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Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 24, 2022

Prof. Alan Guth

QUIZ 3 SOLUTIONS

Quiz Date: December 7, 2022

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins,
and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	20	
3	30	
4	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(2 points)* At the time of recombination, the temperature is

(i) above the proton and electron masses

(ii) below the proton mass but above the electron mass

(iii) $\boxed{\text{below the proton and electron masses}}$

Comment: The temperature of recombination was estimated in Lecture Notes 6 as about 4,000 K, and was calculated by Ryden after Eq. (8.37) as $T_{\rm rec} = 3720\,\rm K$. Ryden gives the equivalent energy in Eq. (8.37) as $kT_{\rm rec} = 0.324$ eV, which is far below the proton mass ($m_p c^2 = 938$ MeV) or the electron mass ($m_e c^2 = 0.511$ MeV).

(b) *(2 points)* At the time of recombination, the universe is

(i) $\boxed{\text{matter-dominated}}$

(ii) radiation-dominated

(iii) vacuum-energy dominated

Comment: In Table 8.1 (p. 157) Ryden gives the time of recombination as 250,000 years, and the time of matter-radiation equality as 50,000 years. (For completeness, we mention that the same table gives the time of photon decoupling as 370,000 years.) In Table 5.2 (p. 96), Ryden gives the time of matter-lambda equality (i.e., the equality of matter and vacuum energy) as 10.2 Gyr. So the time of recombination is unambiguously in the matter-dominated era.

(c) *(2 points)* The reaction that produces the most deuterium in the early universe is

(i) proton-proton fusion ($p + p \to D+$ maybe other elements)

(ii) $\boxed{\text{proton-neutron fusion } (p + n \to D+ \text{ maybe other elements})}$

(iii) neutron-neutron fusion ($n + n \to D+$ maybe other elements)

Comment: This is discussed by Ryden on pp. 172–173.

(d) *(2 points)* For the microcanonical ensemble (describing an isolated system in thermal equilibrium), the probability that the system is in a state i with energy E_i is proportional to

(i) $\Omega_{\text{bath}}(E_{\text{tot}} - E_i)$, where E_{tot} is the energy of the heat bath and the small system and Ω_{bath} is the density of states of the bath

(ii) $\exp(-E_i/kT)$

(iii) $\boxed{1/M \text{ if } E < E_i < E + \delta E\text{, where } M \text{ is the total number of states in this energy range}}$

Comment: This was Eq. (1) in *Notes on Thermal Equilibrium.*

(e) *(2 points)* In the early universe, two important reactions occurred: deuterium production and hydrogen production from proton-electron recombination. How do the binding energies of deuterium and hydrogen compare?

(i) $\boxed{\text{the binding energy of deuterium is larger}}$

(ii) the binding energies of deuterium and hydrogen are (almost) exactly the same

(iii) the binding energy of hydrogen is larger

Comment: On p. 153, Ryden gives the binding energy of hydrogen as 13.6 eV, and on p. 174 she gives the binding energy of deuterium as 2.22 MeV. So the binding energy of deuterium is about 163,000 times larger.

(f) *(2 points)* The universe can be approximated as

(i) homogeneous and isotropic at all scales

(ii) homogeneous and isotropic on small scales ($\lesssim$ 100 Mpc)

(iii) $\boxed{\text{homogeneous and isotropic on large scales } (\gtrsim 100 \text{ Mpc})}$

(iv) inhomogeneous and anisotropic at all scales

Comment: see the first sentence of Ryden's Chaper 11.

(g) *(2 points)* In order to analyze the ionization and recombination of hydrogen as a function of temperature in the early universe, we studied the ratio of the number density of hydrogen atoms n_H to the product of the number densities of the free elements that react to create them: protons, with number density n_p, and electrons, with number density n_e. For 2 points credit, you can EITHER state the name that is given to the equation that describes this ratio, or you can write the equation. (If you try to write the equation, be sure to notice the relevant formulas on the formula sheet.)

This is the *Saha equation*, as given by Ryden as Eq. (8.29) on p. 153:

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right) ,$$

where $Q = (m_p + m_e - m_H)c^2$.

(h) *(2 points)* Gravity tends to make small density perturbations grow rapidly with time. What keeps small density perturbations from creating instabilities in the matter density and leading to collapse? (For example, what stops the air in this room from undergoing gravitational collapse.)

It is *pressure* that opposes gravitational collapse, as Ryden discusses in Sec. 11.2, *The Jeans Length*.

(i) *(2 points)* Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon-baryon fluid. The speed of sound in the baryonic gas is [larger than / equal to / $\boxed{\text{smaller}}$ than] the speed of sound in the photon gas.
Circle one

Comment: On p. 212, Ryden explains that the sound speed before decouping was $c/\sqrt{3}$, while after decoupling it plummets to 5 kilometer per second.

(j) *(2 points)* Baryon acoustic oscillations are due to the interaction of baryons with

(i) $\boxed{\text{photons}}$

(ii) dark matter

(iii) neutrinos

This is discussed by Ryden at the beginning of Sec. 11.6, *Baryon Acoustic Oscillations*.

PROBLEM 2: SHORT ANSWER QUESTIONS: GRAND UNIFIED THEORIES, MAGNETIC MONOPOLES, AND INFLATION *(20 points)*

(a) *(2 points)* A quark is specified by its flavor [u(p), d(own), c(harmed), s(trange), t(op), b(ottom)], its spin [up or down, along any chosen z axis], whether it is a quark or antiquark, and its color. $\boxed{\textbf{T}}$ or **F** .

(b) *(2 points)* Any isolated system of quarks must be a color singlet. A single red quark is an example of a color singlet, since it has only one color. **T** or $\boxed{\textbf{F}}$.

Comment: As discussed in Class 23, 11/28/22, the simplest color singlets — and the only ones known to exist in nature — are 3-quark states (baryons), with equal parts of red, blue, and green, and quark-antiquark states (mesons), with equal parts of each color and its anticolor. A singlet state is one that is invariant under $SU(3)$ gauge transformations.

(c) *(3 points)* Write the definition of the group SU(3). Assume that the reader does not know the meaning of "unitary" or "special", but that they do know the standard mathematical notation of matrix multiplication AB, matrix inversion A^{-1}, matrix transpose A^T, matrix adjoint $A^\dagger$, etc.

$SU(3)$ is the group of 3×3 complex matrices U which are "special," meaning that $\det U = 1$, and also "unitary," meaning that $U^\dagger U = 1$.

In electromagnetism, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be written in terms of the scalar and vector potentials ϕ and $\vec{A}$ as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ .$$

For any function $\Lambda(\vec{x}, t)$, one can define a gauge transformation on ϕ and $\vec{A}$ by

$$\phi'(\vec{x}, t) = \phi(\vec{x}, t) - \frac{\partial \Lambda(\vec{x}, t)}{\partial t} \ , \quad \vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla}\Lambda(\vec{x}, t) \ .$$

(d) *(2 points)* If $\vec{E}'(\vec{x}, t)$ and $\vec{B}'(\vec{x}, t)$ are the new electric and magnetic fields, calculated from $\phi'(\vec{x}, t)$ and $\vec{A}'(\vec{x}, t)$, how are they related to the original $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$? [If you know the answer, you can write it without any derivation.]

$$\vec{E}'(\vec{x}, t) = \vec{E}(\vec{x}, t) + \vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Lambda = \boxed{\vec{E}(\vec{x}, t) \ ,}$$

$$\vec{B}'(\vec{x}, t) = \vec{B}(\vec{x}, t) + \vec{\nabla} \times \vec{\nabla}\Lambda = \boxed{\vec{B}(\vec{x}, t) \ .}$$

(e) *(3 points)* Consider the line integral

$$W = \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} \ ,$$

where P is a closed loop in (three-dimensional) space. If $\vec{A}(\vec{x}, t)$ is replaced by its gauge transform $\vec{A}'(\vec{x}, t)$, how is the resulting W' related to the original W? [Here you should show a short derivation.]

$$\begin{aligned} W' &\equiv \oint_P \vec{A}'(\vec{x}, t) \cdot d\vec{x} \\ &= \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} + \oint_P \vec{\nabla}\Lambda \cdot d\vec{x} \\ &= \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} = \boxed{W \ ,} \end{aligned}$$

since the line integral of a gradient around a closed loop always vanishes.

(f) *(2 points)* The SU(5) grand unified theory relies on the fact that the gauge group of the standard model, SU(3)×SU(2)×U(1), is a subgroup of SU(5). Is SU(3)×SU(2)×U(1) also a subgroup of SU(6)? $\boxed{\textbf{Yes}}$ or **No** .

Comment: SU(5) is clearly a subgroup of SU(6). To see this, note that an SU(6) matrix $U^{(6)}$ corresponding to an element $V^{(5)}$ of an SU(5) subgroup can be constructed explicitly as

$$U^{(6)}_{ij} = \begin{cases} V^{(5)}_{ij} & \text{if } i,\, j \in \{1,2,3,4,5\} \\ 1 & \text{if } i = j = 6 \\ 0 & \text{otherwise,} \end{cases}$$

or more pictorially as

$$\boldsymbol{U^{(6)}} = \left(\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & \boldsymbol{V^{(5)}} & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) .$$

If SU(3)×SU(2)×U(1) is a subgroup of SU(5), and SU(5) is a subgroup of SU(6), it follows that SU(3)×SU(2)×U(1) is a subgroup of SU(6).

(g) *(2 points)* Inflation explains the homogeneity of the universe by modifying the nature of the initial singularity, so the universe is created in a homogeneous state. **T** or $\boxed{\textbf{F}}$.

Comment: Inflation explains the homogeneity of the universe by inserting a period of colossal expansion into the scenario of the early universe, so that the presently observed universe can evolve from a tiny region in which there had been sufficient time for uniformity to have been established by standard thermal equilibrium processes. Inflation in no way affects the initial singularity, if there was an initial singularity.

(h) *(2 points)* Inflation proposes that the early universe went through a period of nearly exponential expansion driven by the repulsive gravity caused by a material with a negative pressure. $\boxed{\textbf{T}}$ or **F** .

(i) *(2 points)* In inflationary models, the enormous expansion tends to smooth out any nonuniformities that may have been present before inflation. Nonetheless, there remain faint ripples in the cosmic microwave background, because there was not enough inflation to smooth the universe completely. **T** or $\boxed{\textbf{F}}$.

Comment: The faint ripples in the cosmic microwave background are believed to have been caused by quantum fluctuations in the inflaton field as inflation was ending. Thus these fluctuations would be present no matter how much inflation took place.

PROBLEM 3: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X *(30 points)*

The following problem was Problem 3 of Quiz 3, 2016, and Problem 10 of the Review Problems for Quiz 3, 2022.

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, X^+, X^-, and X^0, all with masses 0.511 MeV/c^2, exactly the same as the electron. The X^- is the antiparticle of the X^+, and the X^0 is its own antiparticle. Since the X's are spin-1 particles with nonzero mass, each particle has three spin states.

The X's do not interact with neutrinos any more strongly than the electrons and positrons do, so when the X's freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total energy density of the X^+, X^-, and X^0 particles?

(b) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total number density of the X^+, X^-, and X^0 particles?

(c) *(15 points)* The X particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as kT decreases from values that are large compared to 0.511 MeV/c^2 to values that are small compared to it. If the X's, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio T_ν/T_γ, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

(d) *(5 points)* If the mass of the X's was, for example, 0.100 MeV/c^2, so that the electron-positron pairs froze out first, and then the X's froze out, would the final ratio T_ν/T_γ be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

Answers:

(a) *(5 points)* The formula sheet tells us that the energy density of black-body radiation is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ , \tag{S3.1}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases} . \tag{S3.2}$$

Since the X is spin-1, and 1 is an integer, the X particles are bosons and $g = 1$ per spin state. There are 3 species, X^+, X^-, and X^0, and each species we are told has three spin states, so there are a total of 9 spin states, so $g = 9$. Thus,

$$\boxed{u = 9\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3}\ .} \tag{S3.3}$$

Alternatively, one could count the X^+ and X^- as one species with a distinct particle and antiparticle, so $g_{X^+X^-}$ is given by

$$g_{X^+X^-} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 6\ . \tag{S3.4}$$

The X^0 is its own antiparticle, which means that the particle/antiparticle factor is one, so

$$g_{X^0} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{1}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 3\ , \tag{S3.5}$$

so the total g for X^+, X^-, and X^0 is again equal to 9.

(b) *(5 points)* The formula sheet tells us that the number density of particles in black-body radiation is

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ , \tag{S3.6}$$

where

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ . \end{cases} \tag{S3.7}$$

For bosons $g^* = g$, so g^* for the X particles is 9. Then

$$\boxed{n_X = 9\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .} \tag{S3.8}$$

(c) *(15 points)* We are told that, when the X particles freeze out, all of their energy and entropy is given to the photons. We use entropy rather than energy to determine the final temperature of the photons, because the entropy in a comoving volume is simply conserved, while the energy density varies as

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) . \tag{S3.9}$$

Thus, to track the energy, we need to know exactly how p behaves, and the behavior of p during freeze-out is complicated, and we have not calculated it in this course.

The formula sheet tells us that the entropy density of a constituent of black-body radiation is given by

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3} . \tag{S3.10}$$

(At this point I will depart from the solution given in the Review Problems for Quiz 3, using an approach that seems simpler, but the solution in the Review Problems is also correct.)

The quantity s is the entropy per physical volume, so the entropy per *coordinate* volume is given by

$$S = a^3 s . \tag{S3.11}$$

The conservation of entropy means that the total entropy in a coordinate volume is conserved, and that implies that the entropy per coordinate volume is conserved. In this situation, we are told then when the e^+e^- pairs and the X's freeze out, all of their entropy is given to the photons, and none to the neutrinos. If we use the subscripts 1 to denote times before this freeze-out begins, and 2 to denote times after the freeze-out is complete, we can conclude that

$$\begin{aligned} S_{\gamma,2} &= S_{\gamma,1} + S_{e^+e^-,1} + S_{X,1} , \\ S_{\nu,2} &= S_{\nu,1} , \end{aligned} \tag{S3.12}$$

where S_X refers to the total entropy density of X^+, X^-, and X^0. It follows that the ratios of the left-hand sides must equal the ratios of the right-hand sides:

$$\frac{S_{\gamma,2}}{S_{\nu,2}} = \frac{S_{\gamma,1} + S_{e^+e^-,1} + S_{X,1}}{S_{\nu,1}} . \tag{S3.13}$$

But each $S = a^3 s$ is proportional to the value of gT^3 for the relevant particle, with all other factors being particle-independent. Furthermore, before the freeze-outs, all these particles were at the same temperature. Thus, the above equation implies that

$$\frac{g_\gamma T^3_{\gamma,2}}{g_\nu T^3_{\nu,2}} = \frac{g_\gamma + g_{e^+e^-} + g_X}{g_\nu} , \tag{S3.14}$$

which can be solved for the temperature ratio to give

$$\frac{T_{\nu,2}}{T_{\gamma,2}} = \left(\frac{g_\gamma}{g_\gamma + g_{e^+e^-} + g_X} \right)^{1/3} . \tag{S3.15}$$

But $g_\gamma = 2$ and $g_{e^+e^-} = 7/2$, which can be calculated or taken from the formula sheet, so

$$\frac{T_{\nu,2}}{T_{\gamma,2}} = \left(\frac{2}{2 + \frac{7}{2} + 9} \right)^{1/3} = \boxed{\left(\frac{4}{29} \right)^{1/3} .} \tag{S3.16}$$

(d) *(5 points)* The answer would be the same, since it was completely determined by the conservation equation (S3.12) in the above answer. Regardless of the order in which the freeze-outs occurred, the total entropy from the e^+e^- pairs and the X's would ultimately be given to the photons, so the amount of heating of the photons would be the same.

PROBLEM 4: SIZE OF THE HORIZON ON THE SURFACE OF LAST SCATTERING *(30 points)*

In lecture we discussed an approximate calculation of the horizon problem as it relates to the cosmic microwave background, crudely estimating that the radius of the surface of last scattering was about 23 times larger than the horizon distance at the time of last scattering (which is also called the time of decoupling). Here you are asked to write the equations that determine this number accurately, taking into account all the known contributions to the energy density of the universe.

You should assume that the universe is flat, and that you are given the present values of the contributions to Ω of the known components: $\Omega_{\text{vac},0}$ for vacuum energy, $\Omega_{m,0}$ for nonrelativistic matter (dark matter and baryons), and $\Omega_{\text{rad},0}$ for radiation, with

$$\Omega_{\text{vac},0} + \Omega_{m,0} + \Omega_{\text{rad},0} = 1 \ . \tag{P4.1}$$

You are also given the present value H_0 of the Hubble expansion rate, the redshift z_{ls} of the surface of last scattering, and of course the speed of light c. Following Ryden's conventions, you should take $a(t_0) \equiv 1$.

(a) *(15 points)* What is the coordinate radius $\ell_{\text{ls},c}$ of the surface of last scattering? Your answer should take the form of a definite integral, with explicit limits of integration. (Note that the function $a(t)$ is not given, so it should not appear in your final answer.)

(b) *(10 points)* What is the coordinate size $h_{\text{ls},c}$ of the horizon distance at the time of last scattering? Your answer should again take the form of an integral, as in part (a).

(c) *(5 points)* If two points on the surface of last scattering were separated from each other by a horizon distance at the time of last scattering, what will be the angular separation between these two points as seen on Earth today? You may use small angle approximations. Your answer can include any of the given variables, and also the answers $\ell_{\text{ls},c}$ and $h_{\text{ls},c}$ to the previous two parts, whether or not you answered those parts.

Answers:

(a) If we let t_{ls} denote the time at which the CMB photons left the surface of last scattering, then the coordinate distance from the surface of last scattering to us is given by

$$\ell_{\text{ls},c} = \int_{t_{\text{ls}}}^{t_0} \frac{c\,dt}{a(t)} \ , \tag{S4.1}$$

since $c/a(t)$ is the coordinate speed of light. This cannot be our final answer, however, since we don't know $a(t)$ or t_{ls}. However, we can find $a(t_{\text{ls}})$, since

$$1 + z_{\text{ls}} = \frac{a(t_0)}{a(t_{\text{ls}})} \quad \Longrightarrow \quad \boxed{a(t_{\text{ls}}) = \frac{1}{1+z_{\text{ls}}} \ ,} \tag{S4.2}$$

where $z_{\rm ls}$ is given. We can make use of this by changing the variable of integration in Eq. (S4.1) to a. To carry out this change of variable we need to know $\dot{a} \equiv da/dt$, which can be found from the first order Friedmann equation. Using

$$\Omega \equiv \frac{\rho}{\rho_c} \ , \ \text{and} \ \rho_c \equiv \frac{3H^2}{8\pi G} \ , \tag{S4.3}$$

we can write

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho(t) \ . \tag{S4.4}$$

Here $H(t)$ is the Hubble expansion rate and $\rho(t)$ is the total mass density. The mass density of the universe has 3 components: vacuum mass density $\rho_{\rm vac}$ which is independent of time, matter density $\rho_m(t)$ which falls off as $1/a^3$, and radiation density $\rho_{\rm rad}(t)$ which falls off as $1/a^4$. Thus

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3} G\left[\rho_{\rm vac} + \rho_m(t) + \rho_{\rm rad}(t)\right] \\ &= \frac{H_0^2}{\rho_{c,0}}\left[\rho_{\rm vac} + \frac{\rho_{m,0}}{a^3} + \frac{\rho_{\rm rad,0}}{a^4}\right] \\ &= H_0^2\left[\Omega_{\rm vac} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{\rm rad,0}}{a^4}\right] \ .\end{aligned} \tag{S4.5}$$

Thus, the coordinate distance between us and the surface of last scattering is then

$$\begin{aligned}\ell_{{\rm ls},c} &= \int_{t_{\rm ls}}^{t_0} \frac{c\,dt}{a(t)} = \int_{t_{\rm ls}}^{t_0} \frac{c\,a\,\dot{a}\,dt}{a^2\,\dot{a}} = \int_{a(t_{\rm ls})}^{1} \frac{c\,da}{Ha^2} \\ &= \frac{c}{H_0}\int_{a(t_{\rm ls})}^{1} \frac{da}{a^2\sqrt{\frac{\Omega_{\rm rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\rm vac,0}}} \\ &= \boxed{\frac{c}{H_0}\int_{a(t_{\rm ls})}^{1} \frac{da}{\sqrt{\Omega_{\rm rad,0} + \Omega_{m,0}\,a + \Omega_{\rm vac,0}\,a^4}} \ ,}\end{aligned} \tag{S4.6}$$

where $a(t_{\rm ls})$ is given in Eq. (S4.2).

Alternatively, one can use z as the variable of integration, where

$$a = \frac{1}{1+z} \quad \Longrightarrow \quad da = -\frac{dz}{(1+z)^2} \ , \tag{S4.7}$$

which leads to

$$\boxed{\ell_{\text{ls},c} = \frac{c}{H_0} \int_0^{z_{\text{ls}}} \frac{dz}{\sqrt{\Omega_{\text{rad},0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\text{vac},0}}}\ .} \tag{S4.8}$$

Note that the minus sign in Eq. (S4.7) implies that the limits of integration of Eq. (S4.8) are reversed relative to Eq. (S4.6). The lower limit in Eq. (S4.6), $a = a(t_{\text{ls}})$, corresponds to the upper limit, $z = z_{\text{ls}}$, in Eq. (S4.8), while the upper limit in Eq. (S4.6), $a = 1$, corresponds to the lower limit in Eq. (S4.8), $z = 0$.

(b) $h_{\text{ls},c}$ is the total coordinate distance that light could have traveled between $t = 0$ and $t = t_{\text{ls}}$, so the answer is the same as in part (a), except for the limits of integration. So

$$\boxed{h_{\text{ls},c} = \frac{c}{H_0} \int_0^{a(t_{\text{ls}})} \frac{da}{\sqrt{\Omega_{\text{rad},0} + \Omega_{m,0}\, a + \Omega_{\text{vac},0}\, a^4}}\ .} \tag{S4.9}$$

If one prefers to use z as the integration variable, this would be written as

$$\boxed{\ell_{\text{ls},c} = \frac{c}{H_0} \int_{z_{\text{ls}}}^{\infty} \frac{dz}{\sqrt{\Omega_{\text{rad},0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\text{vac},0}}}\ .} \tag{S4.10}$$

(c) Since the comoving diagram describes a flat (Euclidean) space, the angle subtended by a length $h_{\text{ls},c}$ at a distance $\ell_{\text{ls},c}$ is, in small angle approximation, given simply by the ratio

$$\boxed{\theta = \frac{h_{\text{ls},c}}{\ell_{\text{ls},c}}\ .} \tag{S4.11}$$

You were of course not asked to evaluate these quantities numerically, but you might be interested in the answers. Some of the answers were already given in Lecture Notes

8. We use the Planck 2018 parameters*

$$\begin{aligned} H_0 &= 67.36\ \mathrm{km\,s^{-1}\,Mpc^{-1}} \\ \Omega_{m,0} &= 0.3153 \\ \Omega_{\mathrm{vac},0} &= 0.6847 \\ z_{\mathrm{ls}} &= 1089.92\ , \end{aligned} \tag{S4.12}$$

and $\Omega_{\mathrm{rad},0}$ given by

$$\Omega_{\mathrm{rad},0} = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30}\frac{(kT_0)^4}{\hbar^3 c^5}\cdot\frac{8\pi G}{3H_0^2}\ , \tag{S4.13}$$

[See Eq. (6.65) of Lecture Notes 6 and Eq. (3.33) of Lecture Notes 3] with†

$$T_0 = 2.72548\ \mathrm{K}\ . \tag{S4.14}$$

Eq. (S4.13) then gives

$$\Omega_{\mathrm{rad},0} = 9.163 \times 10^{-5}\ . \tag{S4.15}$$

One then finds that the comoving distance (i.e., the coordinate distance with $a(t_0) \equiv 1$) to the surface of last scattering is

$$\ell_{\mathrm{ls},c} = 45.23 \times 10^9\ \text{light-years}\ , \tag{S4.16}$$

and that the comoving horizon distance at the time of last scattering is

$$h_{\mathrm{ls},c} = 0.9150 \times 10^9\ \text{light-years}. \tag{S4.17}$$

The sum of these two numbers is the current horizon distance,

$$\ell_{p,\mathrm{horizon}}(t_0) = 46.15 \times 10^9\ \text{light-years}\ . \tag{S4.18}$$

One finds that the horizon problem for the CMB is actually worse than the very crude calculation that we did in class, by about a factor of 2. In class we found that the radius of the surface of last scattering was about 23 horizon distances, while these numbers tell

* Planck Collaboration, N. Aghanim et al., "Planck 2018 results, VI: Cosmological parameters," Astron. & Astrophys. 641, A6 (2020), arXiv:1807.06209v4 [astro-ph.CO], abstract and column 5 of Tables 1 or 2.

† D.J. Fixsen, "The Temperature of the Cosmic Microwave Background," Astrophys. J. 707, 916–920 (2009), arXiv:0911.1955v2 [astro-ph.CO].

us that the radius of the surface of last scattering was about 49.43 horizon distances. Thus, two points on the CMB sky separated by

$$\theta \approx \frac{1}{49.43} \text{ radian} \approx 1.159^\circ \tag{S4.19}$$

were 1 horizon distance apart on the surface of last scattering, at the time of decoupling. If one treats the quoted uncertainties in the various quantities as being independent, the final result can be written as

$$\theta = 1.159^\circ \pm 0.006^\circ \ . \tag{S4.20}$$

Thus, CMB photons that we receive from points separated by slightly more that 1.16° were last scattered from points that were separated by more than a horizon distance, so neither point could have received a signal from the other. They could, however, have both received a signal from some point in between the two. For 2 points on the CMB sky separated by slightly more than twice this angle, i.e., slightly more than 2.32°, then the two points of last scattering could not have even had any common influences.