Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe October 21, 2022
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 5, 2022

Problem	**Maximum**	**Score**
1	20	
2	50	
3	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(5 points)* Why are there dark regions when we look toward the center of our galaxy using visible wavelength, or at another galaxy edge on?

- (i) There are few stars in those areas.
- (ii) The stars in those areas have a lower brightness than the stars in the halo.
- (iii) Clouds of dust absorb the light of the stars located behind.
- (iv) The gravitational field of the galaxy binds the path of photons away from the edge.

[*Comment:* Absorption by dust clouds is discussed by Weinberg in the captions to photographs in Chapter 2.]

(b) *(5 points)* Using a radio antenna, Penzias and Wilson took about a year to determine that the signal they were receiving was truly of astronomical origin. What made it hard to establish that the signal they received was not just noise?

- (i) It was hard to distinguish it from the electronic noise of the instruments.
- (ii) It was hard to distinguish it from a microwave signal coming from stars or distant galaxies.
- (iii) It was hard to distinguish it from radio communications on Earth.

[*Comment:* The problems of instrument noise are discussed by Weinberg at the start of Chapter 3.]

(c) *(5 points)* The universe contains various types of particles; leptons, such as electrons and neutrinos, and baryons, such as neutrons and protons. Why do we refer to the matter of the universe solely as baryonic?

- (i) There are much more baryons than leptons in the universe.
- (ii) Baryons are heavier than leptons, so they dominate the mass density.
- (iii) Leptons are a subcategory of baryons.
- (iv) Leptons bind together to form baryons.

[*Comment:* Ryden discusses the particle content of the universe in Chapter 2.]

(d) *(5 points)* State whether each of the following statements is true (T) or false (F):

(i) Before the 1840s it was a mystery why stars had visibly different colors, but in 1842 Johann Christian Doppler correctly explained this phenomenon as the consequence of the different velocities of stars. **T** or **[F]**.

(ii) Frauenhofer let light from the Sun pass through a slit and then through a glass prism. The resulting spectrum showed bright lines that could be identified with known elements. **T** or **[F]**.

(iii) In black-body radiation, the number of photons per unit volume is directly proportional to the cube of the temperature. **[T]** or **F** .

(iv) The model of the universe proposed by de Sitter in 1917 predicts a redshift proportional to the distance. **[T]** or **F** .

(v) The horizon distance bears its name because it is the edge of the universe; there are no stars or galaxies beyond it. **T** or **[F]**.

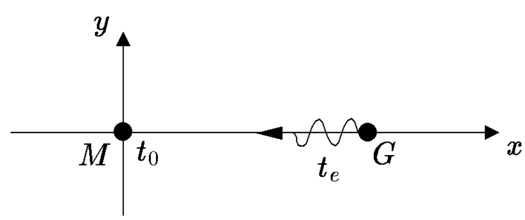
Comments: (i) In Chapter 2, Weinberg explains that the Doppler shift has essentially nothing to do with the color of stars. While the redshift may cause blue light to shift toward the red, at the same time the normally invisible ultraviolet light is shifted into the blue. The visible color differences are really caused by the surface temperature of stars. (ii) In Chapter 2, Weinberg also explains that the spectral lines in the Sun are dark, caused by the selective absorption of radiation by the cooler outer atmosphere of the Sun. (iii) Weinberg discusses this in Chapter 3, pointing out that it is equivalent to saying that the typical separation between photons in black-body radiation is roughly equal to the typical photon wavelength. (iv) Weinberg discusses the de Sitter model in Chapter 2. (v) Ryden explains in Chapter 2 that we cannot see beyond the horizon because light from such sources has not yet had time to reach us.

PROBLEM 2: INTERGALACTIC SIGNALING AND MARKSMANSHIP *(50 points)*

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{3/4} ,$$

where b is a constant. Suppose that we live at cosmic time t_0 on galaxy M (maybe for Milky Way), located at the origin of a comoving coordinate system $(x, y, z) = (0, 0, 0)$. Galaxy G is located somewhere along the positive x axis, as shown in the diagram below:



In all of the following questions, you can express your answer in terms of any or all of the given variables (i.e., any variable that has previously been assigned a symbol) and also any of the symbols that represent answers to the previous parts, whether or not you answered those parts.

(a) *(5 points)* What is the Hubble expansion rate H_0 that we would measure?

Answer: In general

$$H(t) = \frac{1}{a}\frac{da}{dt} ,$$

so in this case

$$\boxed{H_0 = H(t_0) = \frac{1}{bt_0^{3/4}}\left(\frac{3}{4}\right)bt_0^{-1/4} = \frac{3}{4t_0} .}$$

(b) *(7 points)* Suppose that we know that the light that we are receiving now (at time t_0) from G was emitted at time t_e. Given t_e and t_0, and the speed of light c, what is the physical distance $\ell_p(t_0)$ today between us and the galaxy G? What was the physical distance $\ell_p(t_e)$ at the time of emission t_e?

Answer: The coordinate speed of light is $c/a(t)$, so the coordinate distance that light travels between t_e and t_0 is given by

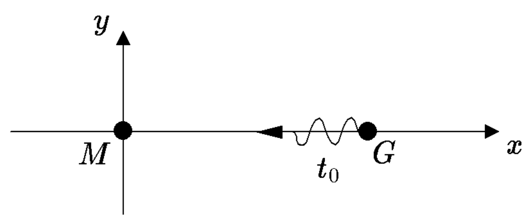
$$\ell_c = \int_{t_e}^{t_0} \frac{c}{bt^{3/4}}\,dt = \frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right) .$$

The physical distance $\ell_p(t_0)$ is obtained by multiplying by the scale factor $a(t_0)$, so

$$\boxed{\ell_p(t_0) = 4ct_0^{3/4}\left(t_0^{1/4} - t_e^{1/4}\right) = 4ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]} .$$

The physical distance at the time of emission $\ell_p(t_e)$ is found by multiplying by the scale factor at the time of emission, so

$$\boxed{\ell_p(t_e) = 4ct_e^{3/4}\left(t_0^{1/4} - t_e^{1/4}\right) = 4ct_e\left[\left(\frac{t_0}{t_e}\right)^{1/4} - 1\right]} .$$



(c) *(8 points)* At the present cosmic time t_0, a light signal is emitted from G toward us, as shown in the diagram above. When we receive this light signal at some point in the future, what will be its redshift z_G?

Answer: To find the redshift, we first find the time t_r at which we will receive the light signal. The coordinate distance that light travels between t_0 and t_r must be the same coordinate distance ℓ_c calculated in part (b), so

$$\frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right) = \int_{t_0}^{t_r} \frac{c}{a(t)}\,dt = \frac{4c}{b}\left(t_r^{1/4} - t_0^{1/4}\right) ,$$

so

$$t_r^{1/4} = 2t_0^{1/4} - t_e^{1/4} .$$

The redshift with which we will receive the light is

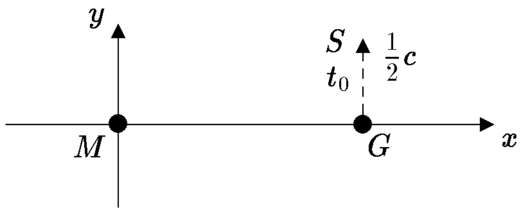
$$1 + z_G = \frac{a(t_r)}{a(t_0)} = \left(\frac{t_r}{t_0}\right)^{3/4} .$$

Using the formula above for $t_r^{1/4}$, this becomes

$$1 + z_G = \left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 ,$$

so

$$\boxed{z_G = \left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1 \ .}$$



(d) *(7 points)* Also at the present cosmic time t_0, a spaceship S is launched from the galaxy G in the y direction, as indicated in the diagram above. This is a science-fiction spaceship (just like one that you saw on the Quiz 1 Review Problems) which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity. Suppose, immediately after its launch, the spaceship emits a light signal directed toward us. When we eventually receive this signal (at M), what will be its redshift z_S? [*Hint:* it may help to imagine an observer at rest on G, who observes this light signal as it leaves the spaceship. You may assume that the redshift observed by this observer will be governed by special relativity. You can think of this observer as a relay station, since the signal will be unchanged if the observer absorbs it and retransmits it.]

Answer: In the frame of the observer on G, the light signal emitted from the spaceship S is directed perpendicular to the velocity of S, so it is a transverse Doppler shift — the Doppler shift is due entirely to time dilation, and is given by

$$1 + z_{\rm relay} = \frac{\lambda_{\rm relay}}{\lambda_S} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2}{\sqrt{3}} \ ,$$

for $v/c = 1/2$. This redshift factor will simply multiply the redshift factor for transmission from G to M, so

$$1 + z_S = \frac{\lambda_M}{\lambda_S} = \frac{\lambda_M}{\lambda_{\rm relay}}\frac{\lambda_{\rm relay}}{\lambda_S} = (1 + z_G)\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 \ .$$

Thus,

$$\boxed{z_S = \frac{2}{\sqrt{3}}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1 \ .}$$

Grading Error: The quizzes were initially graded under the incorrect assumption that the spaceship S was moving along the positive x axis, so the relevant redshift formula would be the one given on the Formula Sheet for motion along a line:

$$\frac{\lambda_{\text{relay}}}{\lambda_S} = 1 + z = \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{3} \ ,$$

for $\beta \equiv v/c = 1/2$. In that case the final answer would be

$$z_S = \sqrt{3}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1 \ .$$

Students who gave this answer were given full credit — those grades will not be revised. Students who gave the correct answer above were marked as wrong, but we expect to have these grading errors fixed at least within the next week. There is no need for you to do anything, since we have the scans and will make the corrections. Apologies for this confusion.

(e) *(7 points)* Find the comoving coordinates $x_c(t)$ and $y_c(t)$ for the spaceship S at an arbitrary time $t > t_0$.

Answer: Since S moves only in the y direction in the comoving coordinate system, $x_c(t)$ does not change. It is always given by

$$\boxed{x_c(t) = \ell_c = \frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right) \ .}$$

The spaceship S moves in the y direction at a physical speed $\frac{1}{2}c$ relative to the comoving coordinate system, so its speed in comoving coordinates is

$$\frac{dy}{dt} = \frac{c}{2a(t)} \ .$$

It starts at $y = 0$ at $t = t_0$, so

$$y_c(t) = \int_{t_0}^{t} \frac{c}{2bt'^{3/4}}\, dt' = \boxed{\frac{2c}{b}\left(t^{1/4} - t_0^{1/4}\right) \ .}$$

(f) *(7 points)* Find the physical distance $\ell_{p,S}(t)$ from S to M at an arbitrary time $t > t_0$.

Answer: Euclidean geometry is valid in a flat universe, so the coordinate distance from S to M at time t is given by the Pythagorean formula

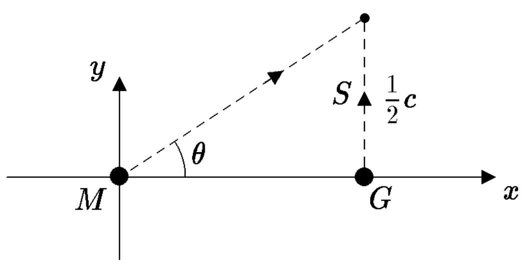
$$\ell_{c,S}(t) = \sqrt{x_c^2(t) + y_c^2(t)} \ ,$$

and then the physical distance is

$$\boxed{\ell_{p,S}(t) = bt^{3/4}\sqrt{x_c^2(t) + y_c^2(t)} \ .}$$

Since we were told that it is okay to answer in terms of symbols that represent the answers to previous parts, the answer above is completely acceptable. If one chooses to substitute for $x_c(t)$ and $y_c(t)$ from part (e), one should find

$$\ell_{p,S}(t) = 2ct^{3/4}t_0^{1/4}\left[5 - 8\left(\frac{t_e}{t_0}\right)^{1/4} + 4\left(\frac{t_e}{t_0}\right)^{1/2} - 2\left(\frac{t}{t_0}\right)^{1/4} + \left(\frac{t}{t_0}\right)^{1/2}\right]^{1/2} \ .$$



(g) *(9 points)* Now suppose that we, at the present time t_0, shoot a directed pulse of light at an angle θ with respect to the x axis, as shown in the diagram above. Our goal is to choose θ so that the light pulse intercepts the spaceship S. Write an equation that would determine the right value of θ to accomplish this interception. Your equation should involve only θ, given quantities, and answers to previous parts. You do *NOT* need to solve this equation (although you may find it easy to solve).

Answer: Probably the easiest way to solve this part is to recognize that both the light pulse and the spaceship start at $t = t_0$ to move at a constant velocity, starting from $y = 0$. When they collide they must also have the same y coordinate, so their y coordinate velocities must be equal. For the spaceship S, we noted in part (e) that

$$\frac{dy_S}{dt} = \frac{c}{2a(t)} \ .$$

For the light pulse, the coordinate speed is $c/a(t)$, and the component in the y direction is then

$$\frac{dy_\gamma}{dt} = \frac{c}{a(t)} \sin\theta \; .$$

Setting these expressions equal to each other, we have

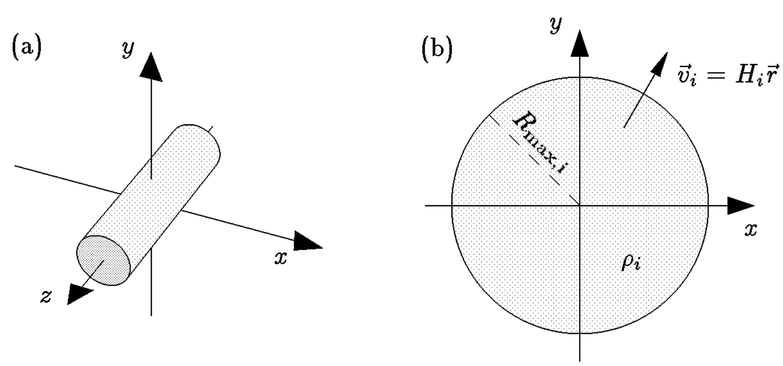
$$\boxed{\sin\theta = \frac{1}{2} \; .}$$

You were not asked to give the value of θ, but it is $\pi/6$, or 30°.

PROBLEM 3: A CYLINDRICAL UNIVERSE *(30 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points. This year it was Problem 1 of Problem Set 3.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



3D Perspective

2D Cross Section (Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \; .$$

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

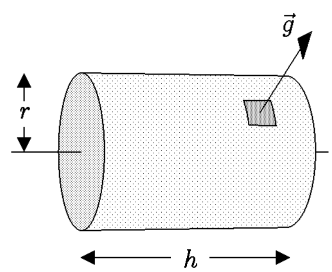
$$\vec{g} = -\frac{A\mu}{r}\hat{r} \; ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

Answer: Gauss's law of gravity states that

$$\oint \vec{g} \cdot d\vec{s} = -4\pi G M \; ,$$

where $\vec{g}$ is the acceleration of gravity, G is Newton's constant, and M is the total mass enclosed inside the volume. Apply this relation to the following slice of the infinite cylinder:



By symmetry $\vec{g}$ points radially outward, so the dot product $\vec{g} \cdot d\vec{s}$ vanishes for the disks that bound the cylindrical slice on the left and right. The only contribution comes from the curved surface of the cylinder, for which the cosine of the dot product is 1. Thus,

$$\oint \vec{g} \cdot d\vec{s} = 2\pi r h g_r \; ,$$

where g_r is the radial component of $\vec{g}$. The mass enclosed, M, is the length times the mass per length, or $h\mu$. (Recall that μ denotes the total mass within the radius r, so it is really a function of r and the mass density ρ.) Thus, Gauss's law gives

$$2\pi r h g_r = -4\pi G h \mu \; ,$$

so

$$g_r = -\frac{2G\mu}{r} \ .$$

Since the other components of $\vec{g}$ vanish by symmetry,

$$\boxed{\vec{g} = -\frac{2G\mu}{r}\,\hat{r} \ ,}$$

and we can read off the constant A,

$$\boxed{A = 2G} \ .$$

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

Answer: $\ddot{r}$ is the acceleration of r. The only force in the problem is gravity, so

$$\ddot{r} = g_r = -\frac{2G\mu}{r} \ ,$$

where I used the gravitational acceleration that was calculated in the previous part of the problem.

The evaluation of μ involves logic that is essentially identical to the spherical case discussed in Lecture Notes 3. If we label each particle by its initial radius r_i, then all the particles for a given r_i will form a cylindrical shell. As the motion proceeds it is conceivable that shells might cross each other, and then the evaluation of μ would become very complicated. However, the Hubble expansion incorporated into the initial conditions implies that initially any two shells are moving apart from each other. Since there are no infinite accelerations, we can count on the fact that there will be at least some nonzero time interval before any crossings can take place. Until such crossings take place, it is easy to find μ: the total mass contained within any shell of particles is independent of time, since the particles that lie at a smaller radius than any given shell at time t are exactly the same particles as those that were at smaller radius at time t_i. Then, even if shells start to cross at some time, the equations that we derive under the no-shell-crossing assumption will be valid until the time that shells cross. Thus, shells can cross only if our equations show the possibility that two shells starting at different values of r_i can come together. However, we will find below that the no-shell-crossing assumption leads to uniform expansion described by an overall scale factor $a(t)$. Since two shells never come together under this uniform expansion, we can conclude that no shell crossings will ever take place.

With no shell crossings, we can evaluate μ for a given shell r_i at the initial time. For definiteness we can consider a length h of the cylinder, so the volume of the cylinder of radius r_i is $\pi r_i^2 h$. The mass per length is then

$$\mu(r_i) = \frac{\pi r_i^2 h \rho_i}{h} = \pi r_i^2 \rho_i \ .$$

Thus,

$$\boxed{\ddot{r} = -\frac{2\pi G r_i^2 \rho_i}{r} \ .}$$

(c) *(7 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \ ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

Answer: The function $u(r_i, t)$ is determined by the differential equation that it obeys, combined with the initial conditions. Using the answer from part (b), the differential equation for u is

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{2\pi G r_i \rho_i}{r} \ ,$$

so

$$\ddot{u} = -\frac{2\pi G \rho_i}{u} \ ,$$

which does not depend on r_i. We still need to check that the initial conditions for u and $\dot{u}$ are independent of r_i. (There are two initial conditions, because the differential equation for u is second order.) Since $r(r_i, t_i) \equiv r_i$, the initial value of u is given by

$$u(r_i, t_i) = 1 \ .$$

Finally, since the initial velocities are set to agree with Hubble's law,

$$\dot{r}(r_i, t_i) = H_i r_i \ ,$$

it follows that

$$\dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \ .$$

Thus, neither the differential equation for $u(r_i, t)$ nor the initial conditions depend on r_i, so the solution will not depend on r_i. Thus, we can define $a(t) \equiv u(r_i, t)$.

Many students showed that the differential equation is independent of r_i, but did not recognize that the initial conditions are important. But they are crucial! If I throw a ball straight up into the air, and neglect air friction, then we know that it obeys the second order equation

$$\ddot{y} = -g \ ,$$

where g is the acceleration of gravity. Does this mean that the trajectory of the ball does not depend on the initial velocity, since the initial velocity does not appear in the differential equation? Of course not! If we specify $y(0)$ and $\dot{y}(0)$, then the solution will be uniquely determined.

(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

Answer: For clarity, we can consider a finite length h of the cylinder. The mass contained inside a cylinder of radius r_i at the initial time t_i is then

$$M(r_i, h) = \pi r_i^2 h \rho_i \ .$$

At time t, this same mass will be uniformly spread in a cylinder of radius $r(r_i, t)$ and length h. The density is therefore

$$\rho(t) = \frac{M(r_i, h)}{\pi r^2 h} = \boxed{\frac{\rho_i}{a^2(t)} \ .}$$

Using this result to replace ρ_i in the differential equation found in (c), we find

$$\boxed{\ddot{a} = -2\pi G \rho a \ .}$$

(e) *(8 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a) \ .$$

What is $V(a)$? Will this universe expand forever, or will it collapse?

Answer: Multiplying the differential equation by $\dot{a}$,

$$\dot{a}\left[\ddot{a} + \frac{2\pi G\rho_i}{a}\right] = 0 \ .$$

Note that I wrote the differential equation in terms of ρ_i rather than ρ, so that $a(t)$ is the only time-dependent quantity in the equation. This expression can be rewritten as

$$\frac{d}{dt}\left[\frac{1}{2}\dot{a}^2 + 2\pi G\rho_i \ln a\right] \equiv \frac{d}{dt}E = 0 \ .$$

In other words, the quantity in square brackets, which we have labeled E, must be constant:

$$E = \frac{1}{2}\dot{a}^2 + 2\pi G\rho_i \ln a \ = \text{const.}$$

We can now identify the rescaled potential energy as

$$\boxed{V(a) = 2\pi G\rho_i \ln a \ .}$$

The potential energy term $V(a)$ grows as $\ln a$ and is hence unbounded. No matter how large the initial value of $\dot{a}^2$, there can never be enough energy to allow the universe to grow to arbitrarily large a. Eventually the $V(a)$ term will grow to be as large as E, at which point $\dot{a}$ will vanish. At that point conservation of energy alone is not sufficient to control what happens next. As far as conservation of energy is concerned, $a(t)$ could remain constant after reaching the maximum value allowed by conservation of energy. (Similarly, if you threw a ball straight up in a uniform gravitational field, conservation of energy would allow you to calculate the speed at any height, but when the ball reaches its maximum height, conservation of energy would not rule out the possibility that the ball could just come to a stop.) To know what happens after $a(t)$ reaches its maximum value, we can look at the second order equation, $\ddot{a} = -2\pi G\rho a$. This equation implies that $\dot{a}$ cannot remain zero, but must become negative, and then conservation of energy would imply that $\dot{a}$ must become more negative as a descreases, so the universe collapses.