____
Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 9, 2022
Prof. Alan Guth

QUIZ 2

BLANK PAGES ARE AT THE END OF THE EXAM.

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins,
and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	15	
3	30	
4	35	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

State whether each of the following statements is true (T) or false (F):

(a) We say that the universe went through a period where it was opaque because there were no photons at that time; they were created later. **T** or **F** .

(b) In thermal equilibrium the number of any type of particle, whose threshold temperature is below the actual temperature (i.e., for which $mc^2 < kT$), is about equal to the number of photons. **T** or **F** .

(c) An electron and a positron can collide and emit 2 photons. **T** or **F** .

(d) Baryon number is not a conserved quantity in the early Universe, since baryons are not elementary particles. **T** or **F** .

(e) The most likely explanation for the apparent predominance of matter over anti-matter in the Universe today is that the Universe segregated into domains of almost pure matter and almost pure anti-matter, and we happen to live in a matter domain. **T** or **F** .

(f) The neutron is an unstable particle and can decay into a proton, an electron, and an anti-neutrino. **T** or **F** .

(g) The first particles to decouple in the early Universe are the electrons because they are the lightest elementary particles. **T** or **F** .

(h) The current mass density of stars, brown dwarfs, and stellar remnants, such as white dwarfs, neutron stars, and black holes is less than 0.5% of the total mass density of the universe. **T** or **F** .

(i) Even if dark matter doesn't absorb, emit, or scatter light, it can still affect the trajectory of photons. **T** or **F** .

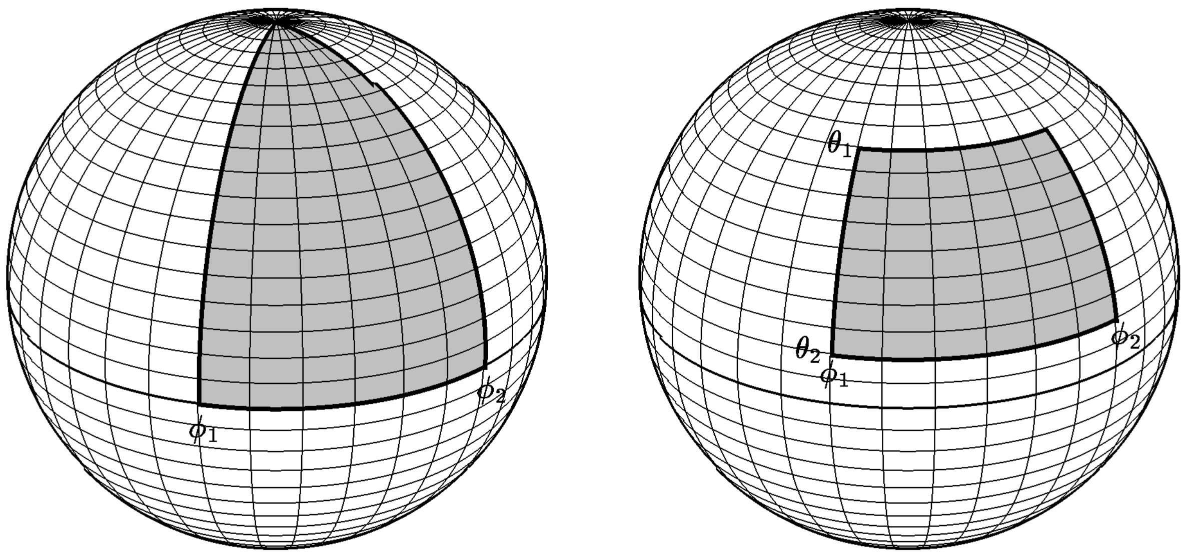
(j) The amounts of the light elements produced in the early Universe depend on the number of neutrino species. (By neutrino species, we mean electron neutrinos, muon neutrinos, etc.) **T** or **F** .

PROBLEM 2: Areas on the Surface of a Sphere *(15 points)*

Consider the surface of a sphere of radius R, with a metric on the surface given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2) .$$

Here θ and ϕ are the usual polar angles, where θ is measured from the north pole, and ϕ increases in the counterclockwise direction as seen from above the north pole.



(a) (b)

(a) *(5 points)* Consider the three-sided region indicated in Fig. (a) above. One vertex is at the north pole, while the other two are on the equator, at the ϕ values ϕ_1 and ϕ_2, with $\phi_1 < \phi_2 < 2\pi$. Find the area of this region. (You can find it by integration, or by relating it to an area that you know.)

(b) *(10 points)* Now consider the four-sided region shown in Fig. (b). It is bounded at the top by the curve $\theta = \theta_1$, at the bottom by $\theta = \theta_2$, on the left by the curve $\phi = \phi_1$, and on the right by $\phi = \phi_2$. Again we assume that $\phi_1 < \phi_2 < 2\pi$, and also that $\theta_1 < \theta_2 < \pi$. Find the area of this region. *[Curiosity: the states of Colorado and Wyoming have boundaries described exactly in this way.]*

PROBLEM 3: WHAT IF THE PARTICLES WERE DIFFERENT? *(30 points)*

Imagine a fictional universe — I will call it Narnia — where the principles of physics are the same as in our world, but the types of particles that exist, and they way in which they interact, are different. Assume that the early history of Narnia is essentially the same as the big bang past that we believe our universe underwent, except for the details of the particles and their interactions.

In particular, suppose that in Narnia there exists two species of spin-$\frac{1}{2}$ fermions, $\psi^{\pm}$ and $\chi^{\pm}$, both with mass m_1. Spin-$\frac{1}{2}$ means that there are two spin states. The ψ^+ and ψ^- are antiparticles of each other, as are the χ^+ and χ^-. In addition there are photons, with exactly the same properties as in our universe. Finally, there are neutrinos, which are massless and left-handed. Unlike our universe, in Narnia there are four flavors of neutrino: ν_e, ν_μ, ν_τ, and ν_σ. There are also antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$, and $\bar{\nu}_\sigma$. No other particles exist.

(a) *(10 points)* When $kT \gg m_1 c^2$, what is the energy density u as a function of the temperature T? (Here k is the Boltzmann constant, and c is the speed of light.)

(b) *(5 points)* Again when $kT \gg m_1 c^2$, what are the particle number densities $n_{\psi^+\psi^-}$, n_γ, and n_ν? Here $n_{\psi^+\psi^-}$ refers to the total number density of ψ^+ and ψ^- particles, n_γ is the number density of photons, and n_ν represents the total number density of neutrinos and antineutrinos.

(c) *(15 points)* Assume, as we do for our universe, that at high temperatures all of the particles were in thermal equilibrium. Unlike our universe, however, in Narnia the photons interact very weakly. As the temperature falls below $m_1 c^2/k$, and the $\psi^{\pm}$ and $\chi^{\pm}$ particles disappear from the thermal equilibrium gas, the photons are interacting so weakly that they obtain none of the energy or entropy that is released by the annihilating $\psi^{\pm}$ and $\chi^{\pm}$ pairs. Instead, all of the energy and entropy goes into heating the neutrinos, heating all 4 species equally. In this universe, after the $\psi^{\pm}$ and $\chi^{\pm}$ particles have disappeared, what is the ratio T_ν/T_γ?

PROBLEM 4: Rotating Frames of Reference *(35 points)*

This problem was on Quiz 2 of 2004, and this year it was Problem 15 in the Review Problems for Quiz 2.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] \,, \tag{4.1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (4.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar t$, $\bar r$, $\bar\phi$, and $\bar z$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar t^{\,2} - \left[\mathrm{d}\bar r^2 + \bar r^2\,\mathrm{d}\bar\phi^2 + \mathrm{d}\bar z^2\right] \,,$$

and then introducing new coordinates t, r, ϕ, and z that are related by

$$\bar t = t, \qquad \bar r = r, \quad \bar\phi = \phi + \omega t, \quad \bar z = z \,,$$

so $\mathrm{d}\bar t = \mathrm{d}t$, $\mathrm{d}\bar r = \mathrm{d}r$, $\mathrm{d}\bar\phi = \mathrm{d}\phi + \omega\,\mathrm{d}t$, and $\mathrm{d}\bar z = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu \,,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \tag{4.2}$$

— *Problem 3 continues on the next page.* —

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_2(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{4.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r. Your answer should have the form

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \dots \ .$$

where $\dots$ is an expression written in terms of r, ω, $\frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$. Any derivatives with respect to the coordinates t, r, θ, or ϕ can be carried out, so the only derivatives that appear in your equation should be derivatives with respect to τ.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ. Again, your answer should be written in terms of r, ω, $\frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$, and the only derivatives should be with respect to τ. It is okay to leave expressions such as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\cdots\Big)$$

in your answer.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for $\mathrm{d}\tau/\mathrm{d}t$, in terms of the quantities indicated, and then ask yourself how this result can be used to find $\mathrm{d}t/\mathrm{d}\tau$.)*