{e^{2\pi\hbar\nu/kT} - 1} d\nu \ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \ .$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} \ ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 = -\frac{\Lambda c^4}{8\pi G} \ .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}\ ,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z}\ ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}\ .$$

Age of universe:

$$\begin{aligned} t_0 &= \frac{1}{H_0}\int_0^1 \frac{x\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}} \\ &= \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}}\ . \end{aligned}$$

Look-back time:

$$t_{\text{look-back}}(z) =$$

$$\frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}}\ .$$

PHYSICAL CONSTANTS:

$$G = 6.674\times10^{-11}\ \text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2} = 6.674\times10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381\times10^{-23}\,\text{joule/K} \\ &= 1.381\times10^{-16}\,\text{erg/K} \\ &= 8.617\times10^{-5}\,\text{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055\times10^{-34}\ \text{joule}\cdot\text{s} \\ &= 1.055\times10^{-27}\ \text{erg}\cdot\text{s} \\ &= 6.582\times10^{-16}\ \text{eV}\cdot\text{s} \end{aligned}$$

$$\begin{aligned} c &= 2.998\times10^{8}\ \text{m/s} \\ &= 2.998\times10^{10}\ \text{cm/s} \end{aligned}$$

$$\hbar c = 197.3\ \text{MeV-fm},\quad 1\ \text{fm} = 10^{-15}\ \text{m}$$

$$1\ \text{yr} = 3.156\times10^{7}\ \text{s}$$

$$1\ \text{eV} = 1.602\times10^{-19}\ \text{joule} = 1.602\times10^{-12}\ \text{erg}$$

$$\begin{aligned} 1\ \text{GeV} = 10^9\ \text{eV} &= 1.783\times10^{-27}\ \ \text{kg (where } c\equiv 1) \\ &= 1.783\times10^{-24}\ \text{g}\ . \end{aligned}$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\begin{aligned}
\ell_P &= \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-35}\text{ m} , \\
&= 1.616 \times 10^{-33}\text{ cm} , \\
t_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44}\text{ s} , \\
m_P &= \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-8}\text{ kg} , \\
&= 2.177 \times 10^{-5}\text{ g} , \\
E_P &= \sqrt{\frac{\hbar c^5}{G}} = 1.221 \times 10^{19}\text{ GeV} .
\end{aligned}$$

We do not have a complete quantum theory of gravity, but we expect the Planck scale to be the scale at which the effects of quantum gravity become significant. That is, we expect the effects of quantum gravity to be important for processes that involve distances of order ℓ_P or less, times of order t_P or less, or particles with masses of order m_P or greater, or energies of order E_P or greater.

CHEMICAL EQUILIBRIUM:

(This topic will NOT be included on Quiz 3, but the formulas are nonetheless included here for logical completeness. They will be relevant to Problem Set 9.)

General Ideal Gas, Relativistic or Not, Bosons or Fermions:

The number density of particles of type i with momenta within a box of size d^3p centered at $\vec{p}$ is given by

$$n_{i,\vec{p}}(\vec{p})\,\mathrm{d}^3p = \frac{\bar{g}_i}{(2\pi\hbar)^3}\,\frac{\mathrm{d}^3p}{\left[\exp\left(\frac{E_i(p)-\mu_i}{kT}\right) \pm 1\right]} ,$$

where

$$\begin{aligned}
\bar{g}_i &= \text{number of spin states of particle} \\
E_i(p) &= \sqrt{m_i^2c^4 + p^2c^2} = \text{energy of particle with momentum } p \\
m_i &= \text{mass of particle} \\
\mu_i &= \text{chemical potential} \\
\pm &= + \text{ for fermions, and } - \text{ for bosons.}
\end{aligned}$$

Note that unlike the quantities g and g^* defined in the section on black-body radiation, $\bar{g}_i$ simply counts spin states, with no correction factor associated with fermions.

Chemical potentials are assigned initially to conserved quantities (e.g., electric charge, baryon number, or lepton number), and are a way of specifying how much of these quantities are present. The chemical potential of any type of particle is the sum of the chemical potentials of its conserved quantities. For example, a proton has one unit of baryon number and one unit of electric charge, so $\mu_p = \mu_{\text{baryon}} + \mu_{\text{charge}}$.

The number density of particle i is given by

$$n_i = \frac{\bar{g}_i}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 \, \mathrm{d}p}{\left[\exp\left(\frac{E(p)-\mu_i}{kT}\right) \pm 1\right]} \ ,$$

and the energy density is given by

$$u_i = \frac{\bar{g}_i}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 \, E(p) \, \mathrm{d}p}{\left[\exp\left(\frac{E(p)-\mu_i}{kT}\right) \pm 1\right]} \ .$$

Ideal Dilute Gas of Nonrelativistic Particles:

The nonrelativistic, dilute gas limit of the formula above for the number density n_i is given by

$$n_i = \bar{g}_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \ .$$

where n_i = number density of particle

$\bar{g}_i$ = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

The formula above assumes that the gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($e^{(\mu_i - m_i c^2)/kT} \ll 1$).

For any reaction that is consistent with all conservation laws, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Consequently, the product of the number densities on the left-hand side, divided by the product of the number densities on the right-hand side, is always independent of all chemical potentials. For example, since $H + \gamma \longleftrightarrow p + e^-$ (hydrogen atom + photon $\longleftrightarrow$ proton + electron) is a possible reaction, $\mu_H + \mu_\gamma = \mu_p + \mu_{e^-}$, and therefore

$$\frac{n_H\, n_\gamma}{n_p\, n_{e^-}}$$

can be evaluated using the formula above for number densities, and all chemical potentials will cancel out. (Photons have no conserved quantities, so $\mu_\gamma \equiv 0$, so $n_H/(n_p\, n_{e^-})$ is also independent of any chemical potentials.)

PROBLEM LIST

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PROBLEM 1: DID YOU DO THE READING (2018)? *(20 points)*

(a) *(5 points)* Which one of the following statements about CMB is NOT correct?

(i) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(ii) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is <T> = 2.725K.

(iii) After the dipole distortion of the CMB is subtracted away, the temperature of the CMB varies by 0.3 microKelvin across the sky.

(iv) The photons of the CMB have mostly been traveling on straight lines since they were last scattered at $t \approx 370,000$ yr, at a location called the surface of last scattering.

(b) *(5 points)* The nonuniformities in the cosmic microwave background allow us to measure the ripples in the mass density of the universe at the time when the plasma combined to form neutral atoms, about 300,000 - 400,000 years after the big bang. These ripples are crucial for understanding what happened later, since they are the seeds which led to the complicated tapestry of galaxies, clusters of galaxies, and voids. Which of the following sentences describes how these ripples are created in the context of inflationary models:

(i) Magnetic monopoles can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(ii) Cosmic strings, which are linelike topological defects, can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(iii) They are generated by quantum fluctuations during inflation.

(iv) Since the early universe was very hot, there were large thermal fluctuations which ultimately evolved into the ripples in the mass density.

(c) *(5 points)* In Chapter 8 of *The First Three Minutes*, Steven Weinberg describes the future of the universe (assuming, as was thought then to be the case, that the cosmological constant is zero). One possibility that he discusses is that the cosmic matter density could be greater than the critical density. Assuming that we live in such a universe, which of the following statements is NOT true?

(i) The universe is finite and its expansion will eventually cease, giving way to an accelerating contraction.

(ii) Three minutes after the temperature reaches a thousand million degrees (10^9 K), the laws of physics guarantee that the universe will crunch, and time will stop.

(iii) During at least the early part of the contracting phase, we will be able to observe both redshifts and blueshifts.

(iv) When the universe has recontracted to one-hundredth its present size, the radiation background will begin to dominate the sky, with a temperature of about 300 K.

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(iv) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(v) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(vi) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

PROBLEM 2: DID YOU DO THE READING? (2016) *(25 points)*

Except for part (d), you should answer these questions by circling the one statement that is correct.

(a) *(5 points)* In the Epilogue of *The First Three Minutes*, Steve Weinberg wrote: "The more the universe seems comprehensible, the more it also seems pointless." The sentence was qualified, however, by a closing paragraph that points out that

(i) the quest of the human race to create a better life for all can still give meaning to our lives.

(ii) if the universe cannot give meaning to our lives, then perhaps there is an afterlife that will.

(iii) the complexity and beauty of the laws of physics strongly suggest that the universe must have a purpose, even if we are not aware of what it is.

(iv) the effort to understand the universe gives human life some of the grace of tragedy.

(b) *(5 points)* In the Afterword of *The First Three Minutes*, Weinberg discusses the baryon number of the universe. (The baryon number of any system is the total number of protons and neutrons (and certain related particles known as hyperons) minus the number of their antiparticles (antiprotons, antineutrons, antihyperons) that are contained in the system.) Weinberg concluded that

(i) baryon number is exactly conserved, so the total baryon number of the universe must be zero. While nuclei in our part of the universe are composed of protons and neutrons, the universe must also contain antimatter regions in which nuclei are composed of antiprotons and antineutrons.

(ii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. Since baryon number is conserved, this can only be explained by assuming that the excess baryons were put in at the beginning.

(iii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. This can be taken as a positive hint that baryon number is not conserved, which can happen if there exist as yet undetected heavy "exotic" particles.

(iv) it is possible that baryon number is not exactly conserved, but even if that is the case, it is not possible that the observed excess of matter over antimatter can be explained by the very rare processes that violate baryon number conservation.

(c) *(5 points)* In discussing the COBE measurements of the cosmic microwave background, Ryden describes a dipole component of the temperature pattern, for which the temperature of the radiation from one direction is found to be hotter than the temperature of the radiation detected from the opposite direction.

(i) This discovery is important, because it allows us to pinpoint the direction of the point in space where the big bang occurred.

(ii) This is the largest component of the CMB anisotropies, amounting to a 10% variation in the temperature of the radiation.

(iii) In addition to the dipole component, the anisotropies also includes contributions from a quadrupole, octupole, etc., all of which are comparable in magnitude.

(iv) This pattern is interpreted as a simple Doppler shift, caused by the net motion of the COBE satellite relative to a frame of reference in which the CMB is almost isotropic.

(d) *(5 points)* (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

(e) *(5 points)* Inflation is driven by a field that is by definition called the *inflaton* field. In standard inflationary models, the field has the following properties:

(i) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its potential energy.

(ii) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its potential energy.

(iii) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(iv) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(v) The inflaton is a tensor field, which is responsible for only a small fraction of the energy density of the universe during inflation.

PROBLEM 3: DID YOU DO THE READING (2013)? *(35 points)*

This was Problem 1 of Quiz 3, 2013.

(a) *(5 points)* Ryden summarizes the results of the COBE satellite experiment for the measurements of the cosmic microwave background (CMB) in the form of three important results. The first was that, in any particular direction of the sky, the spectrum of the CMB is very close to that of an ideal blackbody. The FIRAS instrument on the COBE satellite could have detected deviations from the blackbody spectrum as small as $\Delta\epsilon/\epsilon \approx 10^{-n}$, where n is an integer. To within ± 1, what is n?

(b) *(5 points)* The second result was the measurement of a dipole distortion of the CMB spectrum; that is, the radiation is slightly blueshifted to higher temperatures in one direction, and slightly redshifted to lower temperatures in the opposite direction. To what physical effect was this dipole distortion attributed?

(c) *(5 points)* The third result concerned the measurement of temperature fluctuations after the dipole feature mentioned above was subtracted out. Defining

$$\frac{\delta T}{T}(\theta,\phi) \equiv \frac{T(\theta,\phi) - \langle T\rangle}{\langle T\rangle} ,$$

where $\langle T \rangle = 2.725$ K, the average value of T, they found a root mean square fluctuation,

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} ,$$

equal to some number. To within an order of magnitude, what was that number?

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(e) *(5 points)* The flatness problem refers to the extreme fine-tuning that is needed in Ω at early times, in order for it to be as close to 1 today as we observe. Starting with the assumption that Ω today is equal to 1 within about 1%, one concludes that at one second after the big bang,

$$|\Omega - 1|_{t=1 \text{ sec}} < 10^{-m} ,$$

where m is an integer. To within $\pm$ 3, what is m?

(f) *(5 points)* The total energy density of the present universe consists mainly of baryonic matter, dark matter, and dark energy. Give the percentages of each, according to the best fit obtained from the Planck 2013 data. You will get full credit if the first (baryonic matter) is accurate to $\pm 2\%$, and the other two are accurate to within $\pm 5\%$.

(g) *(5 points)* Within the conventional hot big bang cosmology (without inflation), it is difficult to understand how the temperature of the CMB can be correlated at angular separations that are so large that the points on the surface of last scattering was separated from each other by more than a horizon distance. Approximately what angle, in degrees, corresponds to a separation on the surface last scattering of one horizon length? You will get full credit if your answer is right to within a factor of 2.

PROBLEM 4: DID YOU DO THE READING (2009)? *(25 points)*

This problem was Problem 1, Quiz 3, 2009.

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%).

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB.

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\text{bary}}/n_\gamma$.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct.

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5 (ii) 1/50 (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or F ?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. T or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. T or F ?

PROBLEM 5: DID YOU DO THE READING (2007)? *(25 points)*

The following problem was Problem 1, Quiz 3, in 2007. Each part was worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

(iii) Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

(iv) The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

(iii) matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

∗ PROBLEM 6: TIME EVOLUTION OF A UNIVERSE INCLUDING A HYPOTHETICAL KIND OF MATTER *(30 points)*

The following problem was Problem 2, Quiz 3, 2018.

Suppose that a flat universe includes nonrelativistic matter, radiation, and also mysticium, where the mass density of mysticium behaves as

$$\rho_{\text{myst}} \propto \frac{1}{a^5(t)}$$

as the universe expands. In this problem we will define

$$x(t) \equiv \frac{a(t)}{a(t_0)} \ ,$$

where t_0 is the present time. For the following questions, you need not evaluate any of the integrals that might arise, but they must be integrals of explicit functions with explicit limits of integration; remember that $a(t)$ is not given. You may express your answers in terms of the present value of the Hubble expansion rate, H_0, and the various contributions to the present value of Ω: $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{myst},0}$.

(a) *(7 points)* Write an expression for the Hubble expansion rate $H(t)$.

(b) *(7 points)* Write an expression for the current age of the universe.

(c) *(3 points)* Write an expression for the time $t(x)$ in terms of the value of x.

(d) *(3 points)* Write an expression for the total mass density $\rho(x)$ as a function of x.

(e) *(10 points)* Write an expression for the physical horizon distance, $\ell_{p,\text{hor}}$.

∗ PROBLEM 7: THE CONSEQUENCES OF AN ALT-PHOTON *(25 points)*

Suppose that, in addition to the particles that are known to exist, there also existed an alt-photon, which has exactly the properties of a photon: it is massless, has two spin states (or polarization states), and has the same interactions with other particles that photons do. Like photons, it is its own antiparticle.

(a) *(5 points)* In thermal equilibrium at temperature T, what is the total energy density of alt-photons?

(b) *(5 points)* In thermal equilibrium at temperature T, what is the number density of alt-photons?

(c) *(10 points)* In this situation, what would be the temperature ratios T_ν/T_γ and $T_\nu/T_{\text{alt}\gamma}$ today?

(d) *(5 points)* Would the existence of this particle increase or decrease the abundance of helium, or would it have no effect?

PROBLEM 8: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

PROBLEM 9: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos at temperature T. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

PROBLEM 10: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included.

(For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that $0.75 \text{ MeV} \ll kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of g/cm^3.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

∗ PROBLEM 11: A NEW THEORY OF THE WEAK INTERACTIONS *(40 points)*

This problem was Problem 3, Quiz 3, 2009.

Suppose a New Theory of the Weak Interactions (NTWI) was proposed, which differs from the standard theory in two ways. First, the NTWI predicts that the weak interactions are somewhat weaker than in the standard model. In addition, the theory implies the existence of new spin-$\frac{1}{2}$ particles (fermions) called the R^+ and R^-, with a rest energy of 50 MeV (where 1 MeV $= 10^6$ eV). This problem will deal with the cosmological consequences of such a theory.

The NTWI will predict that the neutrinos in the early universe will decouple at a higher temperature than in the standard model. Suppose that this decoupling takes place at $kT \approx 200$ MeV. This means that when the neutrinos cease to be thermally coupled to the rest of matter, the hot soup of particles would contain not only photons, neutrinos, and e^+-e^- pairs, but also μ^+, μ^-, π^+, π^-, and π^0 particles, along with the R^+-R^- pairs. (The muon is a particle which behaves almost identically to an electron, except that its rest energy is 106 MeV. The pions are the lightest of the mesons, with zero angular momentum and rest energies of 135 MeV and 140 MeV for the neutral and charged pions, respectively. The π^+ and π^- are antiparticles of each other, and the π^0

is its own antiparticle. Zero angular momentum implies a single spin state.) You may assume that the universe is flat.

(a) *(10 points)* According to the standard particle physics model, what is the mass density ρ of the universe when $kT \approx 200$ MeV? What is the value of ρ at this temperature, according to NTWI? Use either g/cm^3 or kg/m^3. (If you wish, you can save time by not carrying out the arithmetic. If you do this, however, you should give the answer in "calculator-ready" form, by which I mean an expression involving pure numbers (no units), with any necessary conversion factors included, and with the units of the answer specified at the end. For example, if asked how far light travels in 5 minutes, you could answer $2.998 \times 10^8 \times 5 \times 60$ m.)

(b) *(10 points)* According to the standard model, the temperature today of the thermal neutrino background should be $(4/11)^{1/3} T_\gamma$, where T_γ is the temperature of the thermal photon background. What does the NTWI predict for the temperature of the thermal neutrino background?

(c) *(10 points)* According to the standard model, what is the ratio today of the number density of thermal neutrinos to the number density of thermal photons? What is this ratio according to NTWI?

(d) *(10 points)* Since the reactions which interchange protons and neutrons involve neutrinos, these reactions "freeze out" at roughly the same time as the neutrinos decouple. At later times the only reaction which effectively converts neutrons to protons is the free decay of the neutron. Despite the fact that neutron decay is a weak interaction, we will assume that it occurs with the usual 15 minute mean lifetime. Would the helium abundance predicted by the NTWI be higher or lower than the prediction of the standard model? To within 5 or 10%, what would the NTWI predict for the percent abundance (by weight) of helium in the universe? (As in part (a), you can either carry out the arithmetic, or leave the answer in calculator-ready form.)

Useful information: The proton and neutron rest energies are given by $m_p c^2 = 938.27$ MeV and $m_n c^2 = 939.57$ MeV, with $(m_n - m_p)c^2 = 1.29$ MeV. The mean lifetime for the neutron decay, $n \to p + e^- + \bar{\nu}_e$, is given by $\tau = 886$ s.

PROBLEM 12: DOUBLING OF ELECTRONS *(10 points)*

The following was on Quiz 3, 2011 (Problem 4):

Suppose that instead of one species of electrons and their antiparticles, suppose there was also another species of electron-like and positron-like particles. Suppose that the new species has the same mass and other properties as the electrons and positrons. If this were the case, what would be the ratio T_ν/T_γ of the temperature today of the neutrinos to the temperature of the CMB photons.

PROBLEM 13: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 14: EVOLUTION OF FLATNESS *(15 points)*

The following problem was Problem 3, Quiz 3, 2004.

The "flatness problem" is related to the fact that during the evolution of the standard cosmological model, Ω is always driven away from 1.

(a) *(9 points)* During a period in which the universe is matter-dominated (meaning that the only relevant component is nonrelativistic matter), the quantity

$$\frac{\Omega - 1}{\Omega}$$

grows as a power of t, provided that Ω is near 1. Show that this is true, and derive the power. (Stating the right power without a derivation will be worth 3 points.)

(b) *(6 points)* During a period in which the universe is radiation-dominated, the same quantity will grow like a different power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will again be worth 3 points.)

In each part, you may assume that the universe was *always* dominated by the specified form of matter.

∗ PROBLEM 15: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

The following problem was Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\rm phys,0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\rm phys,e}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

PROBLEM 16: SECOND HUBBLE CROSSING *(40 points)*

This problem was Problem 3, Quiz 3, 2007. In 2018 we have not yet talked about Hubble crossings and the evolution of density perturbations, so this problem would not be fair as worded. Actually, however, you have learned how to do these calculations, so the problem would be fair if it described in more detail what needs to be calculated.

In Problem Set 9 (2007) we calculated the time $t_{H1}(\lambda)$ of the first Hubble crossing for a mode specified by its (physical) wavelength λ at the present time. In this problem we will calculate the time $t_{H2}(\lambda)$ of the second Hubble crossing, the time at which the growing Hubble length $cH^{-1}(t)$ catches up to the physical wavelength, which is also growing. At the time of the second Hubble crossing for the wavelengths of interest, the universe can be described very simply: it is a radiation-dominated flat universe. However, since λ is defined as the present value of the wavelength, the evolution of the universe between $t_{H2}(\lambda)$ and the present will also be relevant to the problem. We will need to use methods, therefore, that allow for both the matter-dominated era and the onset of the dark-energy-dominated era. As in Problem Set 9 (2007), the model universe that we consider will be described by the WMAP 3-year best fit parameters:

$$\begin{array}{lrcl}
\text{Hubble expansion rate} & H_0 & = & 73.5\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1} \\
\text{Nonrelativistic mass density} & \Omega_m & = & 0.237 \\
\text{Vacuum mass density} & \Omega_{\rm vac} & = & 0.763 \\
\text{CMB temperature} & T_{\gamma,0} & = & 2.725\ \mathrm{K}
\end{array}$$

The mass densities are defined as contributions to Ω, and hence describe the mass density of each constituent relative to the critical density. Note that the model is exactly flat, so you need not worry about spatial curvature. Here you are not expected to give a numerical answer, so the above list will serve only to define the symbols that can appear in your answers, along with λ and the physical constants G, $\hbar$, c, and k.

(a) *(5 points)* For a radiation-dominated flat universe, what is the Hubble length $\ell_H(t) \equiv cH^{-1}(t)$ as a function of time t?

(b) *(10 points)* The second Hubble crossing will occur during the interval

$$30 \text{ sec} \ll t \ll 50,000 \text{ years},$$

when the mass density of the universe is dominated by photons and neutrinos. During this era the neutrinos are a little colder than the photons, with $T_\nu = (4/11)^{1/3} T_\gamma$. The total energy density of the photons and neutrinos together can be written as

$$u_{\text{tot}} = g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{(\hbar c)^3} \ .$$

What is the value of g_1? (For the following parts you can treat g_1 as a given variable that can be left in your answers, whether or not you found it.)

(c) *(10 points)* For times in the range described in part (b), what is the photon temperature $T_\gamma(t)$ as a function of t?

(d) *(15 points)* Finally, we are ready to find the time $t_{H2}(\lambda)$ of the second Hubble crossing, for a given value of the physical wavelength λ today. Making use of the previous results, you should be able to determine $t_{H2}(\lambda)$. If you were not able to answer some of the previous parts, you may leave the symbols $\ell_H(t)$, g_1, and/or $T_\gamma(t)$ in your answer.

PROBLEM 17: THE EVENT HORIZON FOR OUR UNIVERSE *(25 points)*

The following problem was Problem 3 from Quiz 3, 2013.

We have learned that the expansion history of our universe can be described in terms of a small set of numbers: $\Omega_{m,0}$, the present contribution to Ω from nonrelativistic matter; $\Omega_{\text{rad},0}$, the present contribution to Ω from radiation; Ω_{vac}, the present contribution to Ω from vacuum energy; and H_0, the present value of the Hubble expansion rate. The best estimates of these numbers are consistent with a flat universe, so we can take $k = 0$, $\Omega_{m,0} + \Omega_{\text{rad},0} + \Omega_{\text{vac}} = 1$, and we can use the flat Robertson-Walker metric,

$$\mathrm{d}s^2 = -c^2\, dt^2 + a^2(t) \left[\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \ .$$

(a) *(5 points)* Suppose that we are at the origin of the coordinate system, and that at the present time t_0 we emit a spherical pulse of light. It turns out that there is a maximum coordinate radius $r = r_{\max}$ that this pulse will ever reach, no matter how long we wait. (The pulse will never actually reach $r_{\max}$, but will reach all r such that $0 < r < r_{\max}$.) $r_{\max}$ is the coordinate of what is called the *event horizon*: events that happen now at $r \geq r_{\rm max}$ will never be visible to us, assuming that we remain at the origin. Assuming for this part that the function $a(t)$ is a known function, write an expression for $r_{\rm max}$. Your answer should be expressed as an integral, which can involve $a(t)$, t_0, and any of the parameters defined in the preamble. [*Advice:* If you cannot answer this, you should still try part (c).]

(b) *(10 points)* Since $a(t)$ is not known explicitly, the answer to the previous part is difficult to use. Show, however, that by changing the variable of integration, you can rewrite the expression for $r_{\rm max}$ as a definite integral involving only the parameters specified in the preamble, without any reference to the function $a(t)$, except perhaps to its present value $a(t_0)$. You are not expected to evaluate this integral. [*Hint:* One method is to use

$$x = \frac{a(t)}{a(t_0)}$$

as the variable of integration, just as we did when we derived the first of the expressions for t_0 shown in the formula sheets.]

(c) *(10 points)* Astronomers often describe distances in terms of redshifts, so it is useful to find the redshift of the event horizon. That is, if a light ray that originated at $r = r_{\rm max}$ arrived at Earth today, what would be its redshift $z_{\rm eh}$ (eh = event horizon)? You are not asked to find an explicit expression for $z_{\rm eh}$, but instead an equation that could be solved numerically to determine $z_{\rm eh}$. For this part you can treat $r_{\rm max}$ as given, so it does not matter if you have done parts (a) and (b). You will get half credit for a correct answer that involves the function $a(t)$, and full credit for a correct answer that involves only explicit integrals depending only on the parameters specified in the preamble, and possibly $a(t_0)$.

PROBLEM 18: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

The following problem was Problem 2 of Quiz 3, 2016. It was also Problem 2 of Problem Set 7 (2016), except that some numerical constants have been changed, so the answers will not be identical.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^8(t)} \; .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{2}{3}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} \ .$$

Find the power n.

∗ PROBLEM 19: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X *(25 points)*

The following problem was Problem 3 of Quiz 3, 2016.

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, X^+, X^-, and X^0, all with masses 0.511 MeV/c^2, exactly the same as the electron. The X^- is the antiparticle of the X^+, and the X^0 is its own antiparticle. Since the X's are spin-1 particles with nonzero mass, each particle has three spin states.

The X's do not interact with neutrinos any more strongly than the electrons and positrons do, so when the X's freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total energy density of the X^+, X^-, and X^0 particles?

(b) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total number density of the X^+, X^-, and X^0 particles?

(c) *(10 points)* The X particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as kT decreases from values large compared to 0.511 MeV/c^2 to values that are small compared to it. If the X's, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio T_ν/T_γ, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

(d) *(5 points)* If the mass of the X's was, for example, 0.100 MeV/c^2, so that the electron-positron pairs froze out first, and then the X's froze out, would the final ratio T_ν/T_γ be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

PROBLEM 20: THE TIME t_d OF DECOUPLING *(25 points)*

The following problem was Problem 4 of Quiz 3, 2016.

The process by which the photons of the cosmic microwave background stop scattering and begin to travel on straight lines is called *decoupling*, and it happens at a photon temperature of about $T_d \approx 3,000$ K. In Lecture Notes 6 we estimated the time t_d of decoupling, working in the approximation that the universe has been matter-dominated from that time to the present. We found a value of 370,000 years. In this problem we will remove this approximation, although we will not carry out the numerical evaluation needed to compare with the previous answer.

(a) *(5 points)* Let us define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

as on the formula sheets, where t_0 is the present time. What is the value of $x_d \equiv x(t_d)$? Assume that the entropy of photons is conserved from time t_d to the present, and let T_0 denote the present photon temperature.

(b) *(5 points)* Assume that the universe is flat, and that $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ denote the present contributions to Ω from nonrelativistic matter, radiation, and vacuum energy, respectively. Let H_0 denote the present value of the Hubble expansion rate. Write an expression in terms of these quantities for $\mathrm{d}x/\mathrm{d}t$, the derivative of x with respect to t. *Hint: you may use formulas from the formula sheet without derivation, so this problem should require essentially no work. To receive full credit, your answer should include only terms that make a nonzero contribution to the answer.*

(c) *(5 points)* Write an expression for t_d. If your answer involves an integral, you need not try to evaluate it, but you should be sure that the limits of integration are clearly shown.

(d) *(10 points)* Now suppose that in addition to the constituents described in part (b), the universe also contains some of the fictitious material from part (a) of Problem 18 (Quiz 3 Review Problems, 2020), with

$$\rho(t) \propto \frac{1}{a^8(t)} .$$

Denote the present contribution to Ω from this fictitious material as $\Omega_{f,0}$. The universe is still assumed to be flat, so the numerical values of $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ must sum to a smaller value than in parts (b) and (c). With this extra contribution to the mass density of the universe, what is the new expression for t_d?

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING (2018)? *(20 points)*

(a) *(5 points)* Which one of the following statements about CMB is NOT correct?

(i) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(ii) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is <T> = 2.725K.

$\boxed{\text{(iii)}}$ After the dipole distortion of the CMB is subtracted away, the temperature of the CMB varies by 0.3 microKelvin across the sky.

(iv) The photons of the CMB have mostly been traveling on straight lines since they were last scattered at $t \approx 370,000$ yr, at a location called the surface of last scattering.

[Comment: The actual variation is about 30 microKelvin, or maybe a few times that much. Ryden quotes the COBE root mean square fractional variation of the CMB temperature as

$$< \left(\frac{\delta T}{T}\right)^2 >^{1/2} = 1.1 \times 10^{-5}$$

as Eq. (8.8) (2nd Edition), which gives a value of about 30 microKelvin, given that $T \approx 3\,K$*. In Lecture Notes 2 we quoted a value of* 4.14×10^{-5} *computed from Planck data. The root mean square fluctuations increase with better angular resolution, because fluctuations with small angular wavelengths are not seen unless the resolution is high.*

(b) *(5 points)* The nonuniformities in the cosmic microwave background allow us to measure the ripples in the mass density of the universe at the time when the plasma combined to form neutral atoms, about 300,000 - 400,000 years after the big bang. These ripples are crucial for understanding what happened later, since they are the seeds which led to the complicated tapestry of galaxies, clusters of galaxies, and voids. Which of the following sentences describes how these ripples are created in the context of inflationary models:

(i) Magnetic monopoles can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(ii) Cosmic strings, which are linelike topological defects, can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

$\boxed{\text{(iii)}}$ They are generated by quantum fluctuations during inflation.

(iv) Since the early universe was very hot, there were large thermal fluctuations which ultimately evolved into the ripples in the mass density.

(c) *(5 points)* In Chapter 8 of *The First Three Minutes*, Steven Weinberg describes the future of the universe (assuming, as was thought then to be the case, that the cosmological constant is zero). One possibility that he discusses is that the cosmic matter density could be greater than the critical density. Assuming that we live in such a universe, which of the following statements is NOT true?

(i) The universe is finite and its expansion will eventually cease, giving way to an accelerating contraction.

$\boxed{\text{(ii)}}$ Three minutes after the temperature reaches a thousand million degrees (10^9 K), the laws of physics guarantee that the universe will crunch, and time will stop.

(iii) During at least the early part of the contracting phase, we will be able to observe both redshifts and blueshifts.

(iv) When the universe has recontracted to one-hundredth its present size, the radiation background will begin to dominate the sky, with a temperature of about 300 K.

[Comment: Weinberg is very clear no speculations about the end of the universe are guaranteed to be true: "Does time really have to stop some three minutes after the temperature reaches a thousand million degrees? Obviously, we cannot be sure. All the uncertainties that we met in the preceding chapter, in trying to explore the first hundredth of a second, will return to perplex us as we look into the last hundredth of a second."]

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(iv) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

$\boxed{\text{(v)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(vi) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

[Comment: Ryden discusses the Sachs-Wolfe effect on pp. 161–162 (2nd Edition).]

PROBLEM 2: DID YOU DO THE READING (2016)? *(25 points)*

Except for part (d), you should answer these questions by circling the one statement that is correct.

(a) *(5 points)* In the Epilogue of *The First Three Minutes*, Steve Weinberg wrote: "The more the universe seems comprehensible, the more it also seems pointless." The sentence was qualified, however, by a closing paragraph that points out that

(i) the quest of the human race to create a better life for all can still give meaning to our lives.

(ii) if the universe cannot give meaning to our lives, then perhaps there is an afterlife that will.

(iii) the complexity and beauty of the laws of physics strongly suggest that the universe must have a purpose, even if we are not aware of what it is.

$\boxed{\text{(iv)}}$ the effort to understand the universe gives human life some of the grace of tragedy.

(b) *(5 points)* In the Afterword of *The First Three Minutes*, Weinberg discusses the baryon number of the universe. (The baryon number of any system is the total number of protons and neutrons (and certain related particles known as hyperons) minus the number of their antiparticles (antiprotons, antineutrons, antihyperons) that are contained in the system.) Weinberg concluded that

(i) baryon number is exactly conserved, so the total baryon number of the universe must be zero. While nuclei in our part of the universe are composed of protons and neutrons, the universe must also contain antimatter regions in which nuclei are composed of antiprotons and antineutrons.

(ii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. Since baryon number is conserved, this can only be explained by assuming that the excess baryons were put in at the beginning.

$\boxed{\text{(iii)}}$ there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. This can be taken as a positive hint that baryon number is not conserved, which can happen if there exist as yet undetected heavy "exotic" particles.

(iv) it is possible that baryon number is not exactly conserved, but even if that is the case, it is not possible that the observed excess of matter over antimatter can be explained by the very rare processes that violate baryon number conservation.

Explanation: All students were given credit for this part, whether they answered it correctly or not. I was in San Francisco when I made up this quiz, and due to poor planning I did not have my copy of *The First Three Minutes.* So I found a version online, but I could only find the British version, published by Flamingo/Fontana Paperbacks, rather than the US version published by Basic Books. I assumed that the "Afterword" in the two versions would be the same, but I was wrong! So this question was based on a different "Afterword" than the one that you read. 55% of you still got it right, but obviously the question was not fair. Apologies.

(c) *(5 points)* In discussing the COBE measurements of the cosmic microwave background, Ryden describes a dipole component of the temperature pattern, for which the temperature of the radiation from one direction is found to be hotter than the temperature of the radiation detected from the opposite direction.

(i) This discovery is important, because it allows us to pinpoint the direction of the point in space where the big bang occurred.

(ii) This is the largest component of the CMB anisotropies, amounting to a 10% variation in the temperature of the radiation.

(iii) In addition to the dipole component, the anisotropies also include contributions from a quadrupole, octupole, etc., all of which are comparable in magnitude.

$\boxed{\text{(iv)}}$ This pattern is interpreted as a simple Doppler shift, caused by the net motion of the COBE satellite relative to a frame of reference in which the CMB is almost isotropic.

Explanation: (i) is nonsense, since the conventional big bang theory descibes a completely homogeneous universe, which has no single point at which the big bang occurred. (ii) is wrong, because the variations in the temperature of the CMB are much smaller than 10%. The dipole term has a magnitude of about 1/1000 of the mean temperature. (iii) is wrong because the dipole is not comparable to the other terms, because they have magnitudes of only about 1/100,000 of the mean.

(d) *(5 points)* (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

$\boxed{\text{(ii)}}$ After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Explanation: The right value is

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5} \; .$$

(e) *(5 points)* Inflation is driven by a field that is by definition called the *inflaton* field. In standard inflationary models, the field has the following properties:

$\boxed{\text{(i)}}$ The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its potential energy.

(ii) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its potential energy.

(iii) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(iv) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(v) The inflaton is a tensor field, which is responsible for only a small fraction of the energy density of the universe during inflation.

Explanation: These facts were mentioned in both Section 11.5 (*The Physics of Inflation)* of Ryden's book, and also in the article that you were asked to read called *Inflation and the New Era of High-Precision Cosmology,* written by me for the Physics Department 2002 newsletter.

PROBLEM 3: DID YOU DO THE READING (2013)? *(35 points)*

(a) *(5 points)* Ryden summarizes the results of the COBE satellite experiment for the measurements of the cosmic microwave background (CMB) in the form of three important results. The first was that, in any particular direction of the sky, the spectrum of the CMB is very close to that of an ideal blackbody. The FIRAS instrument on the COBE satellite could have detected deviations from the blackbody spectrum as small as $\Delta\epsilon/\epsilon \approx 10^{-n}$, where n is an integer. To within ± 1, what is n?

Answer: $n = 4$

(b) *(5 points)* The second result was the measurement of a dipole distortion of the CMB spectrum; that is, the radiation is slightly blueshifted to higher temperatures in one direction, and slightly redshifted to lower temperatures in the opposite direction. To what physical effect was this dipole distortion attributed?

Answer: The large dipole in the CMB is attributed to the motion of the satellite relative to the frame in which the CMB is very nearly isotropic. (The entire Local Group is moving relative to this frame at a speed of about $0.002c$.)

(c) *(5 points)* The third result concerned the measurement of temperature fluctuations after the dipole feature mentioned above was subtracted out. Defining

$$\frac{\delta T}{T}(\theta, \phi) \equiv \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} \ ,$$

where $\langle T \rangle = 2.725$ K, the average value of T, they found a root mean square fluctuation,

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} \ ,$$

equal to some number. To within an order of magnitude, what was that number?

Answer:

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5} \ .$$

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

$\boxed{\text{(iii)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(e) *(5 points)* The flatness problem refers to the extreme fine-tuning that is needed in Ω at early times, in order for it to be as close to 1 today as we observe. Starting with the assumption that Ω today is equal to 1 within about 1%, one concludes that at one second after the big bang,

$$|\Omega - 1|_{t=1 \text{ sec}} < 10^{-m} ,$$

where m is an integer. To within ± 3, what is m?

Answer: $m = 18$. (See the derivation in Lecture Notes 8.)

(f) *(5 points)* The total energy density of the present universe consists mainly of baryonic matter, dark matter, and dark energy. Give the percentages of each, according to the best fit obtained from the Planck 2013 data. You will get full credit if the first (baryonic matter) is accurate to ±2%, and the other two are accurate to within ±5%.

Answer: Baryonic matter: 5%. Dark matter: 26.5%. Dark energy: 68.5%. The Planck 2013 numbers were given in Lecture Notes 7. To the requested accuracy, however, numbers such as Ryden's Benchmark Model would also be satisfactory.

(g) *(5 points)* Within the conventional hot big bang cosmology (without inflation), it is difficult to understand how the temperature of the CMB can be correlated at angular separations that are so large that the points on the surface of last scattering was separated from each other by more than a horizon distance. Approximately what angle, in degrees, corresponds to a separation on the surface last scattering of one horizon length? You will get full credit if your answer is right to within a factor of 2.

Answer: Ryden gives 1° as the angle subtended by the Hubble length on the surface of last scattering. For a matter-dominated universe, which would be a good model for our universe, the horizon length is twice the Hubble length. Any number from 1° to 5° was considered acceptable.

PROBLEM 4: DID YOU DO THE READING (2009)? *(25 points)*

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%). $\boxed{2.725\,\text{K}}$

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

The dipole anisotropy corresponds to a "peculiar velocity" (that is, velocity which is not due to the expansion of the universe) of $630 \pm 20\,\mathrm{km\,s^{-1}}$*, or in terms of the speed of light,* $\boxed{v/c \approx 2 \times 10^{-3}}$.

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB. $\boxed{1.1 \times 10^{-5}}$

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\rm bary}/n_\gamma$.

The WMAP 5-year value for $\eta = n_b/n_\gamma = \boxed{(6.225 \pm 0.170) \times 10^{-10}}$*, which to closest order of magnitude is* 10^{-9}.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct. $\boxed{\sim 1^\circ}$

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5 $\boxed{\text{(ii) } 1/50}$ (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

This is not a number one has to commit to memory if one can remember the temperature of (re)combination in eV*, or if only in* K *along with the conversion factor (*$k \approx 10^{-4}\,\mathrm{eV\,K^{-1}}$*). One can then calculate that near recombination,* $kT_\gamma/Q_H \approx (10^{-4}\,\mathrm{eV\,K^{-1}})(3000\,\mathrm{K})/(13.6\,\mathrm{eV}) \approx 1/45$.

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

$\boxed{\text{(iii)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

Explanation: Denser regions have a deeper (more negative) gravitational potential. Photons which travel through a spatially varying potential acquire a redshift or blueshift depending on whether they are going up or down the potential, respectively. Photons originating in the denser regions start at a lower potential and must climb out, so they end up being redshifted relative to their original energies.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or $\boxed{\text{F}}$?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. $\boxed{\text{T}}$ or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or $\boxed{\text{F}}$?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or $\boxed{\text{F}}$?

(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. $\boxed{\text{T}}$ or F ?

PROBLEM 5: DID YOU DO THE READING? (2007) *(25 points)*

The following parts are each worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

$\boxed{\text{(ii)}}$ After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Explanation: After subtracting the dipole contribution, the temperature fluctuation is about 1.1×10^{-5}.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

$\boxed{\text{(iii)}}$ Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

Explanation: The North Pole is at sea level. In contrast, the South Pole is nearly 3 kilometers above sea level. BOOMERANG is a balloon-borne experiment launched from Antarctica.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

$\boxed{\text{(iv)}}$ The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

Explanation: These peaks are due to the acoustic oscillations in the photon-baryon fluid.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

$\boxed{\text{(iii)}}$ matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

Explanation: Matter made of top quarks is so unstable that it is seen only fleetingly as a product in high energy particle collisions.

(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

Answers:

(i) Galaxy rotation curves. (I.e., measurements of the orbital speed of stars in spiral galaxies as a function of radius R show that these curves remain flat at radii far beyond the visible stellar disk. If most of the matter were contained in the disk, then these velocities should fall off as $1/\sqrt{R}$.)

(ii) Use the virial theorem to estimate the mass of a galaxy cluster. (For example, the virial analysis shows that only 2% of the mass of the Coma cluster consists of stars, and only 10% consists of hot intracluster gas.

(iii) Gravitational lensing. (For example, the mass of a cluster can be estimated from the distortion of the shapes of the galaxies behind the cluster.)

(iv) CMB temperature fluctuations. (I.e., the analysis of the intensity of the fluctuations as a function of multipole number shows that $\Omega_{\text{tot}} \approx 1$, and that dark energy contributes $\Omega_\Lambda \approx 0.7$, baryonic matter contributes $\Omega_{\text{bary}} \approx 0.04$, and dark matter contributes $\Omega_{\text{dark matter}} \approx 0.26$.)

There are other possible answers as well, but these are the ones discussed by Ryden in Chapters 8 and 9.

PROBLEM 6: TIME EVOLUTION OF A UNIVERSE INCLUDING A HYPOTHETICAL KIND OF MATTER *(30 points)*

Suppose that a flat universe includes nonrelativistic matter, radiation, and also mysticium, where the mass density of mysticium behaves as

$$\rho_{\text{myst}} \propto \frac{1}{a^5(t)}$$

as the universe expands. In this problem we will define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

where t_0 is the present time. For the following questions, you need not evaluate any of the integrals that might arise, but they must be integrals of explicit functions with explicit limits of integration; remember that $a(t)$ is not given. You may express your answers in terms of the present value of the Hubble expansion rate, H_0, and the various contributions to the present value of Ω: $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{myst},0}$.

(a) *(7 points)* Write an expression for the Hubble expansion rate $H(x)$.

(b) *(7 points)* Write an expression for the current age of the universe.

(c) *(3 points)* Write an expression for the time $t(x)$ in terms of the value of x.

(d) *(3 points)* Write an expression for the total mass density $\rho(x)$ as a function of x.

(e) *(10 points)* Write an expression for present value of the physical horizon distance, $\ell_{p,\text{hor}}(t_0)$.

Solution:

(a) Since the universe is flat, the first Friedmann equation becomes

$$H^2 = \frac{8\pi}{3} G\rho ,$$

but then we can write ρ as

$$H^2 = \frac{8\pi}{3} G \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{\text{rad},0} \left[\frac{a(t_0)}{a(t)}\right]^4 + \rho_{\text{myst},0} \left[\frac{a(t_0)}{a(t)}\right]^5 \right\} .$$

Now use

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \text{ and } \Omega \equiv \frac{\rho}{\rho_c} ,$$

so

$$\begin{aligned} H^2 &= \frac{H_0^2}{\rho_{c,0}} \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{\text{rad},0} \left[\frac{a(t_0)}{a(t)}\right]^4 + \rho_{\text{myst},0} \left[\frac{a(t_0)}{a(t)}\right]^5 \right\} \\ &= H_0^2 \left\{ \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \frac{\Omega_{\text{myst},0}}{x^5} \right\} . \end{aligned}$$

Finally,

$$\boxed{H(x) = \frac{H_0}{x^2} \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \frac{\Omega_{\text{myst},0}}{x}} .}$$

(b) To find the current age t_0, we start with

$$H = \frac{\dot a}{a} = \frac{\dot x}{x} \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = xH \quad \Longrightarrow \quad \mathrm{d}t = \frac{\mathrm{d}x}{xH} \ .$$

So t_0 can be found by integrating over the range of x, from 0 to 1:

$$\begin{aligned} t_0 &= \int_0^1 \frac{\mathrm{d}x}{xH(x)} \\ &= \boxed{\frac{1}{H_0}\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}}} \ .} \end{aligned}$$

(c) To find the time t corresponding to some value of x other than 1, one simply integrates $\mathrm{d}t$ from $x' = 0$ to $x' = x$:

$$\begin{aligned} t(x) &= \int_0^x \frac{\mathrm{d}x'}{x'H(x')} \\ &= \boxed{\frac{1}{H_0}\int_0^x \frac{x'\,\mathrm{d}x'}{\sqrt{\Omega_{m,0}x' + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x'}}} \ .} \end{aligned}$$

(d) From the first Friedmann equation,

$$H^2 = \frac{8\pi}{3}G\rho \quad \Longrightarrow \quad \rho = \frac{3}{8\pi G}H^2(x) \ .$$

Given the answer in part (a), this becomes

$$\boxed{\rho(x) = \frac{3}{8\pi G}\frac{H_0^2}{x^4}\left[\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}\right] \ .}$$

(e) The general formula for the physical horizon distance is given on the formula sheet:

$$\ell_{p,\mathrm{hor}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' \ .$$

Here we are not given the function $a(t)$, but we can change the variable of integration to integrate over x:

$$dt' = \frac{dt'}{da}da = \frac{1}{\dot{a}}da = \frac{1}{a}\frac{a}{\dot{a}}da = \frac{da}{aH(x)} \ .$$

So

$$\begin{aligned}
\ell_{p,\text{hor}}(t_0) &= a(t_0)\int_0^{a(t_0)} \frac{c\,da}{a^2 H(a)} \\
&= \int_0^1 \frac{c\,dx}{x^2 H(x)} \\
&= \boxed{\frac{c}{H_0}\int_0^1 \frac{dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \frac{\Omega_{\text{myst},0}}{x}}} \ .}
\end{aligned}$$

PROBLEM 7: THE CONSEQUENCES OF AN ALT-PHOTON *(25 points)*

Suppose that, in addition to the particles that are known to exist, there also existed an alt-photon, which has exactly the properties of a photon: it is massless, has two spin states (or polarization states), and has the same interactions with other particles that photons do. Like photons, it is its own antiparticle.

(a) *(5 points)* In thermal equilibrium at temperature T, what is the total energy density of alt-photons?

(b) *(5 points)* In thermal equilibrium at temperature T, what is the number density of alt-photons?

(c) *(10 points)* In this situation, what would be the temperature ratios T_ν/T_γ and $T_\nu/T_{\text{alt}\gamma}$ today?

(d) *(5 points)* Would the existence of this particle increase or decrease the abundance of helium, or would it have no effect?

Solution:

(a) The energy density will be the same as for photons, since there is no difference. The general formula is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \ ,$$

as given on the formula sheets, and $g = 2$ for alt-photons (or photons), since there are two polarization states, and the particles are bosons. So

$$\boxed{u_{\text{alt}\gamma} = \frac{\pi^2}{15} \frac{(kT)^4}{(\hbar c)^3} .} \tag{4.1}$$

(b) For the number density, the general formula is

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} ,$$

where $g^* = 2$ since again the alt-photons are bosons with two polarization states. So

$$\boxed{n_{\text{alt}\gamma} = 2 \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} .} \tag{4.2}$$

(c) As in the actual scenario, the event that causes a temperature difference is the disappearance of the electron-positron pairs from the thermal equilibrium mix, which occurs as kT changes from values large compared to $m_e c^2 = 0.511$ MeV to values that are small compared to it. The key point is that this disappearance occurs after the neutrinos have decoupled from the other particles, so all of the entropy from the electron-positron pairs is given to the photons, and none is given to the neutrinos. In this case the entropy is given to both the photons and the alt-photons.

The general formula for entropy density is on the formula sheet, and it can be rewritten as

$$s = AgT^3 , \tag{4.3}$$

where

$$A = \frac{2\pi^2}{45} \frac{k^4}{(\hbar c)^3} . \tag{4.4}$$

The value of A will in fact not be needed for this problem.

Since the neutrinos have decoupled by the time the e^+e^- pairs disappear, the entropy of neutrinos and the entropy of everything else will be separately conserved. Entropy conservation means that the entropy per comoving volume does not change. During the period before e^+e^- freeze-out, g is constant, so the constancy of entropy per comoving volume implies that

$$S = sV_{\text{phys}} = gT^3 AV_{\text{phys}} = ga^3 T^3 AV_{\text{coord}} , \tag{4.5}$$

so S/V_{coord} = const implies that a^3T^3 is constant, and so aT is constant. Here T is the common temperature of photons, alt-photons, electrons and positrons, and neutrinos, all of which were in thermal equilibrium during this period. Since aT is constant during this period, we can give the constant a name,

$$aT = [aT]_{\text{before}} \ . \tag{4.6}$$

For the neutrinos, the formula sheet tells us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \ , \tag{4.7}$$

while

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ . \tag{4.8}$$

Thus

$$g_{\text{else}} = g_\gamma + g_{\text{alt}\gamma} + g_{e^+e^-} = 2 + 2 + \frac{7}{2} = \frac{15}{2} \ . \tag{4.9}$$

Thus before the e^+e^- freezeout, the two conserved quantities were

$$\frac{S_\nu}{V_{\text{coord}}} = A g_\nu [aT]^3_{\text{before}} \ , \quad \frac{S_{\text{else}}}{V_{\text{coord}}} = A g_{\text{else}} [aT]^3_{\text{before}} \ . \tag{4.10}$$

After e^+e^- freezeout, the temperature of the neutrinos T_ν will no longer be the same as the temperature T_γ of the photons and alt-photons, and of course e^+e^- pairs will no longer be present. But T_γ and $T_{\text{alt}\gamma}$ will be equal to each other, since they have the same interactions; we know that the interactions of the photons keep them in thermal equilibrium until $t_{\text{decoupling}} \sim 380,000$ years, so both the photons and the alt-photons will remain in thermal equilibrium until long after the era of e^+e^- freezeout, which is of order 1–10 seconds. Thus the two conserved quantities will be

$$\frac{S_\nu}{V_{\text{coord}}} = A g_\nu [aT_\nu]^3_{\text{after}} \ , \quad \frac{S_{\text{else}}}{V_{\text{coord}}} = A (g_\gamma + g_{\text{alt}\gamma}) [aT_\gamma]^3_{\text{after}} \ . \tag{4.11}$$

By equating the values of S_ν/V_{coord} before and after, we see that

$$[aT_\nu]_{\text{after}} = [aT]_{\text{before}} \ , \tag{4.12}$$

and then by equating the values of $S_{\text{else}}/V_{\text{coord}}$ before and after, we see that

$$[aT_\gamma]_{\text{after}} = \left(\frac{g_{\text{else}}}{g_\gamma + g_{\text{alt}\gamma}}\right)^{1/3} [aT]_{\text{before}} = \left(\frac{g_{\text{else}}}{g_\gamma + g_{\text{alt}\gamma}}\right)^{1/3} [aT_\nu]_{\text{after}} \ , \tag{4.13}$$

where we used Eq. (4.12) in the last step. It follows that

$$\left[\frac{T_\nu}{T_\gamma}\right]_{\text{after}} = \left(\frac{g_\gamma + g_{\text{alt}\gamma}}{g_{\text{else}}}\right)^{1/3} = \left(\frac{2+2}{\frac{15}{2}}\right)^{1/3} = \boxed{\left(\frac{8}{15}\right)^{1/3} .} \tag{4.14}$$

(d) It would increase the abundance of helium. The main effect of the alt-photon would be to increase the expansion rate of the universe, which in turn would cause the neutrinos to decouple earlier from the thermal equilibrium mix, which in turn would mean that the ratio n_n/n_p, the ratio of neutrons to protons, would become frozen at a larger value. The increased expansion rate would also mean less time available for free neutron decay, which further increases the number of neutrons that remain when the temperature falls low enough for helium formation to complete. Essentially all the neutrons become bound into helium, so more neutrons implies more helium.

PROBLEM 8: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16} \text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27} \text{erg-sec} \\ c &= 2.998 \times 10^{10} \text{cm/sec} \end{aligned}\right\} \implies \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3 \text{cm}^{-3} .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 .}$$

For the neutrinos,

$$g^*_\nu = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of 3/4 arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g_\nu^* = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11} T_\gamma^3$$

from Problem 8 of Problem Set 3 (2000), one finds

$$\begin{aligned} n_\nu &= \left(\frac{g_\nu^*}{g_\gamma^*}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3} \\ \Longrightarrow &\quad \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}} \end{aligned}$$

PROBLEM 9: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$\begin{aligned} E &= \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{\pi^4}{30\zeta(3)} kT \ .} \end{aligned}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701 \, kT \ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is 7/8 times the number of spin states, and g^* is 3/4 times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)}\,kT\ .}$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{135\zeta(3)}k\ .}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 10: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$n = 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\ \cancel{\text{MeV}}}{6.582\times10^{-16}\ \cancel{\text{eV}}\text{-}\cancel{\text{sec}}\times2.998\times10^{10}\ \cancel{\text{cm}}\text{-}\cancel{\text{sec}}^{-1}}\right)^3 \times\left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3\times\left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}}\right)^3$$

$$= 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\times10^6\times10^2}{6.582\times10^{-16}\times2.998\times10^{10}}\right)^3\ \text{m}^{-3}\ .$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}}\right)^3 \quad .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $a(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{a}}{a} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}\ \text{cm}^3\ \text{sec}^{-2}\ \text{g}^{-1}$, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9\ \text{g/cm}^3$.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion − anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/a$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\rm rest} T^3(t) a^3(t)$$

where $g_{\rm rest} = g_{\rm tot} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 a^3(t) = g_{\rm rest} T^3(t) a^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\rm rest}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 11: A NEW THEORY OF THE WEAK INTERACTIONS *(40 points)*

(a) In the standard model, the black-body radiation at $kT \approx 200$ MeV contains the following contributions:

$$\left.\begin{array}{ll} \text{Photons:} & g = 2 \\ e^+e^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \nu_e, \nu_\mu, \nu_\tau: & g = 6 \times \frac{7}{8} = 5\frac{1}{4} \\ \mu^+\mu^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \pi^+\pi^-\pi^0 & g = 3 \end{array}\right\} g_{\rm TOT} = 17\frac{1}{4}$$

The mass density is then given by

$$\rho = \frac{u}{c^2} = g_{\rm TOT} \frac{\pi^2}{30} \frac{(kT)^4}{\hbar^3 c^5} \quad .$$

In kg/m^3, one can evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right) \frac{\pi^2}{30} \frac{\left[200 \times 10^6 \, \text{eV} \times \dfrac{1.602 \times 10^{-19}\,\text{J}}{\text{eV}}\right]^4}{(1.055 \times 10^{-34}\,\text{J-s})^3 \, (2.998 \times 10^8\,\text{m/s})^5} \quad .$$

Checking the units,

$$[\rho] = \frac{\text{J}^4}{\text{J}^3\text{-s}^3\text{-m}^5\text{-s}^{-5}} = \frac{\text{J-s}^2}{\text{m}^5}$$
$$= \frac{\left(\text{kg-m}^2\text{-s}^{-2}\right)\text{s}^2}{\text{m}^5} = \text{kg/m}^3 \ .$$

So, the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6\times 1.602\times 10^{-19}\right]^4}{(1.055\times 10^{-34})^3\,(2.998\times 10^8)^5}\,\frac{\text{kg}}{\text{m}^3} \ .}$$

You were not expected to evaluate this, but with a calculator one would find

$$\rho = 2.10\times 10^{18}\ \text{kg/m}^3 \ .$$

In g/cm^3, one would evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6\,\text{eV}\times \dfrac{1.602\times 10^{-12}\,\text{erg}}{\text{eV}}\right]^4}{(1.055\times 10^{-27}\,\text{erg-s})^3\,(2.998\times 10^{10}\,\text{cm/s})^5} \ .$$

Checking the units,

$$[\rho] = \frac{\text{erg}^4}{\text{erg}^3\text{-s}^3\text{-cm}^5\text{-s}^{-5}} = \frac{\text{erg-s}^2}{\text{cm}^5}$$
$$= \frac{\left(\text{g-cm}^2\text{-s}^{-2}\right)\text{s}^2}{\text{cm}^5} = \text{g/cm}^3 \ .$$

So, in this case the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6\times 1.602\times 10^{-12}\right]^4}{(1.055\times 10^{-27})^3\,(2.998\times 10^{10})^5}\,\frac{\text{g}}{\text{cm}^3} \ .}$$

No evaluation was requested, but with a calculator you would find

$$\rho = 2.10\times 10^{15}\ \text{g/cm}^3 \ ,$$

which agrees with the answer above.

Note: A common mistake was to leave out the conversion factor 1.602×10^{-19} J/eV (or 1.602×10^{-12} erg/eV), and instead to use $\hbar = 6.582 \times 10^{-16}$ eV-s. But if one works out the units of this answer, they turn out to be eV-sec^2/m^5 (or eV-sec^2/cm^5), which is a most peculiar set of units to measure a mass density.

In the NTWI, we have in addition the contribution to the mass density from R^+-R^- pairs, which would act just like e^+-e^- pairs or μ^+-μ^- pairs, with $g = 3\frac{1}{2}$. Thus $g_{\rm TOT} = 20\frac{3}{4}$, so

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-19}\right]^4}{(1.055 \times 10^{-34})^3\,(2.998 \times 10^8)^5}\frac{\text{kg}}{\text{m}^3}}$$

or

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-12}\right]^4}{(1.055 \times 10^{-27})^3\,(2.998 \times 10^{10})^5}\frac{\text{g}}{\text{cm}^3}\ .}$$

Numerically, the answer in this case would be

$$\rho_{\rm NTWI} = 2.53 \times 10^{18}\ \text{kg/m}^3 = 2.53 \times 10^{15}\ \text{g/cm}^3\ .$$

(b) As long as the universe is in thermal equilibrium, entropy is conserved. The entropy in a given volume of the comoving coordinate system is

$$a^3(t)s\,V_{\rm coord}\quad ,$$

where s is the entropy density and $a^3 V_{\rm coord}$ is the physical volume. So

$$a^3(t)s$$

is conserved. After the neutrinos decouple,

$$a^3 s_\nu \quad \text{and} \quad a^3 s_{\rm other}$$

are separately conserved, where $s_{\rm other}$ is the entropy of everything except neutrinos.

Note that s can be written as

$$s = gAT^3\quad ,$$

where A is a constant. Before the disappearance of the e, μ, R, and π particles from the thermal equilibrium radiation,

$$s_\nu = \left(5\frac{1}{4}\right) AT^3$$

$$s_{\text{other}} = \left(15\frac{1}{2}\right) AT^3 \quad .$$

So

$$\frac{s_\nu}{s_{\text{other}}} = \frac{5\frac{1}{4}}{15\frac{1}{2}} \quad .$$

If $a^3 s_\nu$ and $a^3 s_{\text{other}}$ are conserved, then so is s_ν/s_{other}. By today, the entropy previously shared among the various particles still in equilibrium after neutrino decoupling has been transfered to the photons so that

$$s_{\text{other}} = s_{\text{photons}} = 2AT_\gamma^3 \quad .$$

The entropy in neutrinos is still

$$s_\nu = \left(5\frac{1}{4}\right) AT_\nu^3 \quad .$$

Since s_ν/s_{other} is constant we know that

$$\frac{\left(5\frac{1}{4}\right) T_\nu^3}{2T_\gamma^3} = \frac{s_\nu}{s_{\text{other}}} = \frac{5\frac{1}{4}}{15\frac{1}{2}}$$

$$\Longrightarrow \quad \boxed{T_\nu = \left(\frac{4}{31}\right)^{1/3} T_\gamma} \quad .$$

(c) One can write

$$n = g^* BT^3 \quad ,$$

where B is a constant. Here $g_\gamma^* = 2$, and $g_\nu^* = 6 \times \frac{3}{4} = 4\frac{1}{2}$. In the standard model, one has today

$$\frac{n_\nu}{n_\gamma} = \frac{g_\nu^* T_\nu^3}{g_\gamma^* T_\gamma^3} = \frac{\left(4\frac{1}{2}\right)}{2}\frac{4}{11} = \boxed{\frac{9}{11}\ .}$$

In the NTWI,

$$\frac{n_\nu}{n_\gamma} = \frac{\left(4\frac{1}{2}\right)}{2}\,\frac{4}{31} = \boxed{\frac{9}{31}\;.}$$

(d) At $kT = 200$ MeV, the thermal equilibrium ratio of neutrons to protons is given by

$$\frac{n_{\rm n}}{n_{\rm p}} = e^{-1.29\,{\rm MeV}/200\,{\rm MeV}} \approx 1 \quad .$$

In the standard theory this ratio would decrease rapidly as the universe cooled and kT fell below the p-n mass difference of 1.29 MeV, but in the NTWI the ratio freezes out at the high temperature corresponding to $kT = 200$ MeV, when the ratio is about 1. When kT falls below 200 MeV in the NTWI, the neutrino interactions

$$n + \nu_e \leftrightarrow p + e^- \qquad \text{and} \qquad n + e^+ \leftrightarrow p + \bar{\nu}_e$$

that maintain the thermal equilibrium balance between protons and neutrons no longer occur at a significant rate, so the ratio $n_/n_p$ is no longer controlled by thermal equilibrium. After kT falls below 200 MeV, the only process that can convert neutrons to protons is the rather slow process of free neutron decay, with a decay time τ_d of about 890 s. Thus, when the deuterium bottleneck breaks at about 200 s, the number density of neutrons will be considerably higher than in the standard model. Since essentially all of these neutrons will become bound into He nuclei, the higher neutron abundance of the NTWI implies a

$$\boxed{\text{higher predicted He abundance.}}$$

To estimate the He abundance, note that if we temporarily ignore free neutron decay, then the neutron-proton ratio would be frozen at about 1 and would remain 1 until the time of nucleosynthesis. At the time of nucleosynthesis essentially all of these neutrons would be bound into He nuclei (each with 2 protons and 2 neutrons). For an initial 1:1 ratio of neutrons to protons, all the neutrons and protons can be bound into He nuclei, with no protons left over in the form of hydrogen, so Y would equal 1. However, the free neutron decay process will cause the ratio n_n/n_p to fall below 1 before the start of nucleosynthesis, so the predicted value of Y would be less than 1.

To calculate how much less, note that Ryden estimates the start of nucleosynthesis at the time when the temperature reaches $T_{\rm nuc}$, which is the temperature for which a thermal equilibrium calculation gives $n_D/n_n = 1$. This corresponds to what Weinberg refers to as the breaking of the deuterium bottleneck. The temperature

$T_{\rm nuc}$ is calculated in terms of $\eta = n_B/n_\gamma$ and physical constants, so it would not be changed by the NTWI. The time when this temperature is reached, however, would be changed slightly by the change in the ratio T_ν/T_γ. Since this effect is rather subtle, no points will be taken off if you omitted it. However, to be as accurate as possible, one should recognize that nucleosynthesis occurs during the radiation-dominated era, but long after the e^+-e^- pairs have disappeared, so the black-body radiation consists of photons at temperature T_γ and neutrinos at a lower temperature T_ν. The energy density is given by

$$u = \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3}\left[2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4\right] \equiv g_{\rm eff}\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3}\ ,$$

where

$$g_{\rm eff} = 2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4 .$$

For the standard model

$$g_{\rm eff}^{\rm sm} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{11}\right)^{4/3} ,$$

and for the NTWI

$$g_{\rm eff}^{\rm NTWI} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{31}\right)^{4/3} .$$

The relation between time and temperature in a flat radiation-dominated universe is given in the formula sheets as

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\frac{1}{\sqrt{t}} .$$

Thus,

$$t \propto \frac{1}{g_{\rm eff}^{1/2}T^2} .$$

In the standard model Ryden estimates the time of nucleosynthesis as $t_{\rm nuc}^{\rm sm} \approx 200$ s, so in the NTWI it would be longer by the factor

$$t_{\rm nuc}^{\rm NTWI} = \sqrt{\frac{g_{\rm eff}^{\rm sm}}{g_{\rm eff}^{\rm NTWI}}}\, t_{\rm nuc}^{\rm sm} .$$

While of coure you were not expected to work out the numerics, this gives

$$t_{\rm nuc}^{\rm NTWI} = 1.20\, t_{\rm nuc}^{\rm sm} .$$

Note that Ryden gives $t_{\rm nuc} \approx 200s$, while Weinberg places it at $3\frac{3}{4}$ minutes ≈ 225 s, which is close enough.

To follow the effect of this free decay, it is easiest to do it by considering the ratio neutrons to baryon number, n_n/n_B, since n_B does not change during this period. At freeze-out, when $kT \approx 200$ MeV,

$$\frac{n_n}{n_B} \approx \frac{1}{2} \ .$$

Just before nucleosynthesis, at time $t_{\rm nuc}$, the ratio will be

$$\frac{n_n}{n_B} \approx \frac{1}{2} e^{-t_{\rm nuc}/\tau_d} \ .$$

If free decay is ignored, we found $Y = 1$. Since all the surviving neutrons are bound into He, the corrected value of Y is simply deceased by multiplying by the fraction of neutrons that do not undergo decay. Thus, the prediction of NTWI is

$$\boxed{Y = e^{-t_{\rm nuc}/\tau_d} = \exp\left\{ -\frac{\sqrt{\frac{g^{\rm sm}_{\rm eff}}{g^{\rm NTWI}_{\rm eff}}}\, 200}{890} \right\}} \ ,$$

where $g^{\rm sm}_{\rm eff}$ and $g^{\rm NTWI}_{\rm eff}$ are given above. When evaluated numerically, this would give

$$Y = \text{Predicted He abundance by weight} \approx 0.76 \ .$$

PROBLEM 12: DOUBLING OF ELECTRONS *(10 points)*

The entropy density of black-body radiation is given by

$$\begin{aligned} s &= g \left[\frac{2\pi^2}{45} \frac{k^4}{(\hbar c)^3} \right] T^3 \\ &= g\, C\, T^3 \ , \end{aligned}$$

where C is a constant. At the time when the electron-positron pairs disappear, the neutrinos are decoupled, so their entropy is conserved. All of the entropy from electron-positron pairs is given to the photons, and none to the neutrinos. The same will be true here, for both species of electron-positron pairs.

The conserved neutrino entropy can be described by $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation n^- and n^+ for the new electron-like and positron-like particles, and also the convention that

Primed quantities:	values after $e^+e^-n^+n^-$ annihilation
Unprimed quantities:	values before $e^+e^-n^+n^-$ annihilation.

For the neutrinos,

$$S'_\nu = S_\nu \quad \Longrightarrow \quad g_\nu C\,(a'T'_\nu)^3 = g_\nu C\,(aT_\nu)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\nu = aT_\nu \ .}$$

For the photons, before $e^+e^-n^+n^-$ annihilation we have

$$T_\gamma = T_{e^+e^-n^+n^-} = T_\nu \ ; \qquad g_\gamma = 2,\ g_{e^+e^-} = g_{n^+n^-} = 7/2 \ .$$

When the e^+e^- and n^+n^- pairs annihilate, their entropy is added to the photons:

$$S'_\gamma = S_{e^+e^-} + S_{n^+n^-} + S_\gamma \quad \Longrightarrow \quad 2C\left(a'T'_\gamma\right)^3 = \left(2 + 2\cdot\frac{7}{2}\right)C\left(aT_\gamma\right)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\gamma = \left(\frac{9}{2}\right)^{1/3} aT_\gamma \ ,}$$

so aT_γ increases by a factor of $(9/2)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{2}{9}\right)^{1/3} T'_\gamma \ .}$$

This ratio would remain unchanged until the present day.

PROBLEM 13: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto a^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 14: EVOLUTION OF FLATNESS *(15 points)*

(a) We start with the Friedmann equation from the formula sheet on the quiz:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ .$$

The critical density is the value of ρ corresponding to $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \ .$$

Using this expression to replace H^2 on the left-hand side of the Friedmann equation, and then dividing by $8\pi G/3$, one finds

$$\rho_c = \rho - \frac{3kc^2}{8\pi G a^2} \ .$$

Rearranging,

$$\frac{\rho - \rho_c}{\rho} = \frac{3kc^2}{8\pi G a^2 \rho} \ .$$

On the left-hand side we can divide the numerator and denominator by ρ_c, and then use the definition $\Omega \equiv \rho/\rho_c$ to obtain

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^2}{8\pi G a^2 \rho} \ . \qquad (1)$$

For a matter-dominated universe we know that $\rho \propto 1/a^3(t)$, and so

$$\frac{\Omega - 1}{\Omega} \propto a(t) \ .$$

If the universe is nearly flat we know that $a(t) \propto t^{2/3}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t^{2/3} \ .}$$

(b) Eq. (1) above is still true, so our only task is to re-evaluate the right-hand side. For a radiation-dominated universe we know that $\rho \propto 1/a^4(t)$, so

$$\frac{\Omega - 1}{\Omega} \propto a^2(t) \ .$$

If the universe is nearly flat then $a(t) \propto t^{1/2}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t \ .}$$

PROBLEM 15: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) \ ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/a^3(t)$ and $\rho_\Lambda = \text{const}$, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{a(t)}{a(t_0)} \ ,$$

one has

$$\begin{aligned}\left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3}G\left(\frac{\rho_{m,0}}{x^3}+\rho_\Lambda\right)\\ &= \frac{8\pi}{3}G\rho_{c,0}\left(\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}\right)\\ &= H_0^2\left(\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}\right) .\end{aligned}$$

Here we used the facts that

$$\Omega_{m,0}\equiv\frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0}\equiv\frac{\rho_\Lambda}{\rho_{c,0}} ,$$

and

$$H_0^2=\frac{8\pi}{3}G\rho_{c,0} .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x}=H_0\,x\sqrt{\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}} ,$$

which in turn implies that

$$\mathrm{d}t=\frac{1}{H_0}\frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}}} .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0=\int_0^{t_0}\mathrm{d}t=\boxed{\frac{1}{H_0}\int_0^1\frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}}} .}$$

The answer can also be written as

$$\boxed{t_0=\frac{1}{H_0}\int_0^1\frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x+\Omega_{\Lambda,0}x^4}}}$$

or

$$\boxed{t_0=\frac{1}{H_0}\int_0^\infty\frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} \; ; \qquad \mathrm{d}x = -\frac{\mathrm{d}z}{(1+z)^2} \; .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) \; .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0} \text{ yr} \; ,$$

where $H_0 = 100\,h_0$ km-sec^{-1}-Mpc^{-1}, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9 \text{ yr} \; .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \text{ yr} \; .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} \; ,$$

since the cosmological redshift is given by

$$1 + z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} \; .$$

Using the expression for dt from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}}} \; .$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0}\int_z^{\infty} \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}}} \; .$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0} x_{\max}^3} + \sqrt{\Omega_{\Lambda,0}}\, x_{\max}^{3/2}}{\sqrt{\Omega_m}}\right) \; .$$

Using $x_{\max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr}} \; .$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $a(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\} \; ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{a(t)} \; .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t \; .$$

This is not the final answer, however, because we don't explicitly know $a(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}} \ .$$

So

$$\ell_c = \frac{c}{a(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} \ ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 a(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Finally, then

$$\boxed{\ell_{\text{phys},0} = a(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}} \ ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{0}^{z} \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} \ .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{\mathrm{d}z'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099 \ ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\text{phys},0} = 27.23 \text{ light-yr} \ .}$$

Right again.

(d) $\ell_{\text{phys},e} = a(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{a(t_e)}{a(t_0)}\,\ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z} \ .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr} \ .}$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\,\ell_{\text{phys},0} = c\int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} \ .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} \ .}$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099 \ .}$$

The SDSS people get an A.

PROBLEM 16: SECOND HUBBLE CROSSING *(40 points)*

(a) From the formula sheets, we know that for a flat radiation-dominated universe,

$$a(t) \propto t^{1/2} .$$

Since

$$H = \frac{\dot{a}}{a} ,$$

(which is also on the formula sheets),

$$H = \frac{1}{2t} .$$

Then

$$\ell_H(t) \equiv cH^{-1}(t) = \boxed{2ct\ .}$$

(b) We are told that the energy density is dominated by photons and neutrinos, so we need to add together these two contributions to the energy density. For photons, the formula sheet reminds us that $g_\gamma = 2$, so

$$u_\gamma = 2\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{(\hbar c)^3} .$$

For neutrinos the formula sheet reminds us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} ,$$

so

$$u_\nu = \frac{21}{4}\,\frac{\pi^2}{30}\,\frac{(kT_\nu)^4}{(\hbar c)^3} .$$

Combining these two expressions and using $T_\nu = (4/11)^{1/3}\,T_\gamma$, one has

$$u = u_\gamma + u_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{(\hbar c)^3} ,$$

so finally

$$\boxed{g_1 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} .}$$

(c) The Friedmann equation tells us that, for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho \ ,$$

where in this case $H = 1/(2t)$ and

$$\rho = \frac{u}{c^2} = g_1 \, \frac{\pi^2}{30} \, \frac{(kT_\gamma)^4}{\hbar^3 c^5} \ .$$

Thus

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3} \, g_1 \, \frac{\pi^2}{30} \, \frac{(kT_\gamma)^4}{\hbar^3 c^5} \ .$$

Solving for T_γ,

$$\boxed{T_\gamma = \frac{1}{k}\left(\frac{45\hbar^3 c^5}{16\pi^3 g_1 G}\right)^{1/4} \frac{1}{\sqrt{t}} \ .}$$

(d) The condition for Hubble crossing is

$$\lambda(t) = cH^{-1}(t) \ ,$$

and the first Hubble crossing always occurs during the inflationary era. Thus any Hubble crossing during the radiation-dominated era must be the second Hubble crossing.

If λ is the present physical wavelength of the density perturbations under discussion, the wavelength at time t is scaled by the scale factor $a(t)$:

$$\lambda(t) = \frac{a(t)}{a(t_0)} \, \lambda \ .$$

Between the second Hubble crossing and now, there have been no freeze-outs of particle species. Today the entropy of the universe is still dominated by photons and neutrinos, so the conservation of entropy implies that aT_γ has remained essentially constant between then and now. Thus,

$$\lambda(t) = \frac{T_{\gamma,0}}{T_\gamma(t)} \, \lambda \ .$$

Using the previous results for $cH^{-1}(t)$ and for $T_\gamma(t)$, the condition $\lambda(t) = cH^{-1}(t)$ can be rewritten as

$$kT_{\gamma,0}\left(\frac{16\pi^3 g_1 G}{45\hbar^3 c^5}\right)^{1/4} \sqrt{t}\,\lambda = 2ct \ .$$

Solving for t, the time of second Hubble crossing is found to be

$$\boxed{t_{H2}(\lambda) = (kT_{\gamma,0}\lambda)^2 \left(\frac{\pi^3 g_1 G}{45\hbar^3 c^9}\right)^{1/2} .}$$

Extension: You were not asked to insert numbers, but it is of course interesting to know where the above formula leads. If we take $\lambda = 10^6$ lt-yr, it gives

$$t_{H2}(10^6 \text{ lt-yr}) = 1.04 \times 10^7 \text{ s} = 0.330 \text{ year} .$$

For $\lambda = 1$ Mpc,

$$t_{H2}(1 \text{ Mpc}) = 1.11 \times 10^8 \text{ s} = 3.51 \text{ year} .$$

Taking $\lambda = 2.5 \times 10^6$ lt-yr, the distance to Andromeda, the nearest spiral galaxy,

$$t_{H2}(2.5 \times 10^6 \text{ lt-yr}) = 6.50 \times 10^7 \text{ sec} = 2.06 \text{ year} .$$

PROBLEM 17: THE EVENT HORIZON FOR OUR UNIVERSE *(25 points)*

(a) In a spherical pulse each light ray is moving radially outward, so $\mathrm{d}\theta = \mathrm{d}\phi = 0$. A light ray travels along a null trajectory, meaning that $\mathrm{d}s^2 = 0$, so we have

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}r^2 = 0 . \tag{3.1}$$

from which it follows that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{c}{a(t)} . \tag{3.2}$$

We are interested in a radial pulse that starts at $r = 0$ at time $t = t_0$, so the limiting value of r is given by

$$\boxed{r_{\max} = \int_{t_0}^{\infty} \frac{c}{a(t)}\,\mathrm{d}t .} \tag{3.3}$$

(b) Changing variables of integration to

$$x = \frac{a(t)}{a(t_0)} , \tag{3.4}$$

the integral becomes

$$r_{\max} = \int_1^\infty \frac{c}{a(t)} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{c}{a(t_0)} \int_1^\infty \frac{1}{x} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x \ , \tag{3.5}$$

where we used the fact that $t = t_0$ corresponds to $x = a(t_0)/a(t_0) = 1$. As given to us on the formula sheet, the first-order Friedmann equation can be written as

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2} \ . \tag{3.6}$$

Using this substitution,

$$\boxed{r_{\max} = \frac{c}{a(t_0)H_0} \int_1^\infty \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}} \ ,} \tag{3.7}$$

where we have used $\Omega_{k,0} = 0$, since the universe is taken to be flat.

(c) To find the value of the redshift for the light that we are presently receiving from coordinate distance $r_{\max}$, we can begin by noticing that the time of emission t_e can be determined by the equation which implies that the coordinate distance traveled by a light pulse between times t_e and t_0 must equal $r_{\max}$. Using Eq. (3.2) for the coordinate velocity of light, this equation reads

$$\int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t = r_{\max} \ . \tag{3.8}$$

The "half-credit" answer to the quiz problem would include the above equation, followed by the statement that the redshift z_{eh} can be determined from

$$z = \frac{a(t_0)}{a(t_e)} - 1 \ . \tag{3.9}$$

The "full-credit" answer is obtained by changing the variable of integration as in part (b), so Eq. (3.8) becomes

$$\begin{aligned} r_{\max} &= \int_{x_e}^1 \frac{c}{a(t)} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x \\ &= \frac{c}{a(t_0)} \int_{x_e}^1 \frac{1}{x} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x \ , \end{aligned} \tag{3.10}$$

where x_e is the value of x corresponding to $t = t_e$. Then using Eq. (3.6) with $\Omega_{k,0} = 0$, we find

$$\boxed{r_{\rm max} = \frac{c}{a(t_0)H_0}\int_{x_e}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4}}\ .} \tag{3.11}$$

To complete the answer in this language, we use

$$\boxed{z = \frac{1}{x_e} - 1\ .} \tag{3.12}$$

Eqs. (3.11) and (3.12) constitute a full answer to the question, but one could go further and replace $r_{\rm max}$ using Eq. (3.7), finding

$$\boxed{\begin{aligned}&\int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4}} \\ &\qquad = \int_{x_e}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4}}\ .\end{aligned}} \tag{3.13}$$

In this form the answer depends only on the values of $\Omega_{X,0}$.

You were of course not asked to evaluate this formula numerically, but you might be interested in knowing that the Planck 2013 values $\Omega_{m,0} = 0.315$, $\Omega_{\rm vac,0} = 0.685$, and $\Omega_{\rm rad,0} = 9.2 \times 10^{-5}$ lead to $z_{\rm eh} = 1.87$. Thus, no event that is happening now (i.e., at the same value of the cosmic time) in a galaxy at redshift larger than 1.87 will ever be visible to us or our descendants, even in principle.

PROBLEM 18: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

(a) *(8 points)* This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho c^2 a^3\right) = -p\frac{d}{dt}\left(a^3\right)\ .$$

Setting

$$\rho = \frac{\alpha}{a^8}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{a^5}\right) = -p\frac{d}{dt}\left(a^3\right) \ ,$$

which implies

$$-5\frac{\alpha c^2}{a^6}\frac{da}{dt} = -3pa^2\frac{da}{dt} \ .$$

Thus

$$p = \frac{5}{3}\frac{\alpha c^2}{a^8} = \boxed{\frac{5}{3}\rho c^2 \ .}$$

Alternatively, one may start from the equation for the time derivative of ρ,

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Since $\rho = \frac{\alpha}{a^8}$, we take the time derivative to find $\dot\rho = -8(\dot a/a)\rho$, and therefore

$$-8\frac{\dot a}{a}\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

and therefore

$$p = \frac{5}{3}\,\rho c^2.$$

(b) *(9 points)* For a flat universe, the Friedmann equation reduces to

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

Using $\rho \propto 1/a^8$, this implies that

$$\dot a = \frac{\beta}{a^3} \ ,$$

for some constant β. Rewriting this as

$$a^3\, da = \beta\, dt \ ,$$

we can integrate the equation to give

$$\frac{1}{4}a^4 = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $a = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{a \propto t^{1/4} \ .}$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that "You should be able to determine the function $a(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α that we used in part (a), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then a is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) *(8 points)* We start from the conservation of energy equation in the form

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Substituting $\dot{\rho} = -n(\dot{a}/a)\rho$ and $p = (2/3)\rho c^2$, we have

$$-nH\rho = -3H\left(\frac{5}{3}\rho\right)$$

and therefore

$$\boxed{n = 5.}$$

PROBLEM 19: THE FREEZE-0UT OF A FICTITIOUS PARTICLE X *(25 points)*

(a) *(5 points)* The formula sheet tells us that the energy density of black-body radiation is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases} .$$

Since the X is spin-1, and 1 is an integer, the X particles are bosons and $g = 1$ per spin state. There are 3 species, X^+, X^-, and X^0, and each species we are told has three spin states, so there are a total of 9 spin states, so $g = 9$. Thus,

$$\boxed{u = 9\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3}\ .}$$

Alternatively, one could count the X^+ and X^- as one species with a distinct particle and antiparticle, so $g_{X^+X^-}$ is given by

$$g_{X^+X^-} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 6\ .$$

The X^0 is its own antiparticle, which means that the particle/antiparticle factor is one, so

$$g_{X^0} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{1}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 3\ ,$$

so the total g for X^+, X^-, and X^0 is again equal to 9.

(b) *(5 points)* The formula sheet tells us that the number density of particles in black-body radiation is

$$n = g^* \frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ ,$$

where

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ . \end{cases}$$

For bosons $g^* = g$, so g^* for the X particles is 9. Then

$$\boxed{n_X = 9\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .}$$

(c) *(10 points)* We are told that, when the X particles freeze out, all of their energy and entropy is given to the photons. We use entropy rather than energy to determine the final temperature of the photons, because the entropy in a comoving volume is simply conserved, while the energy density varies as

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Thus, to track the energy, we need to know exactly how p behaves, and the behavior of p during freeze-out is complicated, and we have not calculated it in this course.

The formula sheet tells us that the entropy density of a constituent of black-body radiation is given by

$$s = g\frac{2\pi^2}{45}\,\frac{k^4 T^3}{(\hbar c)^3} .$$

If we consider some fixed coordinate volume $V_{\rm coord}$, the corresponding physical volume is $V_{\rm phys} = V_{\rm coord}\, a^3(t)$, where $a(t)$ is the scale factor. The total entropy of neutrinos in $V_{\rm coord}$ is then

$$S_\nu = g_\nu\frac{2\pi^2}{45}\,\frac{k^4 T_\nu^3(t)}{(\hbar c)^3}\,V_{\rm coord}\, a^3(t) .$$

The quantities $T_\nu(t)$ and $a(t)$ depend on time, but the expression on the right-hand-side does not, since entropy is conserved. For brevity I will write

$$S_\nu = g_\nu A(t) T_\nu^3(t) , \tag{1}$$

where

$$A(t) \equiv \frac{2\pi^2}{45}\,\frac{k^4}{(\hbar c)^3} V_{\rm coord}\, a^3(t) .$$

The e^+e^- pairs and the X's contribute to the black-body radiation only before the freeze-out, when $kT \gg 0.511\ {\rm MeV}/c^2$. Let t_b denote any time before the freeze-out. Before the freeze-out, the total entropy of photons, e^+e^- pairs, and X particles is given by

$$S_{{\rm before},\gamma e X} = (g_\gamma + g_{e^+e^-} + g_X) A(t_b) T_\gamma^3(t_b) . \tag{2}$$

I can call the temperature T_γ, because the e^+e^- pairs and the X's (as well as the neutrinos) are all in thermal equilibrium at this point, so they all have the same temperature.

Using t_a to denote an arbitrary time after the freeze-out, the entropy of the photons during this time period can be written

$$S_{{\rm after},\gamma} = g_\gamma A(t_a) T_\gamma^3(t_a) . \tag{3}$$

But since the e^+e^- pairs and X particles give all their entropy to the photons, we have

$$S_{\text{after},\gamma} = S_{\text{before},\gamma e X} \ . \tag{4}$$

Then using Eqs. (2) and (3) we find

$$g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_b) T_\gamma^3(t_b) \ . \tag{5}$$

We can rewrite the last factor in Eq. (5) by remembering that Eq. (1) holds at all times, and that $T_\nu(t_b) = T_\gamma(t_b)$. So,

$$A(t_b) T_\gamma^3(t_b) = A(t_b) T_\nu^3(t_b) = \frac{S_\nu}{g_\nu} = A(t_a) T_\nu^3(t_a) \ . \tag{6}$$

Substituting Eq. (6) into Eq. (5), we have

$$g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_a) T_\nu^3(t_a) \ ,$$

from which we see that

$$T_\gamma^3(t_a) = \frac{g_\gamma + g_{e^+e^-} + g_X}{g_\gamma} T_\nu^3(t_a) \ ,$$

and therefore

$$\begin{aligned} \frac{T_\nu(t_a)}{T_\gamma(t_a)} &= \left(\frac{g_\gamma}{g_\gamma + g_{e^+e^-} + g_X} \right)^{1/3} \\ &= \left(\frac{2}{2 + \frac{7}{2} + 9} \right)^{1/3} = \boxed{\left(\frac{4}{29} \right)^{1/3}} \ . \end{aligned}$$

(d) *(5 points)* The answer would be the same, since it was completely determined by the conservation equation, Eq. (4) in the above answer. Regardless of the order in which the freeze-outs occurred, the total entropy from the e^+e^- pairs and the X's would ultimately be given to the photons, so the amount of heating of the photons would be the same.

PROBLEM 20: THE TIME t_d OF DECOUPLING *(25 points)*

(a) *(5 points)* If the entropy of photons is conserved, then the entropy density falls as

$$s \propto \frac{1}{a^3(t)} \ .$$

Since $s \propto T^3$, it follows that

$$T \propto \frac{1}{a(t)} \ .$$

Thus, the ratio of the scale factors is equal to the inverse of the ratio temperatures:

$$\boxed{x_d = \frac{T_0}{T_d} \ .}$$

(b) *(5 points)* The formula sheet reminds us that

$$x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \Omega_{k,0} x^2} \ ,$$

where

$$\Omega_{k,0} \equiv - \frac{kc^2}{a^2(t_0) H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} \ .$$

So for a flat universe $\Omega_{k,0} = 0$, and we have

$$\boxed{\frac{dx}{dt} = \frac{H_0}{x} \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4} \ .}$$

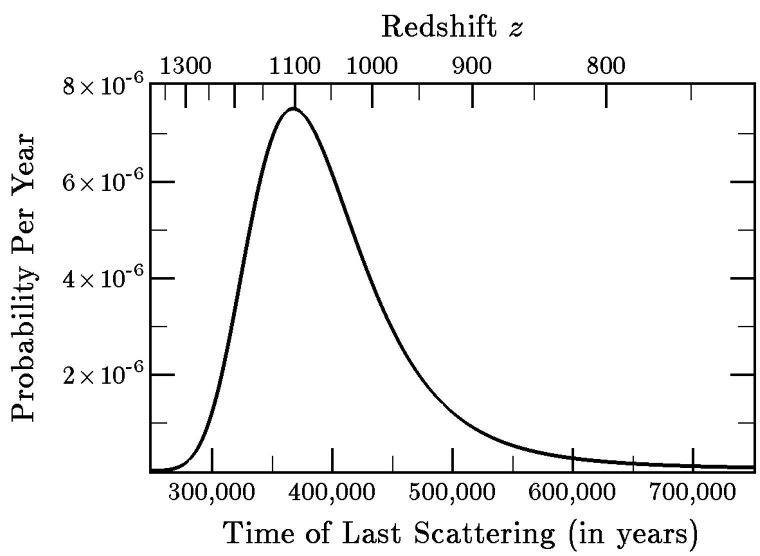
(c) *(5 points)* The answer to part (b) can be rewritten as

$$dt = \frac{x \, dx}{H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4}} \ .$$

t_d is the time that elapses from when the universe has $x = 0$ to when it has $x = x_d$, so

$$\boxed{t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x \, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4}} \ .}$$

You were of course not asked to evaluate this integral numerically, but we will do that now. We take $T_0 = 2.7255$ K from Fixsen et al. (cited in Lecture Notes 6) and the Planck 2015 best fit values of $H_0 = 67.7$ km-s^{-1}-Mpc^{-1}, $\Omega_{m,0} = 0.309$, $\Omega_{\rm vac,0} = 0.691$. The energy density of radiation (photons plus neutrinos) can then be calculated to give $\Omega_{\rm rad,0} = 9.2 \times 10^{-5}$ (see Eq. (6.23) of Lecture Notes 6 and the text of the 2nd paragraph of p. 12 of Lecture Notes 7). To keep our model universe exactly flat, I am modifying $\Omega_{\rm vac,0}$ to set it equal to $0.691 - \Omega_{\rm rad,0}$, which is well within the uncertainties. Numerical integration then gives 366,000 years, very close to our original estimate. Of course this number is still approximate, since we started with $T_d \approx 3000$ K. In any case, the decoupling of the photons in the CMB is actually a gradual process. In 2003 I modified a standard program called CMBFast to calculate the probability distribution of the time of last scattering (published in https://arxiv.org/abs/astro-ph/0306275), with the following results:



The parameters used were $\Omega_{\rm vac,0} = 0.70$, $\Omega_{m,0} = 0.30$, $H_0 = 68$ km-s^{-1}-Mpc^{-1}. The peak of the curve is at 367,000 years, and the median is at 388,000 years.

(d) *(10 points)* The derivation starts with the first-order Friedmann equation. Since we are describing a flat universe, we can start with the Friedmann equation for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho \ .$$

Now we use the facts that $\rho_m \propto 1/a^3$, $\rho_{\rm rad} \propto 1/a^4$, $\rho_{\rm vac} \propto 1$, and $\rho_f \propto 1/a^8$ to write

$$H^2 = \frac{8\pi}{3} G \left[\frac{\rho_{m,0}}{x^3} + \frac{\rho_{\rm rad,0}}{x^4} + \rho_{\rm vac,0} + \frac{\rho_{f,0}}{x^8} \right] \ .$$

Then we use

$$\rho_{m,0} = \rho_c \Omega_{m,0} = \frac{3H_0^2}{8\pi G}\, \Omega_{m,0} ,$$

with similar relations for the other components of the mass density, to rewrite the Friedmann equation as

$$H^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac},0} + \frac{\Omega_{f,0}}{x^8} \right] .$$

Next we rewrite H^2 as

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{x}}{x} \right)^2 ,$$

so

$$\left(\frac{\dot{x}}{x} \right)^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac},0} + \frac{\Omega_{f,0}}{x^8} \right] ,$$

which can be rewritten as

$$x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \frac{\Omega_{f,0}}{x^4}} .$$

From here the derivation is identical to that in part (c), leading to

$$\boxed{t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x\, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \frac{\Omega_{f,0}}{x^4}}} ,}$$

which can also be written more neatly as

$$\boxed{t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x^3\, dx}{\sqrt{\Omega_{m,0} x^5 + \Omega_{\text{rad},0} x^4 + \Omega_{\text{vac},0} x^8 + \Omega_{f,0}}} .}$$