MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 3, 2022
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2

Revised Version*

QUIZ DATE: Wednesday, November 9, 2022.

COVERAGE: Lecture Notes 4, starting with *Evolution of a Closed Universe* on p. 5, Lecture Notes 5, and Lecture Notes 6 through *Relationship between a and T, p. 22.* Problem Sets 4–7. Weinberg, *The First Three Minutes,* Chapters 4 – 8 and *Afterword: Cosmology Since 1977.* In Ryden's *Introduction to Cosmology*, we have read Chapters 4 and Chapter 5 through Section 5.4.1 during this period These chapters, however, parallel what we have done or will be doing in lecture, so you should take them as an aid to learning the lecture material; there will be no questions explicitly based on these sections from Ryden. The quiz will include, however, Ryden's Chapters 7 (*Dark Matter*), Chapter 8 (*The Cosmic Microwave Background*) and Chapter 9 (*Nucleosynthesis and the Early Universe*), with the exceptions of Sections 8.3 (*The Physics of Recombination*), 9.2 (*Neutrons and Protons*), and 9.3 (*Deuterium Synthesis*). These three sections include detailed calculations of chemical equilibrium, which is a subject to which we will return. As I pointed out in the preamble to Problem Set 6, Eqs. (9.11) and (9.12) are not correct. For now, you should read over these three sections with enough understanding to be able to follow the sections that follow, but there will be no questions on the quiz that are based on these sections.

Chapters 4 and 5 of Weinberg's book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. We will not take off for the spelling of names, as long as they are vaguely recognizable. For dates before 1900, it will be sufficient for you to know when things happened to within 100 years. For dates after 1900, it will be sufficient if you can place events within 10 years.

You should expect one 20-point problem based on the readings, and several calculational problems. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the problem sets listed above (extra credit problems included), or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 6, 7, 11, 13, 15, 17, and 18.

QUIZ LOGISTICS: The logistics will be identical to Quiz 1. The quiz will take place during our regular class time, but in a different room, to be announced. The quiz will be closed book, no calculators, no internet, and 80 minutes long.

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* Revised 11/3/22 to fix the bookmarks of the solutions.

PURPOSE OF THE REVIEW PROBLEMS: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz. I thought that 20 review problems would be a good number, but that meant cutting a number of good problems. If by any chance you would like to see more review problems, let me know.

QUIZZES FROM PREVIOUS YEARS: In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, 2011, 2013, 2016, 2018, and 2020. The relevant problems from those quizzes have largely been incorporated into these review problems, but you still may be interested in looking at the quizzes, mainly to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match exactly the coverage from all previous years, but I believe that all these review problems would be fair problems for the upcoming quiz. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

REVIEW SESSION: To help you study for the quiz, Marianne Moore will hold a review session, at a time and place to be announced. Please fill out the Doodle Poll at https://doodle.com/meeting/participate/id/e919wYYe, to communicate your availability.

FUTURE QUIZ: Quiz 3 will be given on Wednesday, December 7, 2022.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" for your reference. For the second quiz, this useful information will be the following:

DOPPLER SHIFT (Definition:)

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} \;,$$

where $\Delta t_{\text{observer}}$ and Δt_{source} are the period of the wave as measured by the observer and by the source, respectively, and $\lambda_{\text{observer}}$ and λ_{source} are the wavelength of the wave, as measured by the observer and by the source, respectively.

DOPPLER SHIFT (For motion along a line):

Nonrelativistic, u = wave speed, source moving at speed v away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed v away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity), $\beta \equiv v/c$, where c is the speed of light and v is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation. A clock that is moving at speed v relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor γ:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \;, \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance ℓ_0 in their own frame, are moving together, in the direction of the line separating them, at speed v relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,

where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,

where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned}\ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\,\rho_r(t_i) \ \ \text{(radiation).}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \ \ \rho_c = \frac{3H^2}{8\pi G} \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$\Omega = 1$.

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$\Omega = \frac{2}{1 + \cos\theta} > 1$,

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3$.

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)$, $\frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)$,

$\Omega = \frac{2}{1 + \cosh\theta} < 1$,

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3$,

$\kappa \equiv -k > 0$.

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

BLACK-BODY RADIATION:

Whenever $kT \gg mc^2$ for any particle, where k is the Boltzmann constant, T is the temperature, and m is the (rest) mass of the particle, in thermal equilibrium there will be a black-body radiation, in which the particle will make the following contributions to the energy density, mass density, pressure, number density, and energy density:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

The values of g and g^* for photons, neutrinos, and electron-positron pairs are as follows:

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \ .$$

PHYSICAL CONSTANTS:

$$G = 6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} = 6.674 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-23} \text{ joule/K} \\ &= 1.381 \times 10^{-16} \text{ erg/K} \\ &= 8.617 \times 10^{-5} \text{ eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-34} \text{ joule} \cdot \text{s} \\ &= 1.055 \times 10^{-27} \text{ erg} \cdot \text{s} \\ &= 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \end{aligned}$$

$$c = 2.998 \times 10^8 \text{ m/s}$$

$$= 2.998 \times 10^{10} \text{ cm/s}$$

$$\hbar c = 197.3 \text{ MeV-fm}, \quad 1 \text{ fm} = 10^{-15} \text{ m}$$

$$1 \text{ yr} = 3.156 \times 10^{7} \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ joule} = 1.602 \times 10^{-12} \text{ erg}$$

$$\begin{aligned} 1 \text{ GeV} = 10^{9} \text{ eV} &= 1.783 \times 10^{-27} \;\text{ kg (where } c \equiv 1) \\ &= 1.783 \times 10^{-24} \text{ g} \;. \end{aligned}$$

PROBLEM LIST

1. Did You Do the Reading? (2020) . . . . 11 (Sol: 34)
2. Did You Do the Reading? (2018) . . . . 13 (Sol: 37)
3. Did You Do the Reading? (2016) . . . . 15 (Sol: 40)
4. Did You Do the Reading? (2013) . . . . 17 (Sol: 43)
5. Did You Do the Reading? (2011) . . . . 17 (Sol: 44)
*6. Evolution of an Open Universe . . . . 18 (Sol: 46)
*7. Tracing Light Rays in a Closed, Matter-Dominated Universe . . . 18 (Sol: 47)
8. Lengths and Areas in a Two-Dimensional Metric . . . . 19 (Sol: 49)
9. Volumes in a Robertson-Walker Universe . . . . 20 (Sol: 51)
10. Geometry in a Closed Universe . . . . 21 (Sol: 52)
*11. The General Spherically Symmetric Metric . . . . 22 (Sol: 53)
12. Geodesics . . . . 22 (Sol: 54)
*13. Geodesics in a Closed Universe . . . . 23 (Sol: 56)
14. Geodesics on the Surface of a Sphere . . . . 24 (Sol: 60)
*15. Rotating Frames of Reference . . . . 25 (Sol: 62)
16. Travel in the Schwarzschild Metric . . . . 27 (Sol: 64)
*17. Gravitational Bending of Light . . . . 29 (Sol: 66)
*18. Pressure and Energy Density of Mysterious Stuff . . . . 31 (Sol: 69)
19. Properties of Black-Body Radiation . . . . 32 (Sol: 70)
20. Doubling of Electrons . . . . 33 (Sol: 72)

PROBLEM 1: DID YOU DO THE READING? (2020) *(30 points)*

(a) *(5 points)* What is a necessary condition for two photons in a head-on collision to be able to produce an electron-positron pair?

(i) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 TeV.

(ii) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 GeV.

(iii) The energy of the photons must be below the "rest energy" of the electron-positron pair, which is roughly 1.02 keV.

(iv) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 MeV.

(v) Two colliding massless objects cannot produce massive particles as an outcome of their collision.

(b) *(5 points)* In Chapter 4, *Recipe for a Hot Universe,* Weinberg states that there are three conserved quantities that must be specified in the recipe for the early universe. What are they? [Grading: one correct = 2 pts; two correct = 4 pts; three correct = 5 pts.]

(i) ________________________

(ii) ________________________

(iii) ________________________

(c) *(5 points)* Why are "complicated" Feynman diagrams needed for calculations of the strong interactions, whereas accurate predictions can be made for electromagnetism with only the "simple" diagrams?

(i) Only the simplest Feynman diagrams in electromagnetism give a non-zero contribution; the more complicated diagrams cancel each other out.

(ii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," about 1/137.036. Complicated diagrams therefore give small contributions. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to one, so complicated diagrams are not suppressed.

(iii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," which is about equal to one, meaning that adding internal lines to a diagram doesn't change its value significantly. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to 137, so complicated diagrams are strongly enhanced.

(iv) Simple Feynman diagrams are mathematically ill-defined in the theory of strong interactions, whereas in electromagnetism they give a sensible numerical result.

(v) Calculations for the strong interactions involve six quarks, while there are only three charged leptons. If there were three more electron-like particles (in addition to the muon and the tau lepton), then electromagnetism would be as complicated as the strong interactions.

(d) *(5 points)* Consider the following observable phenomena:

(A) The motion of stars moving in large-radius circular orbits around the centers of spiral galaxies.

(B) The motion of some galaxies relative to the galaxy clusters they inhabit.

(C) Gravitational lensing of light emitted by stars in nearby galaxies.

Which one of the following combinations constitutes observational evidence that dark matter exists?

(i) Only (A).

(ii) Only (B).

(iii) Only (C).

(iv) (A) and (B).

(v) (A), (B), and (C).

(e) *(5 points)* Consider the following claims about hadrons:

(A) Electrons and neutrinos make up 90% of the Universe's hadronic matter content.

(B) They are the class of particles affected by strong interactions (i.e., by the strong nuclear force).

(C) It is a synonym of "baryons."

(D) They are theorized to be made up of more fundamental particles, called "quarks."

(E) They cannot be made up of more fundamental constituents, because if they were, we should be able to break them up into their building blocks and observe the hypothetical "quarks" directly.

Which combination of these claims is true?

- (i) Only (A).
- (ii) Only (B).
- (iii) (B) and (D).
- (iv) (B) and (E).
- (v) (B), (C), and (D).

(f) *(5 points)* Which of the following is the range of typical binding energies per nucleon in an atomic nucleus? (This is relevant to nuclear fusion and nuclear fission studies, and to the epoch of Big Bang nucleosynthesis.)

- (i) 1 eV – 10 eV.
- (ii) 100 eV – 1 keV.
- (iii) 10 keV – 100 keV.
- (iv) 1 MeV – 10 MeV.
- (v) 100 MeV – 1 GeV.

PROBLEM 2: DID YOU DO THE READING? (2018) *(25 points)*

For clarity, part (b) has been slightly reworded from the original quiz.

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

- (i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.
- (ii) The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.
- (iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.
- (iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

(i) When the temperature of the universe was about 10^{10} °K ($t \sim 1$ sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.

(ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.

(iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first $3\frac{3}{4}$ minutes, and this is why we call this period the era of nucleosynthesis.

(iv) After the first $3\frac{3}{4}$ minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.

(v) The protons and neutrons became decoupled from the photons after the first $3\frac{3}{4}$ minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.

(vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s (iii) 1940s (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

PROBLEM 3: DID YOU DO THE READING? (2016) *(25 points)*

(a) *(5 points)* In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to Ω from clusters of galaxies as

(i) 0.01 (ii) 0.05 (iii) 0.20 (iv) 0.60 (v) 1.00

(b) *(4 points)* One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring

- (i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.
- (ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.
- (iii) the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.
- (iv) the velocity dispersion along the line of sight and also the redshift of the cluster.
- (v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

(c) *(4 points)* Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.

- (i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.
- (ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.
- (iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.
- (iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.
- (v) By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.

(d) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

- (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.
- (ii) There was a breakdown in communication between theorists and experimentalists.
- (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.
- (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.
- (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.
- (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

(e) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

- (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.
- (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.
- (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)
- (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.
- (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

PROBLEM 4: DID YOU DO THE READING? (2013) *(25 points)*

The following problem comes from Quiz 2, 2013.

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity v as a function of radius R. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii?

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

PROBLEM 5: DID YOU DO THE READING? (2011) *(20 points)*

The following problem comes from Quiz 2, 2011.

(a) *(8 points)* During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.

  (i) In *The First Three Minutes,* Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium, He^4. Describe at least one of these two chains.

  (ii) Explain briefly what is the *deuterium bottleneck*, and what is its role during nucleosynthesis.

(b) *(12 points)* In Chapter 4 of *The First Three Minutes*, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures.

In this part of the problem you will explore more quantitatively this statement.

  (i) For a radiation-dominated universe the scale-factor $a(t) \propto t^{1/2}$. Find the cosmic time t as a function of the Hubble expansion rate H.

  (ii) The mass density stored in radiation ρ_r is proportional to the temperature T to the fourth power: i.e., $\rho_r \simeq \alpha T^4$, for some constant α. For a wide range of temperatures we can take $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg} \cdot \mathrm{m}^{-3} \cdot \mathrm{K}^{-4}$. If the temperature is measured in degrees Kelvin (K), then ρ_r has the standard SI units, $[\rho_r] =$

kg $\cdot$ m^{-3}. Use the Friedmann equation for a flat universe ($k = 0$) with $\rho = \rho_r$ to express the Hubble expansion rate H in terms of the temperature T. You will need the SI value of the gravitational constant $G \simeq 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$. What is the Hubble expansion rate, in inverse seconds, at the start of nucleosynthesis, when $T = T_{\rm nucl} \simeq 0.9 \times 10^9\,\mathrm{K}$?

(iii) Using the results in (i) and (ii), express the cosmic time t as a function of the temperature. Your result should agree with Weinberg's claim above. What is the cosmic time, in seconds, when $T = T_{\rm nucl}$?

∗ PROBLEM 6: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time t at which $a/\sqrt{\kappa} = 2\alpha$.

∗ PROBLEM 7: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance ℓ_{phys} at time t. You should leave your answer in the form of a definite integral.

The form of $a(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $a(t)$ is described by the parametric equations

$$\begin{aligned} ct &= \alpha(\theta - \sin\theta) \ , \\ a &= \alpha(1 - \cos\theta) \ , \end{aligned}$$

where

$$\alpha \equiv \frac{4\pi}{3} \frac{G\rho a^3}{c^2} \ .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 8: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

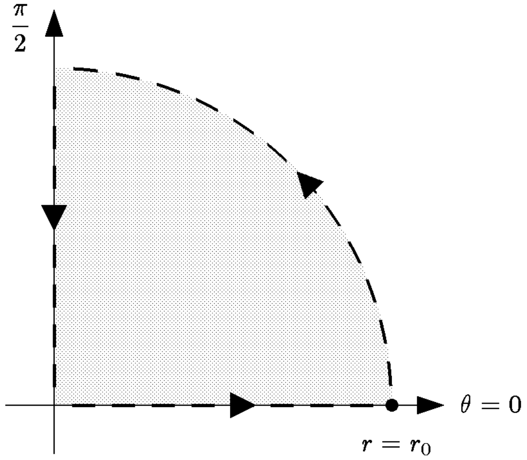
The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2 \, dr^2 + r^2(1 + br)^2 \, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to

$\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

b) *(15 points)* Find the area enclosed by this path.

PROBLEM 9: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \quad .$$

Calculate the volume $V(r_{\rm max})$ of the sphere described by

$$r \le r_{\rm max} \ .$$

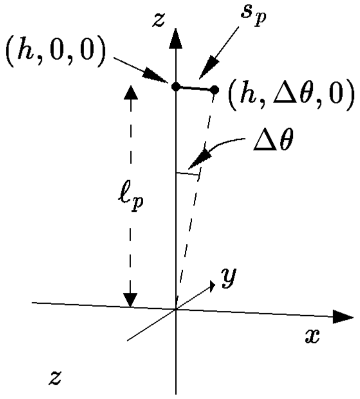
You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

PROBLEM 10: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as a, dropping its explicit t-dependence.

A small rod has one end at the point $(r = h,\ \theta = 0,\ \phi = 0)$ and the other end at the point $(r = h,\ \theta = \Delta\theta,\ \phi = 0)$. Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin ($r = 0$) to the first end $(h, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1-r^2} = \frac{1}{2} \ln \left(\frac{1+r}{1-r} \right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor a, and the coordinates h and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor a.

∗ PROBLEM 11: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2 \ .$$

Express the metric in terms of this new variable.

PROBLEM 12: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$
$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

∗ PROBLEM 13: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

We will assume that this metric is given, and that $a(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt}dt .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed $v_{\rm phys}$ of the object, since it is the speed that would be measured by a comoving observer. Write an expression for $v_{\rm phys}$ as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A\frac{dr}{d\tau}\right] = B\left(\frac{dt}{d\tau}\right)^2 + C\left(\frac{dr}{d\tau}\right)^2 + D\left(\frac{d\theta}{d\tau}\right)^2 + E\left(\frac{d\phi}{d\tau}\right)^2 ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

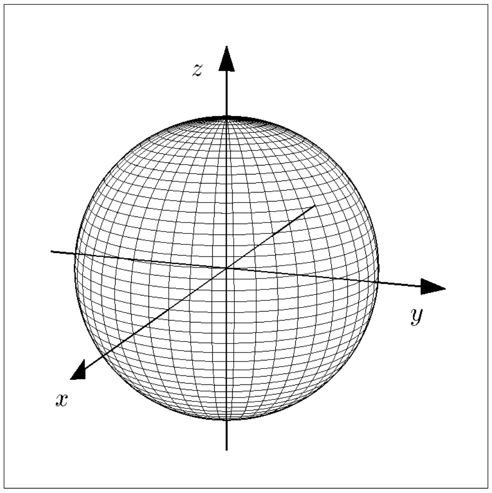
(f) *(5 points EXTRA CREDIT)* On Problem 2 of Problem Set 6 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

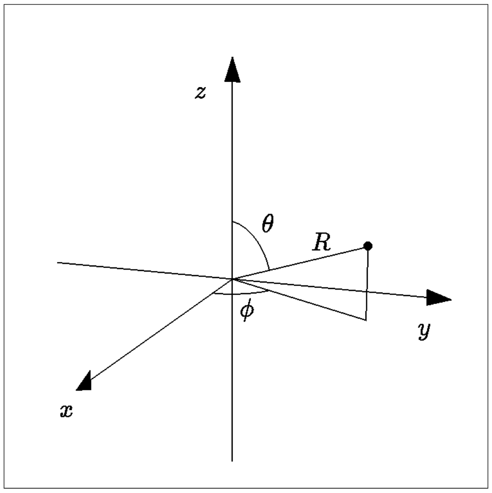
$$p = \frac{mv_{\rm phys}}{\sqrt{1-\frac{v_{\rm phys}^2}{c^2}}} ,$$

falls off as $1/a(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE *(25 points)*

The following problem was Problem 2 on Quiz 2, 2020.





In this problem we will explore the geodesic equation for the metric describing the surface of a sphere. We will describe the sphere as in Lecture Notes 5, with metric given by

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \quad , \tag{14.1}$$

where R is the radius of the sphere. As given on the formula sheet, the geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s} \ . \tag{14.2}$$

(a) *(5 points)* Using the metric of Eq. (14.1), write explicitly the geodesic equation for $i = 1 = \theta$. By "explicitly," we mean that the equation that you give should not contain any instances of g, i, j, or k. All instances of the metric should be replaced using the metric from Eq. (14.1), and all repeated indices should be summed explicitly.

(b) *(5 points)* Again using the metric of Eq. (14.1), write explicitly the geodesic equation for $i = 2 = \phi$.

(c) *(5 points)* Consider a curve that circles the sphere at fixed latitude,

$$\theta(s) = \theta_0 \ , \qquad \phi(s) = \beta s \ , \tag{14.3}$$

where β and θ_0 are constants. Use the geodesic equations you found in parts (a) and (b) to find out if these curves are geodesics (i.e., see if they statisfy the geodesic equations). For what values of θ_0, if any, are these curves geodesics, and for what values of θ_0, if any, are they not?

(d) *(5 points)* Consider a curve that moves along a line of fixed longitude:

$$\theta(s) = \gamma s \ , \qquad \phi(s) = \phi_0 \ , \tag{14.4}$$

where γ and ϕ_0 are constants. Use the equations you found in parts (a) and (b) to find the values of ϕ_0, if any, for which these curves are geodesics, and for what values, if any, they are not.

(e) *(2 points)* The geodesic equations imply that there is a conserved quantity of the form

$$L \equiv F(\theta,\phi)\frac{\mathrm{d}\phi}{\mathrm{d}s} \ . \tag{14.5}$$

(By "conserved," we mean that this quantity is guaranteed to be constant along a geodesic.) Find the function $F(\theta,\phi)$ for which this is true. Note (a) that $F(\theta,\phi)$ is defined by the conservation statement only up to a multiplicative constant; and (b) that a function need not actually vary as its argument is varied (i.e., $q(x) = 5$ is a perfectly valid function of x).

(f) *(3 points)* Use your answers from part (e), (a), and (b) to find an equation for geodesics of the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \text{expression} \ , \tag{14.6}$$

where the "expression" depends only on θ and the conserved quantity L. (This is useful, since differential equations involving just one variable are generally easier to solve than differential equations involving two variables.) You may leave $F(\theta,\phi)$ in your answer, if you have not evaluated it.

∗ PROBLEM 15: ROTATING FRAMES OF REFERENCE *(35 points)*

The following problem was Problem 3, Quiz 2, 2004.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] \ , \tag{P15.1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (P15.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, and $\bar{z}$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] \ ,$$

and then introducing new coordinates t, r, ϕ, and z that are related by

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z \ ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, *and* $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu \ ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = \,? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \,? \\ g_{22} &\equiv g_{\phi\phi} = \,? \\ g_{33} &\equiv g_{zz} = \,? \end{aligned} \tag{P15.2}$$

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{P15.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

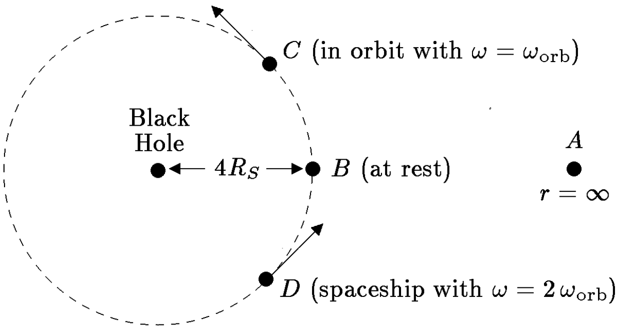
explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for* $\mathrm{d}\tau/\mathrm{d}t$*, in terms of the quantities indicated, and then ask yourself how this result can be used to find* $\mathrm{d}t/\mathrm{d}\tau$*.)*

PROBLEM 16: TRAVEL IN THE SCHWARZSCHILD METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 2020.



Consider a black hole of mass M, described by the Schwarzschild metric as written on the formula sheets:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ , \tag{P16.1}$$

In this problem we will consider a set of different observers in this spacetime, labeled by A, B, C, and D, as shown in the diagram above and described below.

(a) *(5 points)* Let observer A be stationary at an arbitrarily large value of r (you can take $r_A = \infty$), and let observer B be stationary at

$$r_B = 4R_S \ , \tag{P16.2}$$

where R_S is the Schwarzschild radius, $R_S = 2GM/c^2$. (Note that observer B needs to be using rocket propulsion to prevent herself from falling into the black hole.) Suppose that observer B is sending out pulses of electromagnetic radiation, at evenly spaced intervals $\Delta\tau_B$, as measured on B's local clock. What is the time interval $\Delta\tau_A$ that observer A will measure between the pulses that he receives?

(b) *(5 points)* Observer C is in orbit around the black hole, also at radius

$$r_C = 4R_S \ . \tag{P16.3}$$

The orbit is in the equatorial plane, with θ fixed at $\theta = \frac{\pi}{2}$, and ϕ changing with time as

$$\phi(t) = \omega_{\text{orb}}\, t \ . \tag{P16.4}$$

In Problem 4 of Problem Set 6 you showed that the orbital angular velocity ω_{orb}, for a circular orbit, must satisfy

$$r\omega^2_{\text{orb}} = \frac{GM}{r^2} \ . \tag{P16.5}$$

As measured on C's local clock, how long does it take for C to complete one orbit (i.e., for ϕ to change by 2π)?

(c) *(5 points)* Now consider an observer D on a spaceship traveling in a circle at the same radius,

$$r_D = 4R_S \ , \tag{P16.6}$$

also in the equatorial plane, $\theta = \frac{\pi}{2}$. Using its rocket power, the spaceship travels twice as fast as observer C, with

$$\phi(t) = \omega_D\, t = 2\,\omega_{\text{orb}}\, t \ . \tag{P16.7}$$

As measured on D's local clock, how long does it take for D to make one full circle (i.e., for ϕ to change by 2π).

(d) *(5 points)* If a tape measure, at rest with respect to the Schwarzschild coordinates, were stretched around the circle at radius $r = 4R_S$, what circumference would it measure?

∗ PROBLEM 17: GRAVITATIONAL BENDING OF LIGHT *(30 points)*

When a light ray passes by a massive object, general relativity predicts that it will be bent. Since most celestial objects are nearly spherical, we can use the Schwarzschild metric to calculate the bending. Furthermore, since we are usually interested in objects that are not black holes or anywhere nearly as dense, we can obtain an accurate answer by carrying out the calculation in a weak-field approximation. For a photon that grazes the Sun, for example, the value of $R_{\text{Sch}}/R_{\odot}$, the Schwarzschild radius over the radius of the Sun, is about 4×10^{-6}.

Starting with the Schwarzschild metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{R_{\text{Sch}}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{R_{\text{Sch}}}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \ , \qquad \text{(P17.1)}$$

where $R_{\text{Sch}} = 2GM/c^2$, we can expand in powers of R_{Sch}/r and keep only the first order terms:

$$\mathrm{d}s^2 = -\left(1 - \frac{R_{\text{Sch}}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 + \frac{R_{\text{Sch}}}{r}\right) \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \ . \qquad \text{(P17.2)}$$

For this problem it is useful to switch to Cartesian-like coordinates, defined in terms of r, θ, and ϕ by the usual Cartesian formulas,

$$\begin{aligned} x &= r \, \sin\theta \, \cos\phi \ , \\ y &= r \, \sin\theta \, \sin\phi \ , \\ z &= r \cos\theta \ . \end{aligned} \qquad \text{(P17.3)}$$

General relativity allows us to make any coordinate redefinitions that we might want, as long as we calculate the metric in terms of the new coordinates. It is useful to continue to use the quantity r, but now it will be thought of as a function of the coordinates x, y, and z:

$$r = (x^2 + y^2 + z^2)^{1/2} \ . \qquad \text{(P17.4)}$$

The metric can then be rewritten as the Minkowski metric of special relativity, plus small corrections:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \frac{R_{\text{Sch}}}{r} c^2 \mathrm{d}t^2 + \frac{R_{\text{Sch}}}{r} (\mathrm{d}r)^2 \ , \qquad \text{(P17.5)}$$

where from Eq. (4) one can see that

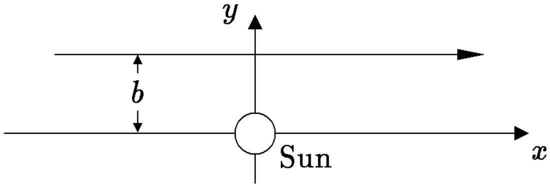
$$\mathrm{d}r = \frac{1}{r}(x \, \mathrm{d}x + y \, \mathrm{d}y + z \, \mathrm{d}z) \ . \qquad \text{(P17.6)}$$

(a) *(6 points)* For the metric as approximated by Eqs. (P17.5) and (P17.6), write the expressions for g_{tt}, g_{xx}, and g_{xy}.

The trajectory of the photon is lightlike, so we cannot use τ to parameterize the trajectory, because proper time intervals along a lightlike trajectory are zero. Nonetheless, it can be shown that one can use an "affine parameter" λ, for which the geodesic equation has the usual form:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right\} = \frac{1}{2}\left[\partial_\mu g_{\sigma\tau}\right]\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\tau}{\mathrm{d}\lambda} \ . \qquad \text{(P17.7)}$$

To obtain an answer that is accurate to first order in G, we begin by considering the *unperturbed* photon trajectory — the trajectory it would have if G were taken as zero, so $R_{\mathrm{Sch}} = 2GM/c^2 = 0$. This would be a straight line in the (x, y, z) coordinates, as shown in the diagram below:



Here b is called the *impact parameter*. We can parameterize this path by

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = b \ , \qquad z(\lambda) = 0 \ , \qquad t(\lambda) = \lambda/c \ . \qquad \text{(P17.8)}$$

We will calculate the deflection (to first order in G) by assuming that the photon path is accurately described by Eq. (P17.8), and we will calculate the y-velocity that the photon acquires due to the gravitational attraction of the Sun.

(b) *(9 points)* With the goal of calculating $\mathrm{d}^2y/\mathrm{d}\lambda^2$, we evaluate the geodesic equation for $\mu = y$. Start here by evaluating the left-hand side of Eq. (P17.7) for $\mu = y$, to first order in G. Expand the derivative with respect to λ using the product rule, working out explicitly the derivatives of the relevant $g_{\mu\nu}$ with respect to λ. In parts (b) and (c), you may assume that $x(\lambda)$, $y(\lambda)$, $z(\lambda)$, and $t(\lambda)$, as well as $\mathrm{d}x/\mathrm{d}\lambda$, $\mathrm{d}y/\mathrm{d}\lambda$, $\mathrm{d}z/\mathrm{d}\lambda$, and $\mathrm{d}t/\mathrm{d}\lambda$, are all given to sufficient accuracy by Eq. (P17.8) and its derivatives with respect to λ. (Be careful: it is likely that there are more terms than you will at first notice.)

(c) *(9 points)* Evaluate the right-hand side of Eq. (P17.7) for $\mu = y$, to first order in G. Carry out all derivatives explicitly. (It always pays to be careful.)

(d) *(2 points)* Use your answers to parts (c) and (d) to find an equation for $\mathrm{d}^2y/\mathrm{d}\lambda^2$.

(e) *(4 points)* If the photon starts out on the unperturbed trajectory, its initial value of $\mathrm{d}y/\mathrm{d}\lambda$ will be zero. The final value of $\mathrm{d}y/\mathrm{d}\lambda$ will then be

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right|_{\text{final}} = \int_{-\infty}^{\infty}\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda \ . \qquad \text{(P17.9)}$$

Use this fact to express the deflection angle α, to first order in G, as an explicit integral. You need not carry out the integral, but you may wish to use the table of integrals given below to carry it out so that you can check your answer. The correct final answer is

$$\alpha = \frac{4GM}{c^2 b\ .} \tag{P17.10}$$

TABLE OF INTEGRALS:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{b} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^{3/2}}\,\mathrm{d}x = \frac{2}{b^2} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{\pi}{2b^3}$$

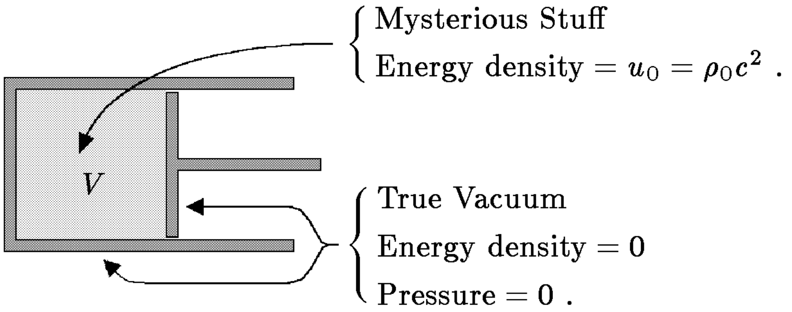
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{\pi}{2b} \qquad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^{5/2}}\,\mathrm{d}x = \frac{2}{3b^2} \qquad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^3}\,\mathrm{d}x = \frac{\pi}{8b^3}$$

∗ PROBLEM 18: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

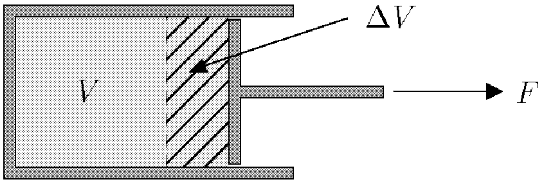
The following problem was Problem 3, Quiz 3, 2002.

In Lecture Notes 6, with further calculations in Problem 1 of Problem Set 7, a thought experiment involving a piston was used to show that $p = \frac{1}{3}\rho c^2$ for radiation. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

PROBLEM 19: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos at temperature T. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

PROBLEM 20: DOUBLING OF ELECTRONS *(10 points)*

The following was on Quiz 3, 2011 (Problem 4):

Suppose that instead of one species of electrons and their antiparticles, suppose there was also another species of electron-like and positron-like particles. Suppose that the new species has the same mass and other properties as the electrons and positrons. If this were the case, what would be the ratio T_ν/T_γ of the temperature today of the neutrinos to the temperature of the CMB photons.

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? (2020) *(30 points)*

(a) *(5 points)* What is a necessary condition for two photons in a head-on collision to be able to produce an electron-positron pair?

- (i) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 TeV.
- (ii) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 GeV.
- (iii) The energy of the photons must be below the "rest energy" of the electron-positron pair, which is roughly 1.02 keV.
- (iv) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 MeV.
- (v) Two colliding massless objects cannot produce massive particles as an outcome of their collision.

[*Comment:* The condition in item (iv) is necessary, but it is not always sufficient. A total energy exceeding 1.02 MeV will be sufficient to produce an electron-positron pair in a head-on collision if the photons have equal energy. Otherwise one would have to calculate the energy in the center-of-mass frame of reference (also called the zero-momentum frame). If the total center-of-mass energy exceeds 1.02 MeV, then there is enough energy to produce an electron-positron pair.]

(b) *(5 points)* In Chapter 4, *Recipe for a Hot Universe,* Weinberg states that there are three conserved quantities that must be specified in the recipe for the early universe. What are they? [Grading: one correct = 2 pts; two correct = 4 pts; three correct = 5 pts.]

- (i) electric charge
- (ii) baryon number
- (iii) lepton number

[*Comment:* Some students included energy and/or momentum on this list. By looking at Weinberg's Chapter 4, it is easy to verify that these items were not on his list. But it is less obvious why they do not belong on this list.

In the context of general relativity, the energy density and momentum density of everything except the gravitational field are well-defined, but they are obviously not conserved. As in Newtonian mechanics, matter can exchange energy and momentum with the gravitational field. To invoke energy or momentum conservation, we have to be able to include the energy and momentum of the gravitational field. For localized systems the metric approaches the Minkowski metric at large distances, and for such systems it is known how to construct a total energy and momentum, including gravity, that is conserved. But for our cosmological model, the metric does not approach the Minkowski metric anywhere.

We have already constructed the most general possible homogeneous and isotropic universe, described by the Robertson-Walker metric and evolving according to the Friedmann equations. We did not have to specify either the value of the energy or the value of the momentum. Since the Friedmann equations fully describe the evolution, if there was a conserved energy or momentum, the conservation would have to be a consequence of the Friedmann equations. However, the Friedmann equations can in fact be used to show that, if one does not specify the pressure, then any closed universe can evolve into any other, any open universe can evolve into any other, and any flat universe can evolve into any other. So if there is a conserved energy, then all closed universes must have the same energy, all open universes must have the same energy, and all flat universes must have the same energy. Some physicists, including Landau and Lifshitz (*The Classical Theory of Fields, p. 335*), adopt a definition of energy that implies that the total energy of any closed universe is zero. Misner, Thorne, and Wheeler, on the other hand, argue that the total energy of a closed universe is undefined. I (A.G.) side with Landau and Lifshitz.]

(c) *(5 points)* Why are "complicated" Feynman diagrams needed for calculations of the strong interactions, whereas accurate predictions can be made for electromagnetism with only the "simple" diagrams?

(i) Only the simplest Feynman diagrams in electromagnetism give a non-zero contribution; the more complicated diagrams cancel each other out.

(ii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," about 1/137.036. Complicated diagrams therefore give small contributions. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to one, so complicated diagrams are not suppressed.

(iii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," which is about equal to one, meaning that adding internal lines to a diagram doesn't change its value significantly. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to 137, so complicated diagrams are strongly enhanced.

(iv) Simple Feynman diagrams are mathematically ill-defined in the theory of strong interactions, whereas in electromagnetism they give a sensible numerical result.

(v) Calculations for the strong interactions involve six quarks, while there are only three charged leptons. If there were three more electron-like particles (in addition to the muon and the tau lepton), then electromagnetism would be as complicated as the strong interactions.

(d) *(5 points)* Consider the following observable phenomena:

(A) The motion of stars moving in large-radius circular orbits around the centers of spiral galaxies.

(B) The motion of some galaxies relative to the galaxy clusters they inhabit.

(C) Gravitational lensing of light emitted by stars in nearby galaxies.

Which one of the following combinations constitutes observational evidence that dark matter exists?

(i) Only (A).

(ii) Only (B).

(iii) Only (C).

(iv) (A) and (B).

(v) $\boxed{\text{(A), (B), and (C).}}$

[*Comment:* Since (A), (B), and (C) are all correct, answer (v) received full credit. But since the other answers are partially correct, answers (i), (ii), and (iii) received 1 point, and answer (iv) received 3 points.]

(e) *(5 points)* Consider the following claims about hadrons:

(A) Electrons and neutrinos make up 90% of the Universe's hadronic matter content.

(B) They are the class of particles affected by strong interactions (i.e., by the strong nuclear force).

(C) It is a synonym of "baryons."

(D) They are theorized to be made up of more fundamental particles, called "quarks."

(E) They cannot be made up of more fundamental constituents, because if they were, we should be able to break them up into their building blocks and observe the hypothetical "quarks" directly.

Which combination of these claims is true?

(i) Only (A).

(ii) Only (B).

(iii) $\boxed{\text{(B) and (D).}}$

(iv) (B) and (E).

(v) (B), (C), and (D).

[*Comment:* Since (B) and (D) are both correct, answer (iii) received full credit. But since answer (ii) is partially correct, it received 2 points.]

(f) *(5 points)* Which of the following is the range of typical binding energies per nucleon in an atomic nucleus? (This is relevant to nuclear fusion and nuclear fission studies, and to the epoch of Big Bang nucleosynthesis.)

(i) 1 eV – 10 eV.

(ii) 100 eV – 1 keV.

(iii) 10 keV – 100 keV.

(iv) $\boxed{\text{1 MeV – 10 MeV.}}$

(v) 100 MeV – 1 GeV.

PROBLEM 2: DID YOU DO THE READING? (2018) *(25 points)*

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

(i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.

$\boxed{\text{(ii)}}$ The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.

(iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.

(iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

[Comment: The most promising way to find the primordial value of the deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium in ***intergalactic gas clouds*** *that lie between the quasars and us.]*

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

$\boxed{\text{(i)}}$ When the temperature of the universe was about 10^{10} °K (t $\sim$ 1 sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.

(ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.

(iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first $3\frac{3}{4}$ minutes, and this is why we call this period the era of nucleosynthesis.

$\boxed{\text{(iv)}}$ After the first $3\frac{3}{4}$ minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.

(v) The protons and neutrons became decoupled from the photons after the first $3\frac{3}{4}$ minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.

(vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

[Comment: The statement (i) is described in Chapter 5 of Weinberg's book, and it has also been discussed in class. The correctness of statement (iv) can also be seen from Weinberg's Chapter 5, which says that the fraction of neutrons after the first three minutes is around 14%, and it goes down to around 13% "a little later", which he explains is probably about $3\frac{3}{4}$ minutes, depending on the ratio of photons to nuclear particles. Weinberg also explains that most of the neutrons present at this time immdiately combined with protons to form helium, which causes the ratio to be nearly constant until today. If you did not remember Weinberg's numbers, the statement that the fraction by weight of primordial helium is 26% should allow you to determine the fraction of neutrons, provided that you remember that the helium

nucleus consists of 2 protons and two neutrons, that the mass of the proton and neutron are about equal, and that the remaining 74% of the matter is essentially hydrogen, with no neutrons. Thus helium is very nearly half protons and half neutrons by weight, so the neutrons must be about 13% of the matter in the universe. (Note that we are talking about fractions of the total "baryonic" matter, which does not include the dark matter.) (ii) is clearly false, because the temperature of neutrinos becomes lower than that of photons. (iii) is clearly false, because most atoms heavier than helium were made much later in the history of the universe, in the interiors of stars. (v) is false because protons did not decouple from photons until about 350,000 years, and it happened because the plasma of protons and electrons combined to form neutral hydrogen. (vi) is false because most of the helium in the universe today is primordial. Ryden points out, for example, that the abundance of helium in the Sun's atmosphere is only about 28%. Weinberg states, near the end of Chapter 5, that "the 20-30 percent helium abundance could not have been created recently without liberating enormous amounts of radiation that we do not observe."]

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s [✓ (iii) 1940s] (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

Answer:

The neutron decays through the reaction

$$n \to p + e^- + \bar{\nu}_e \ .$$

The quantum numbers of these particles can be described by the following table:

Particle	Charge	Baryon Number	Lepton Number
Neutron (n)	0	1	0
Proton (p)	$+e$	1	0
Electron (e^-)	-e	0	1
Anti-electron-neutrino ($\bar{\nu}_e$)	0	0	-1

Thus, the total charge of the final state is zero, the total baryon number is 1, and the total lepton number is zero, in all cases matching the initial value of these quantities.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

Answer: You should have given two of the following three items.

1) *Virial theorem: The virial theorem relates the total kinetic energy of a steady-state cluster to is gravitational potential energy. Since the kinetic energy is proportional to the mass M of the cluster, while the potential energy is proportional to M^2, the relation will hold for only one value of M. By measuring the velocity dispersion (root mean square of the radial galaxy velocities relative to the mean radial velocity) and the size of the cluster, the mass can be inferred.*

2) *Hot, x-ray emitting gases: By measuring the x-rays emitted by the cluster, it is possible to model the density, temperature, and composition of the hot gas within the cluster (intracluster gas). By assuming that the gas is in hydrostatic equilibrium — i.e., by assuming that the pressure gradients balance the gravitational forces — one can infer the gravitational field, and hence the total mass.*

3) *Gravitational lensing: If one can find a galaxy behind the cluster, so that the image of the galaxy is gravitationally lensed, then the mass of the galaxy can be inferred by the degree to which the galaxy is lensed.*

PROBLEM 3: DID YOU DO THE READING? (2016) *(25 points)*

(a) *(5 points)* In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to Ω from clusters of galaxies as

(i) 0.01 (ii) 0.05 [✓ (iii) 0.20] (iv) 0.60 (v) 1.00

(b) *(4 points)* One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring

(i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.

(ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.

[✓ (iii)] the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.

(iv) the velocity dispersion along the line of sight and also the redshift of the cluster.

(v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

Explanation: The virial theorem relates the kinetic energy to the potential energy. The key relationship is

$$\frac{1}{2}M{<}v^2{>} = \frac{\alpha}{2}\frac{GM^2}{r_h} \ ,$$

where M is the mass of the cluster, ${<}v^2{>}$ is the average squared velocity of its galaxies, and r_h is the radius containing half the total mass, which is estimated by the radius containing half the luminosity. α is a numerical factor depending on the structure of the cluster, estimated at 0.4 based on observed clusters. Velocities along the line of sight are measured by the spread in Doppler shifts, while velocities perpendicular to the line of sight are essentially impossible to measure, eliminating answers (ii) and (v). Since r_h is needed, neither (i) nor (iv) include enough information. (iii) is exactly right.

(c) *(4 points)* Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.

(i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.

(ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.

(iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.

(iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.

[✓ (v)] By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.

Explanation: The dominant component of the mass is apparently dark matter, so the hot intracluster gas is only a small fraction, and we have no direct way of knowing what fraction. But the gas settles into a state of hydrostatic equilibrium which is determined by pressures and gravitational forces. The gas can be mapped by measuring its x-rays, which allows astronomers to estimate the gravitational forces, and hence the mass.

(d) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

Answer: The correct answers were $\boxed{\text{(ii), (v) and (vi).}}$ The others were incorrect for the following reasons:

(i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe".

(e) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

[✓ (ii)] In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

[✓ (iv)] In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

PROBLEM 4: DID YOU DO THE READING? (2013) *(25 points)*

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity v as a function of radius R. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii?

Answer: If stars contributed most of the mass, then at large radii the mass would appear to be concentrated as a spherical lump at the center, and the orbits of the stars would be "Keplerian," i.e., orbits in a $1/r^2$ gravitational field. Then $\vec{F} = m\vec{a}$ implies that

$$\frac{1}{R^2} \propto \frac{v^2}{R} \quad \Longrightarrow \quad v \propto \frac{1}{\sqrt{R}} \ .$$

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

Answer: $v(R)$ looks nearly flat at large radii.

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

Answer: For a gravitationally bound system in equilibrium,

$$\text{Kinetic energy} = -\frac{1}{2}\,(\text{Gravitational potential energy}) \ .$$

(The equality holds whenever $\ddot{I} \approx 0$, where I is the moment of inertia.)

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

Answer: When kT was large compared to 150 MeV, the excess of quarks over antiquarks was tiny: only about 3 extra quarks for every 10^9 antiquarks. But there was massive quark-antiquark annihilation as kT fell below 150 MeV, so that today we see the excess quarks, bound into baryons, and almost no sign of antiquarks.

PROBLEM 5: DID YOU DO THE READING? (2011) *(20 points)*[†]

(a) *(8 points)*

(i) *(4 points)* We will use the notation X^A to indicate a nucleus,[*] where X is the symbol for the element which indicates the number of protons, while A is the mass number, namely the total number of protons and neutrons. With this notation H^1, H^2, H^3, He^3 and He^4 stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + p \to He^4 + \gamma,}$$

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + n \to He^4 + \gamma.}$$

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place (in general with different probabilities). For example:

$$p + n \to H^2 + \gamma \qquad H^2 + H^2 \to He^4 + \gamma,$$

$$p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + H^2 \to He^4 + n,$$

$$p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + H^2 \to He^4 + p,$$

$$\ldots$$

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[*] Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.

Students who described chains different from those of Weinberg, but that can still take place, got full credit for this part. Also, notice that photons in the reactions above carry the additional energy released. However, since the main point was to describe the nuclear reactions, students who didn't include the photons still received full credit.

(ii) *(4 points)* The *deuterium bottleneck* is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium (H^2) plays a crucial role in nucleosynthesis, since it is the starting point for all the chains. However, the deuterium nucleus is extremely loosely bound compared to H^3, He^3, or especially He^4. So, there will be a range of temperatures which are low enough for H^3, He^3, and He^4 nuclei to be bound, but too high to allow the deuterium nucleus to be stable. This is the temperature range where the *deuterium bottleneck* is in action: even if H^3, He^3, and He^4 nuclei could in principle be stable at those temperatures, they do not form because deuterium, which is the starting point for their formation, cannot be formed yet. Nucleosynthesis cannot proceed at a significant rate until the temperature is low enough so that deuterium nuclei are stable; at this point the deuterium bottleneck has been passed.

(b) *(12 points)*

(i) *(3 points)* If we take $a(t) = bt^{1/2}$, for some constant b, we get for the Hubble expansion rate:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \quad \Longrightarrow \quad \boxed{t = \frac{1}{2H}.}$$

(ii) *(6 points)* By using the Friedmann equation with $k = 0$ and $\rho = \rho_r = \alpha T^4$, we find:

$$H^2 = \frac{8\pi}{3} G\rho_r = \frac{8\pi}{3} G\alpha T^4 \quad \Longrightarrow \quad \boxed{H = T^2 \sqrt{\frac{8\pi}{3} G\alpha}\ .}$$

If we substitute the given numerical values $G \simeq 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$ and $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg \cdot m^{-3} \cdot K^{-4}}$ we get:

$$\boxed{H \simeq T^2 \times 5.03 \times 10^{-21}\,\mathrm{s^{-1} \cdot K^{-2}}\ .}$$

Notice that the units correctly combine to give H in units of $\mathrm{s^{-1}}$ if the temperature is expressed in degrees Kelvin (K). In detail, we see:

$$[G\alpha]^{1/2} = (\mathrm{N \cdot m^2 \cdot kg^{-2} \cdot kg \cdot m^{-3} \cdot K^{-4}})^{1/2} = \mathrm{s^{-1} \cdot K^{-2}}\ ,$$

where we used the fact that 1 N = 1 kg · m · s^{-2}. At $T = T_{\text{nucl}} \simeq 0.9 \times 10^9$K we get:

$$\boxed{H \simeq 4.07 \times 10^{-3}\text{s}^{-1}.}$$

(iii) *(3 points)* Using the results in parts (i) and (ii), we get

$$\boxed{t = \frac{1}{2H} \simeq \left(\frac{9.95 \times 10^{19}}{T^2}\right) \text{s} \cdot \text{K}^2 \ .}$$

To good accuracy, the numerator in the expression above can be rounded to 10^{20}. The above equation agrees with Weinberg's claim that, for a radiation dominated universe, time is proportional to the inverse square of the temperature. In particular for $T = T_{\text{nucl}}$ we get:

$$\boxed{t_{\text{nucl}} \simeq 123 \text{ s} \approx 2 \text{ min.}}$$

†Solution written by Daniele Bertolini.

∗ PROBLEM 6: EVOLUTION OF AN OPEN UNIVERSE

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh\theta - \theta)$$

$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \ .$$

Evaluating the second of these equations at $a/\sqrt{\kappa} = 2\alpha$ yields a solution for θ:

$$2\alpha = \alpha(\cosh\theta - 1) \quad \Longrightarrow \quad \cosh\theta = 3 \quad \Longrightarrow \quad \theta = \cosh^{-1}(3) \ .$$

We can use these results in the first equation to solve for t. Noting that

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{8} = 2\sqrt{2} \ ,$$

we have

$$\boxed{t = \frac{\alpha}{c}\left[2\sqrt{2} - \cosh^{-1}(3)\right] \ .}$$

Numerically, $t \approx 1.06567\,\alpha/c$.

∗ PROBLEM 7: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\, dt^2 + a^2(t) d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{a^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{a(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{a(t')} dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} a(t)\, d\psi = \boxed{a(t) \int_0^t \frac{c}{a(t')} dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1 - \cos\theta)}{a(t)} \ .$$

Then using $a = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $a = \alpha(1 - \cos\theta)$, one can see that a returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 5. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 \ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 8: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1+ar)^2\, dr^2$, or $ds = (1+ar)\, dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\, dr = r_0 + \frac{1}{2} a r_0^2 \ .$$

Along the second segment $dr = 0$, so $ds = r(1 + br)\, d\theta$, where $r = r_0$. So the length of the second segment is

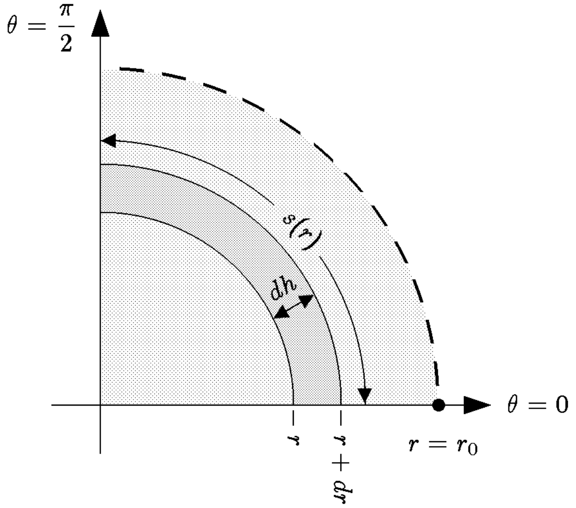
$$S_2 = \int_0^{\pi/2} r_0(1 + br_0)\, d\theta = \frac{\pi}{2} r_0(1 + br_0) \ .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$S = 2S_1 + S_2 = 2\left(r_0 + \frac{1}{2} a r_0^2\right) + \frac{\pi}{2} r_0 (1 + br_0)$$

$$= \boxed{\left(2 + \frac{\pi}{2}\right) r_0 + \frac{1}{2}(2a + \pi b) r_0^2 \ .}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar)\, dr \\ &= \frac{\pi}{2} \int_0^{r_0} [r + (a + b)r^2 + abr^3]\, dr \\ &= \boxed{\frac{\pi}{2} \left[\frac{1}{2} r_0^2 + \frac{1}{3}(a + b) r_0^3 + \frac{1}{4} abr_0^4\right]} \end{aligned}$$

PROBLEM 9: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = a(t)\frac{dr}{\sqrt{1-kr^2}} \times a(t) r d\theta \times a(t) r \sin\theta d\phi$$

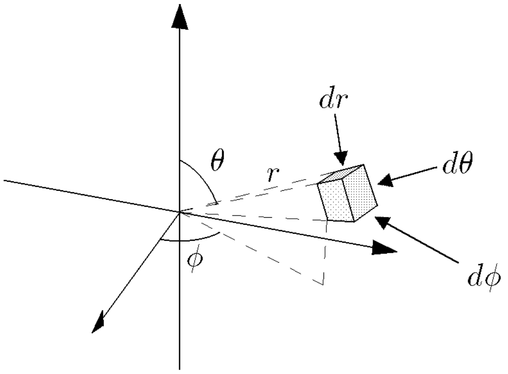
The total volume is then

$$V = \int dV = a^3(t) \int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi a^3(t) \int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}} \ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r + dr$, between θ and $\theta + d\theta$, and between ϕ and $\phi + d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$\begin{aligned} ds_1 &= a(t)\frac{dr}{\sqrt{1-kr^2}} \\ ds_2 &= a(t) r \, d\theta \\ ds_3 &= a(t) r \sin\theta \, d\theta \ . \end{aligned}$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\, d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\sqrt{k}\, r = \sin\psi \quad (\text{if } k > 0)$$
$$\sqrt{-k}\, r = \sinh\psi \quad (\text{if } k > 0) .$$

The answer is

$$V = \begin{cases} 2\pi a^3(t) \left[\dfrac{\sin^{-1}\left(\sqrt{k}\, r_{\rm max}\right)}{k^{3/2}} - \dfrac{\sqrt{1 - k r_{\rm max}^2}}{k} \right] & (\text{if } k > 0) \\ 2\pi a^3(t) \left[\dfrac{\sqrt{1 - k r_{\rm max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\, r_{\rm max}\right)}{(-k)^{3/2}} \right] & (\text{if } k < 0) \quad .] \end{cases}$$

PROBLEM 10: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(h, 0, 0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{a\, dr}{\sqrt{1 - r^2}} \ .$$

So

$$\boxed{\ell_p = \int_0^h \frac{a\, dr}{\sqrt{1 - r^2}} = a \sin^{-1} h \ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = ar\, d\theta \ ,$$

so

$$\boxed{s_p = ah\, \Delta\theta \ .}$$

(c) From part (a), one has

$$h = \sin(\ell_p / a) \ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{a \sin(\ell_p / a)} \ .}$$

Note that as $a \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p / \ell_p$.

∗ PROBLEM 11: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from} \\ \text{origin to } r_0}} ds \ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0 \ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\, d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0) \sin\theta \, d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\, d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0) \sin\theta \, d\theta \, d\phi \ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0) \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \\ &= \rho^2(r_0)(2\pi) \left(-\cos\theta \Big|_0^{\pi} \right) \\ \Longrightarrow & \quad \boxed{A = 4\pi\rho^2(r_0) \ .} \end{aligned}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 2 of Problem Set 5, we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\,dr \ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi \int_0^{r_0} \rho^2(r)\,dr \ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma} \ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma} \ .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma}\,)$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}\,)\left[d\theta^2 + \sin^2\theta\,d\phi^2\right] \ .}$$

PROBLEM 12: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda} \ .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij} dx^i dx^j = dr^2 + r^2\, d\theta^2 \ ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 \ , \qquad g_{r\theta} = g_{\theta r} = 0 \ .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{ g_{rr} \frac{dr}{d\lambda} \right\} = \frac{1}{2} \left(\partial_r g_{\theta\theta}\right) \frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda} \ ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{ \frac{dr}{d\lambda} \right\} = \frac{1}{2} \left(\partial_r r^2\right) \left(\frac{d\theta}{d\lambda} \right)^2 \ ,$$

or

$$\boxed{\frac{d^2 r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda} \right)^2 \ .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i, j). So

$$\boxed{\frac{d}{d\lambda}\left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 \ .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$\begin{aligned} r(\lambda) &= \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ , \\ \theta(\lambda) &= \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ . \end{aligned}$$

Calculating the needed derivatives,*

$$\begin{aligned}\frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2+1}}\\ \frac{d^2r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3}\\ \frac{d\theta}{d\lambda} &= -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2}\frac{1}{\lambda^2} = -\frac{1}{r^2}\ .\end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \Longleftrightarrow \frac{1}{r^3} = r\left(-\frac{1}{r^2}\right)^2\ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \Longleftrightarrow \frac{d}{d\lambda}\left\{r^2\left(-\frac{1}{r^2}\right)\right\} = 0\ ,$$

which also checks.

∗ PROBLEM 13: GEODESICS IN A CLOSED UNIVERSE

(a) *(7 points)* For purely radial motion, $d\theta = d\phi = 0$, so the line element reduces do

$$-c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2}\right\}\ .$$

Dividing by dt^2,

$$-c^2\left(\frac{d\tau}{dt}\right)^2 = -c^2 + \frac{a^2(t)}{1-r^2}\left(\frac{dr}{dt}\right)^2\ .$$

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* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi/\cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = (1+\tan^2\phi)d\phi\ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2}\ .$$

Rearranging,

$$\boxed{\frac{d\tau}{dt} = \sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}\ .}$$

(b) *(3 points)*

$$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}} = \boxed{\frac{1}{\sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}}\ .}$$

(c) *(10 points)* During any interval of clock time dt, the proper time that would be measured by a clock moving with the object is given by $d\tau$, as given by the metric. Using the answer from part (a),

$$d\tau = \frac{d\tau}{dt}\,dt = \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .$$

Integrating to find the total proper time,

$$\boxed{\tau = \int_{t_1}^{t_2}\sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .}$$

(d) *(10 points)* The physical distance $d\ell$ that the object moves during a given time interval is related to the coordinate distance dr by the spatial part of the metric:

$$d\ell^2 = ds^2 = a^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \quad\Longrightarrow\quad d\ell = \frac{a(t)}{\sqrt{1-r^2}}\,dr\ .$$

Thus

$$\boxed{v_{\rm phys} = \frac{d\ell}{dt} = \frac{a(t)}{\sqrt{1-r^2}}\,\frac{dr}{dt}\ .}$$

Discussion: A common mistake was to include $-c^2\,dt^2$ in the expression for $d\ell^2$. To understand why this is not correct, we should think about how an observer would measure $d\ell$, the distance to be used in calculating the velocity of a passing object.

The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval $dt_{\rm meas}$. Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the ***same*** time t. Thus, when we compute the distance between the two marks, we set $dt = 0$. To compute the speed she would then divide the distance by $dt_{\rm meas}$, which is nonzero.

(e) *(10 points)* We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \ .$$

This formula is true for each possible value of μ, while the Einstein summation convention implies that the indices ν, λ, and σ are summed. We are trying to derive the equation for r, so we set $\mu = r$. Since the metric is diagonal, the only contribution on the left-hand side will be $\nu = r$. On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when $\lambda = \sigma$. The term will vanish unless $dx^\lambda/d\tau$ is nonzero, so λ must be either r or t (i.e., there is no motion in the θ or ϕ directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r} \ .$$

Since $g_{tt} = -c^2$, the derivative with respect to r will vanish. Thus, the only nonzero contribution on the right-hand side arises from $\lambda = \sigma = r$. Using

$$g_{rr} = \frac{a^2(t)}{1-r^2} \ ,$$

the geodesic equation becomes

$$\frac{d}{d\tau}\left\{g_{rr}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left(\partial_r g_{rr}\right)\frac{dr}{d\tau}\frac{dr}{d\tau} \ ,$$

or

$$\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left[\partial_r\left(\frac{a^2}{1-r^2}\right)\right]\frac{dr}{d\tau}\frac{dr}{d\tau} \ ,$$

or finally

$$\boxed{\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = a^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2 \ .}$$

This matches the form shown in the question, with

$$A = \frac{a^2}{1-r^2} \ , \text{ and } C = a^2 \frac{r}{(1-r^2)^2} \ ,$$

with $B = D = E = 0$.

(f) *(5 points EXTRA CREDIT)* The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for p. Using the answer from (d),

$$p = \frac{m v_{\text{phys}}}{\sqrt{1 - \frac{v_{\text{phys}}^2}{c^2}}} = \frac{m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt}}{\sqrt{1 - \frac{a^2}{c^2(1-r^2)} \left(\frac{dr}{dt}\right)^2}} \ .$$

Using the answer from (b), this simplifies to

$$p = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt} \frac{dt}{d\tau} = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{d\tau} \ .$$

Multiply the geodesic equation by m, and then use the above result to rewrite it as

$$\frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} = ma^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\tau} \right)^2 \ .$$

Expanding the left-hand side,

$$\begin{aligned} LHS = \frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ap \frac{r}{(1-r^2)^{3/2}} \frac{dr}{d\tau} \\ &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ma^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\tau} \right)^2 \ . \end{aligned}$$

Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} = 0 \ .$$

Multiplying by $\sqrt{1-r^2}$, one has the desired result:

$$\frac{d}{d\tau} \{ap\} = 0 \quad \Longrightarrow \quad \boxed{p \propto \frac{1}{a(t)} \ .}$$

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE *(25 points)*

(a) Taking $i = \theta$ in Eq. (14.1), and remembering that the metric is diagonal and that only $g_{\phi\phi}$ depends on θ, the equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial\theta}\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} \ .$$

Using $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2\sin^2\theta$, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{R^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial(R^2\sin^2\theta)}{\partial\theta}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 \ .$$

Simplifying,

$$\boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \sin\theta\,\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 \ .}$$

(b) Taking $i = \phi$ and remembering that the metric is diagonal and that none of the components of $g_{\phi\phi}$ depend on ϕ, the geodesic equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 \ .$$

Again using $g_{\phi\phi} = R^2\sin^2\theta$, this becomes

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{R^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 \ .}$$

The equation above is a perfectly acceptable answer. One might also recognize that R^2 can be factored out, giving

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 \ .}$$

If one wishes, one could expand the derivative using the product rule, giving

$$\boxed{\sin^2\theta\,\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\sin\theta\,\cos\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\,\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \ .}$$

(c) The ϕ equation is trivially satisfied, since

$$\sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}s} = \sin^2(\theta_0)\,\beta \ ,$$

which is constant, so its derivative vanishes. The θ equation becomes

$$0 = \beta^2 \sin\theta_0 \,\cos\theta_0 \ ,$$

which is satisfied for $\theta_0 = \pi/2$. Thus,

$$\boxed{\text{the curves are geodesics if and only if } \theta_0 = \pi/2.}$$

The geodesic equations are formally satisfied if $\theta_0 = 0$ or $\theta_0 = \pi$, since then $\sin\theta_0$ vanishes. These are not actually geodesics, however, due to the singularity of the polar coordinate system at $\theta = 0$ and $\theta = \pi$. For these values of θ, ϕ is undefined, and the curve described by Eq. (14.3) degenerates into a point. (Although $\theta_0 = 0$ or $\theta_0 = \pi$ are not actually geodesics, no points were taken off for including them in the answer.)

(d)

$$\boxed{\text{The curves are geodesics for all values of } \phi_0.}$$

The θ equation is satisfied, since $\mathrm{d}^2\theta/\mathrm{d}s^2 = 0$ and $\mathrm{d}\phi/\mathrm{d}s = 0$, while $\mathrm{d}\phi/\mathrm{d}s = 0$ alone is sufficient to imply that the ϕ equation is satisfied.

(e) The ϕ equation already has this form,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}L}{\mathrm{d}s} = 0 \ ,$$

for

$$L = \sin^2\theta\,\frac{\mathrm{d}\phi}{\mathrm{d}s} \ ,$$

so

$$\boxed{F(\theta,\phi) = \sin^2\theta \ .}$$

It would of course also be correct to use $F(\theta,\phi) = R^2\sin^2\theta$, or more generally $F(\theta,\phi)$ can be any constant times $\sin^2\theta$.

(f) Using the definition of L chosen in the solution above for part (e), we can write

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{L}{\sin^2\theta} \ .$$

Substituting this into the geodesic equation for θ, we find

$$\boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \frac{\cos\theta}{\sin^3\theta}\,L^2 \ .}$$

If L were defined with a different multiplicative constant, then this equation would be modified accordingly.

∗ PROBLEM 15: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\phi + \omega\,\mathrm{d}t\right)^2 + \mathrm{d}z^2\right] ,$$

and the metric coefficients are then just read off from this expression:

$$\begin{aligned}
&g_{11} \equiv g_{rr} = 1\\
&g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = -c^2 + r^2\omega^2\\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = r^2\omega\\
&g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = r^2\\
&g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = 1 \ .
\end{aligned}$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by $1/2$, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\,dx^\mu\,dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\,g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned}
RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .
\end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 ,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (P15.1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)} ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}}\ .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\ ,$$

adapted to the rotating cylindrical coordinate system.

PROBLEM 16: TRAVEL IN THE SCHWARZSCHILD METRIC *(20 points)*

(a) For B, the coordinate time interval between pulses, Δt_B, is related to the proper time interval between pulses, $\Delta\tau_B$, by the metric relation

$$-c^2\,\Delta\tau_B^2 = -\left(1 - \frac{2GM}{r_B c^2}\right)c^2\Delta t_B^2\ .$$

The other terms vanish, since B is stationary, so $\Delta r = \Delta\theta = \Delta\phi = 0$. Using $r_B = 4R_S$, it follows after a litle algebra that

$$\Delta t_B = \sqrt{\frac{4}{3}}\,\Delta\tau_B\ .$$

$\Delta\tau_B$ is the time interval that would be measured on B's local clock. (If this is not clear, it may help to read again the section *The Generalization from Space to Spacetime*, in Lecture Notes 5. Note also that B's clock is not free-falling, since B is using a rocket to hold her position. In the section on *Accelerating Clocks* in Lecture Notes 1, we discuss why it is reasonable to assume an "ideal clock" for which any corrections due to acceleration can be neglected. If the corrections are not negligible, then they depend on the detailed construction of the clock.)

Since the metric does not depend on time, each pulse traveling from B to A will take the same amount of coordinate time for the trip. So the coordinate time interval at A will be $\Delta t_A = \Delta t_B$, and hence $\Delta t_A = \sqrt{\frac{4}{3}}\,\Delta\tau_B$. Finally, we note that for A, at $r = \infty$, the metric reduces to the Minkowski metric, and since A is stationary, this gives $\Delta\tau_A = \Delta t_A$. So finally,

$$\boxed{\Delta\tau_A = \sqrt{\frac{4}{3}}\,\Delta\tau_B\ .}$$

(b) From Eq. (16.4), it is clear that the coordinate time interval for one orbit is

$$\Delta t_{\rm orb} = \frac{2\pi}{\omega_{\rm orb}} \ ,$$

so the problem is reduced to finding the relation between coordinate time and proper time for C. For C, the metric relation becomes

$$-c^2 \Delta\tau_C^2 = -\left(1 - \frac{2GM}{r_C c^2}\right) c^2 \Delta t_C^2 + r_C^2 \sin^2\theta_C \, \Delta\phi_C^2 \ .$$

Using $\Delta\phi_C = \omega_{\rm orb} \, \Delta t_C$ and then

$$r_C^2 \, \omega_{\rm orb}^2 = \frac{GM}{r_C} = \frac{1}{8} c^2 \ ,$$

and also using $\sin^2\theta_C = 1$, one finds

$$-c^2 \Delta\tau_C^2 = -\frac{3}{4} c^2 \, \Delta t_C^2 + \frac{1}{8} c^2 \, \Delta t_C^2 = -\frac{5}{8} c^2 \, \Delta t_C^2 \ ,$$

so $\Delta\tau_C = \sqrt{\frac{5}{8}} \, \Delta t_C$. Thus

$$\Delta\tau_C = \sqrt{\frac{5}{2}} \frac{\pi}{\omega_{\rm orb}} \ .$$

Using Eq. (16.5) and $r = 4R_S$, one finds that

$$\omega_{\rm orb} = \frac{c^3}{16\sqrt{2}\, GM} \ ,$$

which can be used to replace $\omega_{\rm orb}$ in the previous equation, resulting in

$$\boxed{\Delta\tau_C = 16\sqrt{5}\, \pi \frac{GM}{c^3} \ .}$$

(c) The clock time for one orbit is clearly

$$\Delta t_D = \frac{\pi}{\omega_{\rm orb}} \ ,$$

but we need to convert to proper time. For D, they are related by

$$\begin{aligned} -c^2 \Delta\tau_D^2 &= -\left(1 - \frac{2GM}{r_D c^2}\right) c^2 \Delta t_D^2 + r_D^2 \sin^2\theta_D \, \Delta\phi_D^2 \\ &= -\frac{3}{4} c^2 \Delta t_D^2 + r_D^2 (2\omega_{\rm orb} \, \Delta t_D)^2 \ . \end{aligned}$$

Using $r_D^2\omega_{\rm orb}^2 = \frac{1}{8}\,c^2$, this becomes

$$-c^2\Delta\tau_D^2 = -\frac{3}{4}c^2\Delta t_D^2 + \frac{1}{2}c^2\Delta t_D^2 = -\frac{1}{4}c^2\Delta t_D^2 \ .$$

So

$$\Delta\tau_D = \frac{1}{2}\,\Delta t_D \ ,$$

and then for one orbit,

$$\Delta\tau_D = \frac{\pi}{2\omega_{\rm orb}} \ .$$

Using the previous equation for $\omega_{\rm orb}$, this becomes

$$\boxed{\Delta\tau_D = \frac{8\sqrt{2}\,\pi GM}{c^3} \ .}$$

(d) For this case the metric reduces to

$$ds^2 = r^2\,{\rm d}\phi^2 \ ,$$

so

$$\Delta s = 2\pi r = 8\pi R_S = \boxed{\frac{16\pi GM}{c^2} \ .}$$

* PROBLEM 17: GRAVITATIONAL BENDING OF LIGHT *(30 points)*

(a) *(6 points)* Note that

$$\begin{aligned}{\rm d}r^2 &= \frac{1}{r^2}\left(x\,{\rm d}x + y\,{\rm d}y + z\,{\rm d}z\right)^2 \\ &= \frac{1}{r^2}\left(x^2\,{\rm d}x^2 + y^2\,{\rm d}y^2 + z^2\,{\rm d}z^2 + 2xy\,{\rm d}x\,{\rm d}y + 2xz\,{\rm d}x\,{\rm d}z + 2yz\,{\rm d}y\,{\rm d}z\right) \ .\end{aligned} \tag{S17.1}$$

By using this expression for $({\rm d}r)^2$ in Eq. (P17.5), we have the full expression for ${\rm d}s^2$ written out, from which we can read off the components of $g_{\mu\nu}$:

$$\boxed{\begin{aligned}g_{tt} &= \text{coefficient of } {\rm d}t^2 = -c^2\left(1 - \frac{R_{\rm Sch}}{r}\right) \\ g_{xx} &= \text{coefficient of } {\rm d}x^2 = 1 + \frac{R_{\rm Sch}}{r^3}\,x^2 \\ g_{xy} &= \frac{1}{2} \text{ of coefficient of } {\rm d}x\,{\rm d}y = \frac{R_{\rm Sch}}{r^3}\,xy \ .\end{aligned}} \tag{S17.2}$$

A number of people missed the factor of 1/2 in the value of g_{xy}. It arises because the general formula is written as $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, which when expanded becomes

$$\mathrm{d}s^2 = g_{xx}\mathrm{d}x^2 + g_{yy}\mathrm{d}y^2 + g_{zz}\mathrm{d}z^2 + g_{tt}\mathrm{d}t^2 + g_{xy}\mathrm{d}x\,\mathrm{d}y + g_{yx}\mathrm{d}y\,\mathrm{d}x + \ldots\ .$$

Since $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}y\,\mathrm{d}x$, the coefficient of $\mathrm{d}x\,\mathrm{d}y$ is $g_{xy} + g_{yx} = 2g_{xy}$.

(b) *(9 points)* It will be useful to know the derivatives of r:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}\\ &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{x}{r}\ .\end{aligned} \tag{S17.3}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \text{and} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}\ , \tag{S17.4}$$

and

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\lambda} &= \frac{\partial r}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{\partial r}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\lambda} + \frac{\partial r}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}\lambda}\\ &= \frac{x}{r}\ .\end{aligned} \tag{S17.5}$$

In the 2nd line I used the value of $\partial r/\partial x$ from Eq. (S17.3), and the derivatives

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda} = 1\ , \qquad \frac{\mathrm{d}y}{\mathrm{d}\lambda} = \frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0 \tag{S17.6}$$

that can be found from Eq. (P17.8).

Now, to expand the left-hand side of the geodesic equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right\} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{yy}\frac{\mathrm{d}y}{\mathrm{d}\lambda} + g_{yx}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right\}\\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\left[1 + \frac{R_{\mathrm{Sch}}}{r^3}y^2\right]\frac{\mathrm{d}y}{\mathrm{d}\lambda} + \frac{R_{\mathrm{Sch}}}{r^3}xy\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right\}\\ &= \frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2} - 3\frac{R_{\mathrm{Sch}}}{r^4}\frac{x}{r}xy\frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{R_{\mathrm{Sch}}}{r^3}\frac{\mathrm{d}x}{\mathrm{d}\lambda}y\frac{\mathrm{d}x}{\mathrm{d}\lambda}\\ &= \boxed{\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2} - 3\frac{R_{\mathrm{Sch}}b}{r^5}x^2 + \frac{R_{\mathrm{Sch}}b}{r^3}\ .}\end{aligned} \tag{S17.7}$$

Note that I dropped a term

$$\frac{R_{\mathrm{Sch}}y^2}{r^3}\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2}$$

and a term

$$\frac{R_{\rm Sch}}{r^3} x y \frac{{\rm d}^2 x}{{\rm d}\lambda^2} \ ,$$

which is justified because the acceleration $\frac{{\rm d}^2 y}{{\rm d}\lambda^2}$ will be proportional to G, and $R_{\rm Sch}$ is proportional to G, so this term is 2nd order in G. The problem stated that we are to work to first order in G. No points were taken off, however, from students who retained these or other negligible terms.

Note, however, that ${\rm d}^2 y \ {\rm d}\lambda^2$ is not negligible, and appears in the answer. This is because ${\rm d}y/{\rm d}\lambda$ is not actually zero, but is of order G. ${\rm d}y/{\rm d}\lambda$ is zero for the unperturbed path, but in reality the photon picks up a small velocity in the y-direction, caused by the gravitational attraction of the Sun and proportional to G. ${\rm d}^2 y/{\rm d}\lambda^2$ will also be proportional to G. When ${\rm d}y/{\rm d}\lambda$ multiplies a factor proportional to $R_{\rm Sch}$, the product is of order G^2 and hence negligible. But ${\rm d}^2 y/{\rm d}\lambda^2$ by itself is of order G and is not negligible.

Note on propagation of errors: I normally do not take off points for propagating errors, so for example a student who forgot the factor of 1/2 in determining g_{xy} would get full credit on part (b), even though the answer would contain terms that are wrong by a factor of 1/2. However, it seems right to me to make an exception to this rule in cases where an error on part (a) causes the consequent answer on a later part to become trivial. For example, if a student described a metric in part (a) which had no dependence on r, then many of the terms in parts (b) and (c) would not be present. In such cases I still took off points in parts (b) and (c), because it didn't seem fair to me to give such a student credit for calculating these terms, when the student exhibited no such capability.

(c) *(9 points)*

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial y}\left(g_{\sigma\tau}\right)\frac{{\rm d}x^\sigma}{{\rm d}\lambda}\frac{{\rm d}x^\tau}{{\rm d}\lambda} &= \frac{1}{2}\frac{\partial}{\partial y}(g_{xx})\left(\frac{{\rm d}x}{{\rm d}\lambda}\right)^2 + \frac{1}{2}\frac{\partial}{\partial y}(g_{tt})\left(\frac{{\rm d}t}{{\rm d}\lambda}\right)^2 \\ &= \frac{1}{2}\frac{\partial}{\partial y}\left(1+\frac{R_{\rm Sch}}{r^3}x^2\right) - \frac{1}{2}c^2\frac{\partial}{\partial y}\left(1-\frac{R_{\rm Sch}}{r}\right)\left(\frac{1}{c^2}\right) \\ &= -\frac{1}{2}\left(3\frac{R_{\rm Sch}}{r^4}\frac{y}{r}x^2\right) - \frac{1}{2}\left(\frac{R_{\rm Sch}}{r^2}\frac{y}{r}\right) \\ &= \boxed{-\frac{3}{2}\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{1}{2}\frac{R_{\rm Sch}b}{r^3} \ .} \end{aligned} \tag{S17.8}$$

(d) *(2 points)* Combining Eqs. (S17.7) and (S17.8), we find

$$\begin{aligned} \frac{{\rm d}^2 y}{{\rm d}\lambda^2} &= -\frac{3}{2}\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{1}{2}\frac{R_{\rm Sch}b}{r^3} + 3\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{R_{\rm Sch}b}{r^3} \\ &= \boxed{\frac{3}{2}R_{\rm Sch}b\left[\frac{x^2}{r^5} - \frac{1}{r^3}\right] \ .} \end{aligned} \tag{S17.9}$$

(e) *(4 points)* The final value of $\mathrm{d}y/\mathrm{d}\lambda$ is given by Eq. (P17.9), while the final value of $\mathrm{d}x/\mathrm{d}\lambda$ will be equal to 1, at least up to possible corrections proportional to G. Thus, the final velocity will make an angle α relative to the horizontal, where

$$\begin{aligned}\tan\alpha &= \frac{\mathrm{d}y/d\lambda|_{\text{final}}}{\mathrm{d}x/d\lambda|_{\text{final}}} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda\ .\end{aligned}$$

Since $\tan\alpha$ will be proportional to G, the small angle approximation $\tan\alpha = \alpha$ will apply, and

$$\boxed{\alpha \approx \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda\ .} \qquad \text{(S17.10)}$$

Then, using Eqs. (S17.9) and combining with Eqs. (P17.4) and (P17.8),

$$\begin{aligned}\alpha &= \frac{3}{2} R_{\text{Sch}} b \int_{-\infty}^{\infty} \left[\frac{x^2}{r^5} - \frac{1}{r^3}\right] \mathrm{d}\lambda \\ &= \boxed{\frac{3}{2} R_{\text{Sch}} b \int_{-\infty}^{\infty} \left[\frac{\lambda^2}{(\lambda^2+b^2)^{5/2}} - \frac{1}{(\lambda^2+b^2)^{3/2}}\right] \mathrm{d}\lambda\ .}\end{aligned} \qquad \text{(S17.11)}$$

You were not asked to carry out these integrals, but using the table of integrals given with the problem, one finds

$$\alpha = \frac{3}{2} R_{\text{Sch}} b \left[\frac{2}{3b^2} - \frac{2}{b^2}\right] = -\frac{2R_{\text{Sch}}}{b} = -\frac{4GM}{c^2 b}\ . \qquad \text{(S17.12)}$$

The minus sign indicates that the deflection is downward, as one would expect.

∗ PROBLEM 18: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V+\Delta V) = u_0 \sqrt{\frac{V}{V+\Delta V}}\ .$$

(The above expression is proportional to $1/\sqrt{V+\Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1+\frac{\Delta V}{V}}} = \frac{u_0}{1+\frac{1}{2}\frac{\Delta V}{V}} = u_0\left(1 - \frac{1}{2}\frac{\Delta V}{V}\right)\ .$$

The total energy is the energy density times the volume, so

$$U = u(V+\Delta V) = u_0\left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V\left(1 + \frac{\Delta V}{V}\right) = U_0\left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V}U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V}U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}u_0 \ .}$$

PROBLEM 19: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}kT \ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT \ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{2\pi^4}{45\zeta(3)} k \ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is 7/8 times the number of spin states, and g^* is 3/4 times the number of spin states. So the average energy per particle is

$$E = \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{180\zeta(3)} kT \ .}$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$\begin{aligned} S &= \frac{g\dfrac{2\pi^2}{45}\,\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \frac{\dfrac{7}{8}\,\dfrac{2\pi^2}{45}\,\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\,\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{7\pi^4}{135\zeta(3)}\,k\ .} \end{aligned}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 20: DOUBLING OF ELECTRONS *(10 points)*

The entropy density of black-body radiation is given by

$$\begin{aligned} s &= g\left[\frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3}\right]T^3 \\ &= g\,C\,T^3\ , \end{aligned}$$

where C is a constant. At the time when the electron-positron pairs disappear, the neutrinos are decoupled, so their entropy is conserved. All of the entropy from electron-positron pairs is given to the photons, and none to the neutrinos. The same will be true here, for both species of electron-positron pairs.

The conserved neutrino entropy can be described by $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation n^- and n^+ for the new electron-like and positron-like particles, and also the convention that

Primed quantities: values after $e^+e^-n^+n^-$ annihilation

Unprimed quantities: values before $e^+e^-n^+n^-$ annihilation.

For the neutrinos,

$$S'_\nu = S_\nu \quad \Longrightarrow \quad g_\nu C \left(a' T'_\nu\right)^3 = g_\nu C \left(a T_\nu\right)^3 \quad \Longrightarrow$$

$$\boxed{a' T'_\nu = a T_\nu \ .}$$

For the photons, before $e^+e^-n^+n^-$ annihilation we have

$$T_\gamma = T_{e^+e^-n^+n^-} = T_\nu \ ; \qquad g_\gamma = 2, \ g_{e^+e^-} = g_{n^+n^-} = 7/2 \ .$$

When the e^+e^- and n^+n^- pairs annihilate, their entropy is added to the photons:

$$S'_\gamma = S_{e^+e^-} + S_{n^+n^-} + S_\gamma \quad \Longrightarrow \quad 2C \left(a' T'_\gamma\right)^3 = \left(2 + 2 \cdot \frac{7}{2}\right) C \left(a T_\gamma\right)^3 \quad \Longrightarrow$$

$$\boxed{a' T'_\gamma = \left(\frac{9}{2}\right)^{1/3} a T_\gamma \ ,}$$

so aT_γ increases by a factor of $(9/2)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{2}{9}\right)^{1/3} T'_\gamma \ .}$$

This ratio would remain unchanged until the present day.