MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 29, 2022
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Wednesday, October 5, 2022.

QUIZ COVERAGE

Lecture Notes 1, 2, 3, and the beginning of Lecture Notes 4, though the section on *The Horizon Distance*; Problem Sets 1, 2, and 3; Weinberg, Chapters 1, 2, and 3; In Ryden, Chapters 1, 2, and 3 have been assigned, and I recommend that you read all three chapters. But only Chapter 2 will be fair game for reading questions on the quiz. (Chapter 1 contains interesting introductory material; Chapter 3 should be treated as an aid to understanding topics discussed in lecture, although some of those topics will be discussed in lecture later.) On Problem Set 3 it said that none of Lecture Notes 4 would be included, so I apologize for that omission.

Quiz Logistics: The quiz will take place during the regular class time, 11:05 am to 12:25 pm. The quiz will be closed book, no calculators, no internet, and 80 minutes long. We may use a different classroom, so watch for announcements.

New information: One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 6, 7, 12, 13, 14, and 18. The starred problems do not include any reading questions, but parts of the reading questions in these Review Problems may also recur on the upcoming quiz. For the homework problems, extra credit problems are eligible to be the problem used on the quiz.

PURPOSE OF THE REVIEW PROBLEMS: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, 2011, 2013, 2016, 2018, and 2020. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the original quizzes, just to see how much material has been included in each quiz. Since the schedule and the number of quizzes has varied over the years, the coverage of this quiz will not necessarily be the same as Quiz 1 from all previous years. In fact, however, the first quiz this year covers essentially the same material as the first quiz in any year from 2009 onwards.

REVIEW SESSION: To help you study for the quiz, there will be a review session led by Marianne Moore, on Sunday October 2, at 6:00 pm, in our usual classroom, Room 4-265.

FUTURE QUIZZES: The other quiz dates this term will be Wednesday, November 9, and Wednesday, December 7, 2022.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the back of the quiz. For the first quiz, this useful information will be the following:

DOPPLER SHIFT (Definition:)

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} \ ,$$

where $\Delta t_{\text{observer}}$ and Δt_{source} are the period of the wave as measured by the observer and by the source, respectively, and $\lambda_{\text{observer}}$ and λ_{source} are the wavelength of the wave, as measured by the observer and by the source, respectively.

DOPPLER SHIFT (For motion along a line):

Nonrelativistic, u = wave speed, source moving at speed v away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed v away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity), $\beta \equiv v/c$, where c is the speed of light and v is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

COSMOLOGICAL REDSHIFT:

$$1 + z = \frac{a(t_{\text{observer}})}{a(t_{\text{source}})}$$

SPECIAL RELATIVITY:

Time Dilation. A clock that is moving at speed v relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor γ:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance ℓ_0 in their own frame, are moving together, in the direction of the line separating them, at speed v relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,
where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,
where $\ell_p(t)$ is the physical distance between any two objects at time t, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)}\dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with coordinate speed

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \quad \ddot{a} = -\frac{4\pi}{3} G\rho a ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} ,$$

where H is the Hubble expansion rate, $a(t)$ is the scale factor, an overdot (as in $\dot{a}$) denotes a derivative with respect to cosmic time t, G is Newton's gravitational constant, $\rho(t)$ is the mass density, $\rho_c(t)$ is the critical density, and k is a constant. Here t_i is "initial" time at which we defined the properties of the model universe, but any time can be taken as the initial time t_i; the equation for $\rho(t)$ on the second line above holds for *any* two times t and t_i.

For a flat ($k = 0$) universe,

$$a(t) \propto t^{2/3} , \; \Omega = 1 .$$

PROBLEM LIST

PROBLEM 1: DID YOU DO THE READING (2020)? *(30 points)*

(a) *(5 points)* In what way do spectral lines help us understand the properties of nearby stars? Consider the following options:

(A) We can determine their chemical composition.

(B) We can determine their speed relative to us.

(C) We can determine their acceleration relative to us.

(D) We can determine their mass.

(E) We can determine their volume.

Which one of the following combinations is correct:

(i) (A) and (B)

(ii) (A) and (C)

(iii) (B) and (C)

(iv) (D) and (E)

(v) (A), (B), (C), (D), and (E)

(b) *(5 points)* What is the cosmological principle?

(i) We live in the center of the universe.

(ii) The universe is homogeneous and isotropic on large distance scales.

(iii) The universe is isotropic.

(iv) All galaxies are evenly spaced throughout the universe.

(v) Newtonian gravity applies on large distance scales.

(c) *(5 points)* Roughly, what is the size of the Milky Way? And, at what length scale does the cosmological principle become valid?

(i) 10^7 light years and 100 Mpc

(ii) 10^5 light years and 1 Mpc

(iii) 10^5 light years and 100 Mpc

(iv) 10^3 light years and 1 Mpc

(v) 10^3 light years and 100 Mpc

(d) *(5 points)* Which one of the following statements is a valid description of the history of what is now known as the "Cosmic Microwave Background"?

(i) The CMB radiation initially had a temperature less than 2.7 degrees Kelvin, but it increased gradually to its current value as it absorbed the energy of the light from stars.

(ii) Plasma instabilities in the early universe generated large fluctuations in electromagnetic radiation, which we now see as a "background" to astronomical observations.

(iii) The universe was originally filled with microwave radiation, and it persists until today.

(iv) It is a primordial kind of radiation with a blackbody spectrum of 2.7 degrees Kelvin at the time it was generated.

(v) At some point in the history of our universe, photons, which were once in thermal equilibrium with other particles, stopped interacting because electrons and atomic nuclei combined into atoms. The photons then free-streamed throughout the universe, from then until now.

(e) *(5 points)* Why is the night sky not uniformly bright due to the universe being homogeneously populated with stars (or other bright objects)? Consider the following options:

(A) Space is not transparent.

(B) The universe is not infinitely large.

(C) The universe is not infinitely old.

(D) The absolute brightness of stars depends on their distance away from us.

(E) The cosmological redshift makes stars look dimmer and dimmer as they are further away from us.

Which one of the following combinations is correct?

(i) (A) and (C)

(ii) (B) and (D)

(iii) (C) and (E)

(iv) (A), (C), and (E)

(v) (B), (C), and (D)

(f) *(5 points)* Which of the following statements best describes Einstein's equivalence principle?

(i) Physical laws do not depend on the place of the universe where we make measurements.

(ii) General relativity is equivalent to special relativity when only massless particles are considered.

(iii) An observer on the surface of the Earth experiences the same external forces as a free-falling observer.

(iv) An observer on the surface of the Earth, without looking through a window, cannot tell whether they are actually on Earth or if they are on a rocket ship accelerating at 9.8 m/s^2 in empty space.

(v) Any calculation done in the framework of General Relativity must be equivalent to another in Newtonian gravity in the appropriate limit.

PROBLEM 2: DID YOU DO THE READING (2018)? *(30 points)*

(a) *(5 points)* After telescopes became available, more and more extended objects in the sky, called nebulae, were discovered, but those were thought as members of our galaxy. Who is the person who first proposed that some of the nebulae are galaxies like our own located outside our galaxy?

(i) Isaac Newton

(ii) Immanuel Kant

(iii) Edwin Hubble

(iv) Albert Einstein

(b) *(5 points)* Before 1923, questions of the nature of the spiral and elliptical nebulae could not be settled without some reliable method of determining how far away they are. In 1923, Edwin Hubble was for the first time able to resolve the Andromeda Nebula (galaxy) into separate stars and estimated the distance to the Andromeda Nebula. What observational quantity did he measure to estimate the distance?

(i) the radial velocity of individual stars in the Adromeda Nebula

(ii) the radial velocity of the Andromeda Nebula itself

(iii) the periods of variation of a class of stars in the Andromeda Nebula

(iv) the parallax of bright stars in the Adromeda Nebula

(c) *(5 points)* In 1917, a year after the completion of Einstein's general theory of relativity, __________ looked specifically for a solution that would be homogeneous, isotropic, and static, and thus was forced to mutilate the equations by introducing a term, the so-called cosmological constant. In the same year, another solution of the modified theory was found by the Dutch astronomer __________. Although this solution appeared to be static, it had the remarkable property of predicting a red-shift proportional to the distance. In 1922, the general homogeneous and isotropic solution of the original Einstein equations was found by the Russian mathematician __________, which provides a mathematical background for the most modern cosmological theories. Which is the right answer to fill in the blanks in turn?

(i) Friedmann — Einstein — de Sitter

(ii) Friedmann — de Sitter — Einstein

(iii) Einstein — Friedmann — de Sitter

(iv) Einstein — de Sitter — Friedmann

(v) de Sitter — Einstein — Friedmann

(vi) de Sitter — Friedmann — Einstein

(d) *(5 points)* After radio noises with the equivalent temperature of about 3.5° K were detected, Penzias, Wilson, Dicke, Peebles, Roll, and Wilkinson decided to publish a pair of companion letters in the Astrophysical Journal, in which Penzias and Wilson would announce their observations, and Dicke, Peebles, Roll, and Wilkinson would explain the cosmological interpretation. What is the title of the paper written by Penzias and Wilson?

(i) "A Measurement of Excess Antenna Temperature at 4,080 Mc/s"

(ii) "Cosmic Black-Body Radiation"

(iii) "Origin of the Microwave Radio Background"

(iv) "Three Degrees Above Zero: Bell Labs in the Information Age"

(e) *(5 points)* The universe contains different types of particles. Which of the following statements is NOT true?

(i) A baryon is defined as a particle made of three quarks.

(ii) Electrons and neutrinos are leptons.

(iii) There are three types of neutrinos and they all have zero charge.

(iv) The component of the universe made of ions, atoms, and molecules is generally referred to as baryonic matter, since only the baryons (protons and neutrons) contribute significantly to the mass density.

(v) About three-fourths of the baryonic matter in the universe is currently in the form of helium.

(f) *(5 points)* If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1000 Mpc. (1 Mpc $= 10^6$ pc, 1 pc $= 3.086 \times 10^{16}$ m $= 3.262$ light-year).

(ii) 100 Mpc.

(iii) 1 Mpc.

(iv) 100 kpc (1 kpc $=$ 1000 pc).

(v) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

PROBLEM 3: DID YOU DO THE READING (2016)? *(35 points)*

(a) *(5 points)* The Milky Way has been known since ancient times as a band of light stretching across the sky. We now recognize the Milky Way as the galaxy of stars in which we live, with a large collection of stars, including our sun, arranged in a giant disk. Since the individual stars are mostly too small for our eyes to resolve, we observe the collective light from these stars, concentrated in the plane of the disk. The idea that the Milky Way is actually a disk of stars was proposed by

(i) Claudius Ptolemy, in the 2nd century AD.

(ii) Johannes Kepler, in 1610.

(iii) Isaac Newton, in 1695.

(iv) Thomas Wright, in 1750.

(v) Immanuel Kant, in 1755.

(vi) Edwin Hubble, in 1923.

(b) *(5 points)* Once it was recognized that we live in a galaxy, it was initially assumed that ours was the only galaxy. The suggestion that some of the patches of light known as nebulae might actually be other galaxies like our own was made by

(i) Claudius Ptolemy, in the 2nd century AD.

(ii) Johannes Kepler, in 1610.

(iii) Isaac Newton, in 1695.

(iv) Thomas Wright, in 1750.

(v) Immanuel Kant, in 1755.

(vi) Edwin Hubble, in 1923.

(c) *(5 points)* The first firm evidence that there is more than one galaxy stemmed from the ability to observe the Andromeda Nebula with high enough resolution to distinguish its individual stars. In particular, the observation of Cepheid variable stars in Andromeda allowed a distance estimate that place it well outside the Milky Way. The observation of Cepheid variable stars in Andromeda was first made by

(i) Johannes Kepler, in 1610.

(ii) Isaac Newton, in 1695.

(iii Thomas Wright, in 1750.

(iv) Immanuel Kant, in 1755.

(v) Henrietta Swan Leavitt and Harlow Shapley in 1915.

(vi) Edwin Hubble, in 1923.

(d) *(5 points)* The first hint that the universe is filled with radiation with an effective temperature near 3 K, although not recognized at the time, was an observation of absorption lines in cyanogen (CN) by Adams and McKellar in 1941. They observed dark spectral lines which they interpreted as absorption by the cyanogen of light coming from the star behind the gas cloud. Explain in a few sentences how these absorption lines can be used to make inferences about the cosmic background radiation bathing the cyanogen gas cloud.

(e) *(5 points)* As the universe expands, the temperature of the cosmic microwave background

(i) goes up in proportion to the scale factor $a(t)$.

(ii) stays constant.

(iii) goes down in proportion to $1/a(t)$.

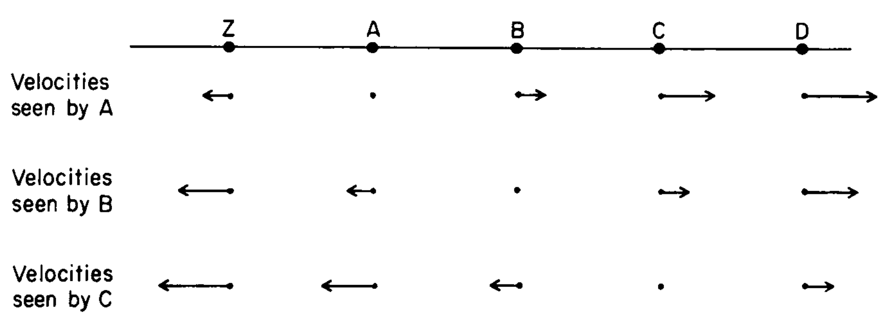
(iv) goes down in proportion to $1/a^2(t)$.

(f) *(5 points)* When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) *(5 points)* Explain in a few sentences what is meant by the equivalence principle?

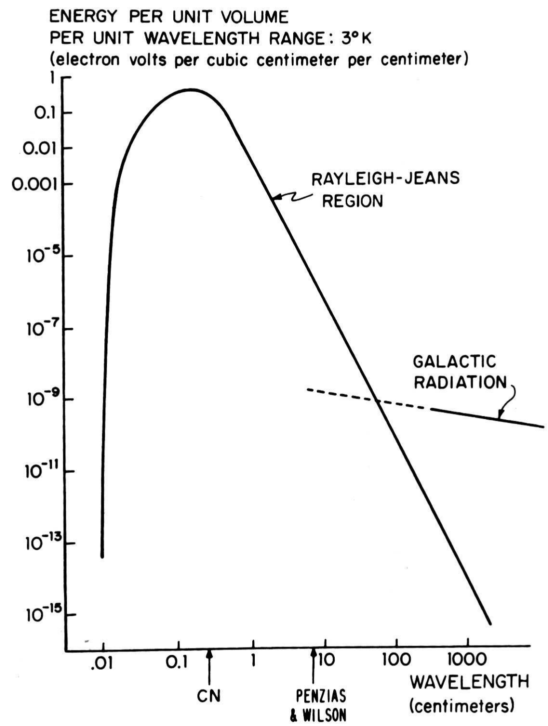
PROBLEM 4: DID YOU DO THE READING (2013)? *(25 points)*

(a) *(5 points)*



Weinberg used the diagram above to explain an important property of the Hubble expansion law. What property was it. Explain in one or a few sentences how this diagram illustrates the property in question.

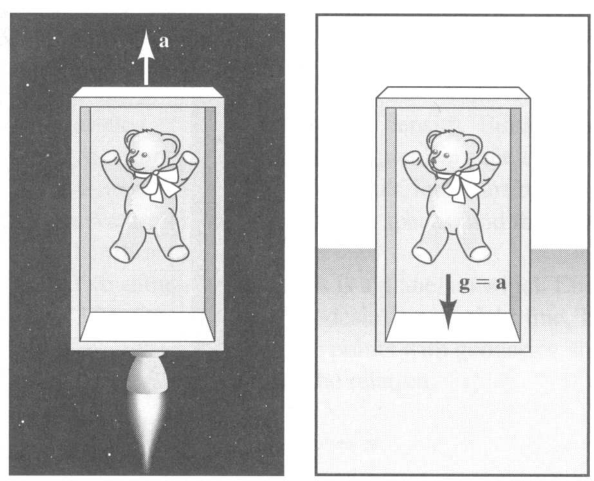
(b) *(5 points)* The diagram on the right shows the Planck spectrum for black-body radiation at 3.0 K, as used in Weinberg's book. Weinberg explained that the intensity of black-body radiation falls off at long wavelengths because "it is hard to fit radiation into any volume whose dimensions are smaller than the wavelength." In one or a few sentences, what is is reason for the suppression in intensity at short wavelengths?



(c) *(5 points)* The label CN on the diagram at the right refers to cyanogen, a radical consisting of one carbon and one nitrogen atom. In one or a few sentences, what does cyanogen have to do with black-body radiation?

(d) *(5 points)* Another label in the diagram above is "Penzias & Wilson." What discovery did they make *(3 points)*, and where were they employed *(2 points)*?

(e) *(5 points)* Ryden used the teddy-bear diagram at the right to illustrate an important property of general relativity. What property was it? Explain in one or a few sentences how this diagram illustrates the property in question.



PROBLEM 5: DID YOU DO THE READING (2011)? *(25 points)*

(a) *(10 points)* Hubble's law relates the distance of galaxies to their velocity. The Doppler effect provides an accurate tool to measure velocity, while the measure of cosmic distances is more problematic. Explain briefly the method that Hubble used to estimate the distance of galaxies in deriving his law.

(b) *(5 points)* One expects Hubble's law to hold as a consequence of the Cosmological Principle. What does the Cosmological Principle state?

(c) *(10 points)* Give a brief definition for the words *homogeneity* and *isotropy*. Then say for each of the following two statements whether it is true or false. If true explain briefly why. If false give a counter-example. You should assume Euclidean geometry (which Weinberg implicitly assumed in his discussion).

(i) If the universe is isotropic around one point then it has to be homogeneous.

(ii) If the universe is isotropic around two or more distinct points then it has to be homogeneous.

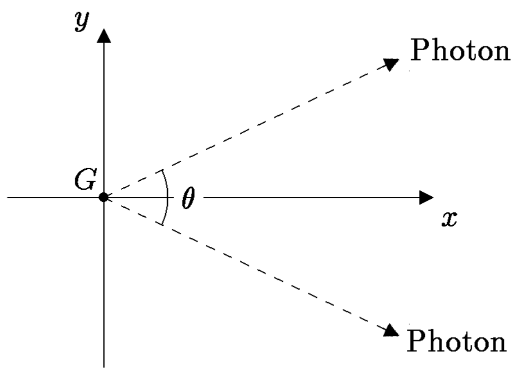
(d) Bonus question: *(2 points extra credit)* If we allow curved (i.e., non-Euclidean) spaces, is it true that a universe which is isotropic around two distinct points has to be homogeneous? If true explain briefly why, and otherwise give a counter-example.

∗ PROBLEM 6: LIGHT RAYS TRAVELING THROUGH A MATTER-DOMINATED FLAT UNIVERSE *(40 points)*

Consider a flat, matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Now consider a galaxy G in this universe which at time t_1 emits two photons, with an angular separation θ between their paths, as shown in the diagram:

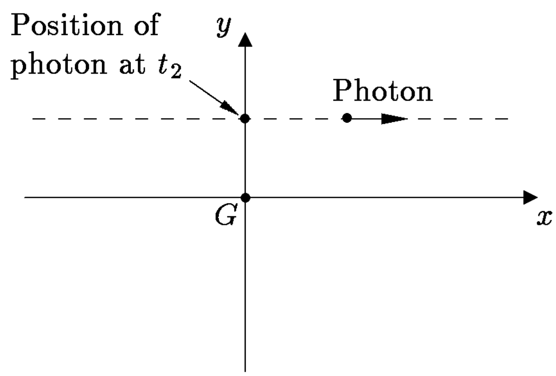


(a) *(10 points)* At cosmic time t (for $t > t_1$), what is the physical distance $\ell_{1,\text{phys}}(t)$ of each of these photons from the galaxy G?

(b) *(5 points)* If the frequency of the photons was ν_1 when they were emitted, what is their frequency $\nu(t)$ at cosmic time t (for $t > t_1$)? $\nu(t)$ should be the frequency as it would be measured by a comoving observer, i.e. an observer at rest with respect to the matter at the same location.

(c) *(10 points)* What is the physical distance $\ell_{2,\text{phys}}(t)$ between the two photons at time t (for $t > t_1$)?

Now consider a different situation, but in the same universe. This time we consider a photon that travels past the galaxy G, traveling in the x direction, in the x-y plane, as shown in the diagram below. We are told that the photon crosses the y axis at time t_2, and at that time the photon is a physical distance h from the galaxy.



(d) *(10 points)* What is the physical distance $\ell_{3,\text{phys}}(t)$ between the photon and the galaxy G at arbitrary time t, which might be earlier or later than t_2?

(e) *(5 points)* At time t_2, what is the recessional speed $d\ell_{3,\text{phys}}(t)/dt$ of the photon from the galaxy. *Hint:* if you are clever, this can be done with very little calculation.

∗ PROBLEM 7: THE STEADY-STATE UNIVERSE THEORY *(30 points)*

The following problem was Problem 2, Quiz 1, 2000, and also Problem 3, Quiz 1, 2018.

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties were confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density does not fall. Like the conventional theory, the steady-state theory describes a homogeneous, isotropic, expanding universe, so the same comoving coordinate formulation can be used.

a) *(15 points)* The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $a(t)$ which is consistent with this hypothesis.

b) *(15 points)* Suppose that the mass density of the universe is ρ_0, which of course does not change with time. In terms of the general form for $a(t)$ that you found in part (a), calculate the rate at which new matter must be created for ρ_0 to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. *[If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $a(t)$.]*

PROBLEM 8: OBSERVING A DISTANT GALAXY IN A MATTER-DOMINATED FLAT UNIVERSE

The following problem was taken from Quiz 1, 2016, where it counted 40 points.

Suppose that we are living in a matter-dominated flat universe, with a scale factor given by

$$a(t) = bt^{2/3} \ ,$$

where b is a constant. The present time is denoted by t_0.

(a) *(5 points)* If we measure time in seconds, distance in meters, and coordinate distances in notches, what are the units of b?

(b) *(5 points)* Suppose that we observe a distant galaxy which is one half of a "Hubble length" away, which means that the physical distance today is $\ell_p = \frac{1}{2}cH_0^{-1}$, where c is the speed of light and H_0 is the present value of the Hubble expansion rate. What is the proper velocity $v_p \equiv \frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t}$ of this galaxy relative to us?

(c) *(5 points)* What is the coordinate distance ℓ_c between us and the distant galaxy?

If you did not answer the previous part, you may still continue with the following parts, using the symbol ℓ_c for the coordinate distance to the galaxy.

(d) *(5 points)* At what time t_e was the light that we are now receiving from the galaxy emitted?

(e) *(5 points)* What is the redshift z of the light that we are now receiving from the distant galaxy?

(f) *(10 points)* Consider a light pulse that leaves the distant galaxy at time t_e, as calculated in part (d), and arrives here at the present time, t_0. Calculate the physical distance $r_p(t)$ between the light pulse and us. Find $r_p(t)$ as a function of t for all t between t_e and t_0.

(g) *(5 points)* If we send a radio message now to the distant galaxy, at what time t_r will it be received?

PROBLEM 9: A RADIATION-DOMINATED FLAT UNIVERSE

We have learned that a matter-dominated homogeneous and isotropic universe can be described by a scale factor $a(t)$ obeying the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ .$$

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that the mass density of nonrelativistic matter falls off as $1/a^3(t)$ as the universe expands; the mass of each particle remains constant, and the density of particles falls off as $1/a^3(t)$ because the volume increases as $a^3(t)$. For the photon-dominated universe, the density of photons falls of as $1/a^3(t)$, but in addition the frequency (and hence the energy) of each photon redshifts in proportion to $1/a(t)$. Since mass and energy are equivalent, the mass density of the gas of photons falls off as $1/a^4(t)$.

For a flat (i.e., $k = 0$) matter-dominated universe we learned that the scale factor $a(t)$ is proportional to $t^{2/3}$. How does $a(t)$ behave for a photon-dominated universe?

PROBLEM 10: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)*

The following problem appeared on Quiz 1 of 2011.

Consider again a flat matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Let t_0 denote the current time.

(a) *(5 points)* What is the current value of the physical horizon distance $\ell_{p,\text{horizon}}(t_0)$? That is, what is the present distance of the most distant matter that can be seen, limited only by the speed of light.

(b) *(5 points)* Consider a photon that is arriving now from an object that is just at the horizon. Our goal is to trace the trajectory of this object. Suppose that we set up a coordinate system with us at the origin, and the source of the photon along the positive x-axis. What is the coordinate x_0 of the photon at $t = 0$?

(c) *(5 points)* As the photon travels from the source to us, what is its coordinate $x(t)$ as a function of time?

(d) *(5 points)* What is the physical distance $\ell_p(t)$ between the photon and us as a function of time?

(e) *(5 points)* What is the maximum physical distance $\ell_{p,\max}(t)$ between the photon and us, and at what time $t_{\max}$ does it occur?

PROBLEM 11: SIGNAL PROPAGATION IN A FLAT MATTER-DOMINATED UNIVERSE *(55 points)*

The following problem was on Quiz 1, 2009.

Consider a flat, matter-dominated universe, with scale factor

$$a(t) = bt^{2/3} ,$$

where b is an arbitrary constant. For the following questions, the answer to any part may contain symbols representing the answers to previous parts, whether or not the previous part was answered correctly.

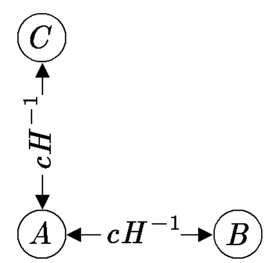
(a) *(10 points)* At time $t = t_1$, a light signal is sent from galaxy A. Let $\ell_{p,sA}(t)$ denote the physical distance of the signal from A at time t. (Note that $t = 0$ corresponds to the origin of the universe, not to the emission of the signal.) (i) Find the speed of separation of the light signal from A, defined as $\mathrm{d}\ell_{p,sA}/\mathrm{d}t$. What is the value of this speed (ii) at the time of emission, t_1, and (iii) what is its limiting value at arbitrarily late times?

(b) *(5 points)* Suppose that there is a second galaxy, galaxy B, that is located at a physical distance cH^{-1} from A at time t_1, where $H(t)$ denotes the Hubble expansion rate and c is the speed of light. (cH^{-1} is called the Hubble length.) Suppose that the light signal described above, which is emitted from galaxy A at time t_1, is directed toward galaxy B. At what time t_2 does it arrive at galaxy B?

(c) *(10 points)* Let $\ell_{p,sB}(t)$ denote the physical distance of the light signal from galaxy B at time t. (i) Find the speed of approach of the light signal towards B, defined as $-\mathrm{d}\ell_{p,sB}/\mathrm{d}t$. What is the value of this speed (ii) at the time of emission, t_1, and (iii) at the time of reception, t_2?

(d) *(10 points)* If an astronomer on galaxy A observes the light arriving from galaxy B at time t_1, what is its redshift z_{BA}?

(e) *(10 points)* Suppose that there is another galaxy, galaxy C, also located at a physical distance cH^{-1} from A at time t_1, but in a direction orthogonal to that of B. If galaxy B is observed from galaxy C at time t_1, what is the observed redshift z_{BC}? Recall that this universe is flat, so Euclidean geometry applies.



(f) *(10 points)* Suppose that galaxy A, at time t_1, emits electromagnetic radiation spherically symmetrically, with power output P. (P might be measured, for example, in watts, where 1 watt = 1 joule/second.) What is the radiation energy flux J that is received by galaxy B at time t_2, when the radiation reaches galaxy B? (J might be measured, for example, in watts per meter^2. Units are mentioned here only to help clarify the meaning of these quantities — your answer should have no explicit units, but should be expressed in terms of any or all of the given quantities t_1, P, and c, plus perhaps symbols representing the answers to previous parts.)

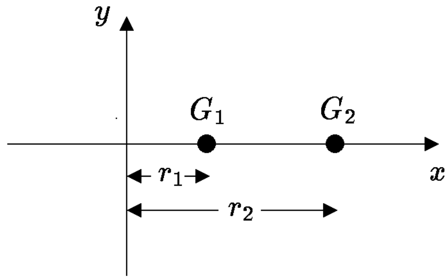
∗ PROBLEM 12: INTERGALACTIC SIGNALING AND TRAVEL

The following problem was Problem 2 on Quiz 1, 2020, where it was worth 30 points.

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{2/3} ,$$

where b is a constant. Suppose that galaxy G_1 is located at comoving coordinates $(x, y, z) = (r_1, 0, 0)$ and galaxy G_2 is located at $(x, y, z) = (r_2, 0, 0)$, as shown in the following diagram:



(a) *(10 points)* At cosmic time $t = t_1$, a light signal is sent from G_1, in the direction of G_2. At what time t_2 does the light signal reach G_2? Your answer should be expressed in terms of some or all of the given variables t_1, b, r_1, r_2, and the speed of light c.

(b) *(6 points)* When the light signal arrives at G_2, what is its redshift z? Your answer can be expressed in terms of any or all of the variables mentioned in part (a), and also t_2, the answer to part (a).

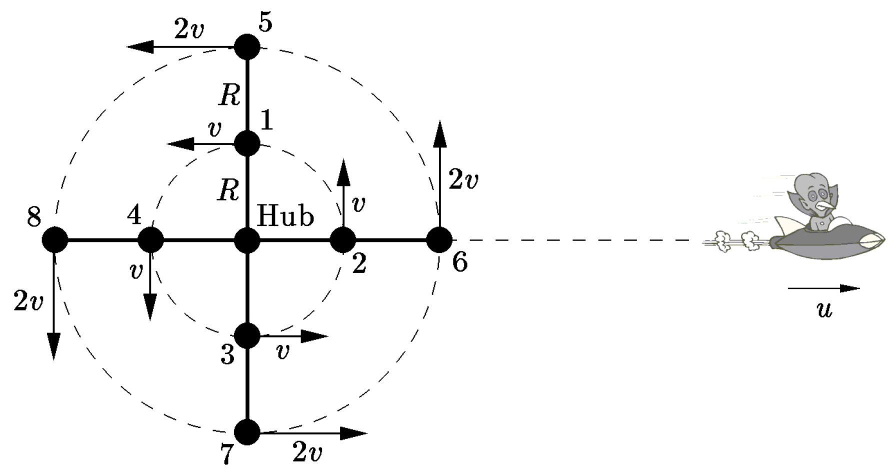
(c) *(7 points)* Now suppose that at time t_1, the galaxy G_1 also launches a starship towards G_2. This is a science-fiction starship, which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity, and no time to decelerate at the end of its journey. At what time t_3 does the starship arrive at G_2? Your answer may depend on any of the allowed variables in part (b).

(d) *(7 points)* We have not discussed how the clocks on such a starship would behave, but general relativity implies that, as the starship passes any comoving observer, the comoving observer would measure the clocks on the starship to be time-dilated exactly as they would be in special relativity, for the same relative velocity. Given this fact, what time will the clocks on the starship read when the ship arrives at G_2?

∗ PROBLEM 13: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND *(15 points)*

This problem was Problem 3 on Quiz 1, 2007.

Consider a high-speed merry-go-round which is similar to the one discussed in Problem 3 of Problem Set 1, but which has two levels. That is, there are four evenly spaced cars which travel around a central hub at speed v at a distance R from a central hub, and also another four cars that are attached to extensions of the four radial arms, each moving at a speed $2v$ at a distance $2R$ from the center. In this problem we will consider only light waves, not sound waves, and we will assume that v is not negligible compared to c, but that $2v < c$.



We learned in Problem Set 1 that there is no redshift when light from one car at radius R is received by an observer on another car at radius R.

(a) *(5 points)* Suppose that cars 5–8 are all emitting light waves in all directions. If an observer in car 1 receives light waves from each of these cars, what redshift z does she observe for each of the four signals?

(b) *(10 points)* Suppose that a spaceship is receding to the right at a relativistic speed u along a line through the hub, as shown in the diagram. Suppose that an observer in car 6 receives a radio signal from the spaceship, at the time when the car is in the position shown in the diagram. What redshift z is observed?

∗ PROBLEM 14: AN EXPONENTIALLY EXPANDING FLAT UNIVERSE *(20 points)*

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems. The problem also appeared as Problem 2 on Quiz 1, 2007.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$a(t) = a_0 e^{\chi t} ,$$

where a_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift z. Express the time t_r at which we receive the pulse in terms of z, χ, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of z, χ, and any relevant physical constants.

PROBLEM 15: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE

The following problem was taken from Problem 3, Quiz 1, 2005, where it counted 25 points.

Consider a flat universe that expands with a scale factor

$$a(t) = bt^{1/2} ,$$

where b is a constant. We will learn later that this is the behavior of the scale factor for a radiation-dominated universe.

(a) *(5 points)* At an arbitrary time $t = t_f$, what is the physical horizon distance? (By "physical," I mean as usual the distance in physical units, such as meters or centimeters, as measured by a sequence of rulers, each of which is at rest relative to the comoving matter in its vicinity.)

(b) *(3 points)* Suppose that a photon arrives at the origin, at time t_f, from a distant piece of matter that is precisely at the horizon distance at time t_f. What is the time t_e at which the photon was emitted?

(c) *(2 points)* What is the **coordinate** distance from the origin to the point from which the photon was emitted?

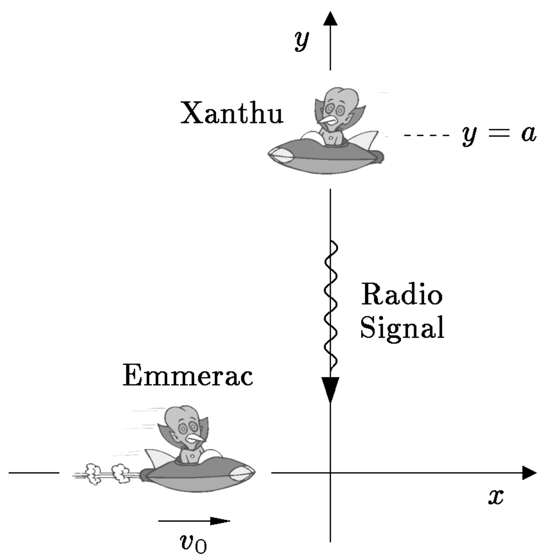
(d) *(10 points)* For an arbitrary time t in the interval $t_e \leq t \leq t_f$, while the photon is traveling, what is the **physical** distance $\ell_p(t)$ from the origin to the location of the photon?

(e) *(5 points)* At what time $t_{\max}$ is the physical distance of the photon from the origin at its largest value?

PROBLEM 16: TRANSVERSE DOPPLER SHIFTS

The following problem was taken from Problem 4, Quiz 1, 2005, where it counted 20 points.

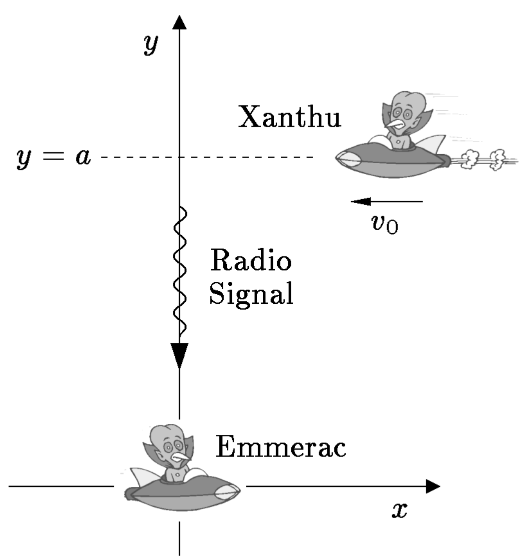
(a) *(8 points)* Suppose the spaceship Xanthu is at rest at location $(x=0, y=a, z=0)$ in a Cartesian coordinate system. (We assume that the space is Euclidean, and that the distance scales in the problem are small enough so that the expansion of the universe can be neglected.) The spaceship Emmerac is moving at speed v_0 along the x-axis in the positive direction, as shown in the diagram, where v_0 is comparable to the speed of light. As the Emmerac crosses the origin, it receives a radio signal that had been sent some time earlier from the Xanthu. Is the radiation received redshifted or blueshifted? What is the redshift z (where negative values of z can be used to describe blueshifts)?



(b) *(7 points)* Now suppose that the Emmerac is at rest at the origin, while the Xanthu is moving in the negative x-direction, at $y = a$ and $z = 0$, as shown in the diagram. That is, the trajectory of the Xanthu can be taken as

$$(x = -v_0 t, y = a, z = 0) \ .$$

At $t = 0$ the Xanthu crosses the y-axis, and at that instant it emits a radio signal along the y-axis, directed at the origin. The radiation is received some time later by the Emmerac. In this case, is the radiation received redshifted or blueshifted? What is the redshift z (where again negative values of z can be used to describe blueshifts)?

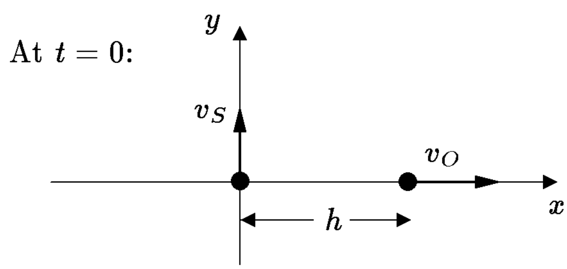


(c) *(5 points)* Is the sequence of events described in (b) physically distinct from the sequence described in (a), or is it really the same sequence of events described in a reference frame that is moving relative to the reference frame used in part (a)? Explain your reasoning in a sentence or two. *(Hint: note that there are three objects in the problem: Xanthu, Emmerac, and the photons of the radio signal.)*

PROBLEM 17: THE NONRELATIVISTIC TRANSVERSE DOPPLER SHIFT

The following problem was taken from Problem 4, Quiz 1, 2020, where it counted 10 points.

At time $t = 0$, a source of sound waves is located at the origin of a coordinate system, and an observer is located at a distance h along the positive x axis, as shown in the diagram below. The source is moving at constant speed v_S along the positive y axis, and the observer is moving at constant speed v_O along the positive x axis. Denote the speed of sound by u.



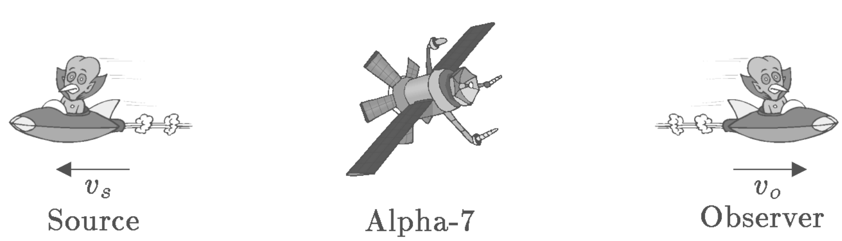
(a) *(6 points)* At $t = 0$, the sound source emits a wave crest. At what time t_1 does it arrive at the observer?

(b) *(4 points)* Under the usual assumption that the wavelength of the wave is very short compared to any other distance, what is the Doppler shift of the sound wave, as received by the observer?

∗ PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 20 points.

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with a speed v_s relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7 with speed v_o relative to Alpha-7.



(a) *(10 points)* Calculate the Doppler shift z of the radio wave as received by the observer. (Recall that radio waves are electromagnetic waves, just like light except that the wavelength is longer.)

(b) *(10 points)* Use the results of part (a) to determine v_{tot}, the velocity of the source spaceship as it would be measured by the observer spaceship. *(8 points will be given for the basic idea, whether or not you have the right answer for part (a), and 2 points will be given for the algebra.)*

PROBLEM 19: A FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

The following problem was Problem 3, Quiz 1, 2000.

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{\gamma} ,$$

where b and γ are constants. *[This universe differs from the matter-dominated universe described in the lecture notes in that ρ is not proportional to $1/a^3(t)$. Such behavior is possible when pressures are large, because a gas expanding under pressure can lose energy (and hence mass) during the expansion.]* For the following questions, any of the answers may depend on γ, whether it is mentioned explicitly or not.

a) *(5 points)* Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by a distant object. In terms of these quantities, find the value of the redshift parameter z with which the light is received.

b) *(5 points)* Find the "look-back" time as a function of z and t_0. The look-back time is defined as the length of the interval in cosmic time between the emission and observation of the light.

c) *(10 points)* Express the present value of the physical distance to the object as a function of H_0, z, and γ, where H_0 is the present value of the Hubble expansion rate.

d) *(10 points)* At the time of emission, the distant object had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this object at the earth today? Express your answer in terms of P, H_0, z, and γ. *[Energy flux (which might be measured in erg-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow.]*

e) *(10 points)* Suppose that the distant object is a galaxy, moving with the Hubble expansion. Within the galaxy a supernova explosion has hurled a jet of material directly towards Earth with a speed v, measured relative to the galaxy, which is comparable to the speed of light c. Assume, however, that the distance the jet has traveled from the galaxy is so small that it can be neglected. With what redshift z_J would we observe the light coming from this jet? Express your answer in terms of all or some of the variables v, z (the redshift of the galaxy), t_0, H_0, and γ, and the constant c.

PROBLEM 20: ANOTHER FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The following problem was Problem 3, Quiz 2, 1988:

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as

$$a(t) = bt^{1/3} \; .$$

Here b denotes a constant.

(a) If a light pulse is emitted at time t_e and observed at time t_o, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that t_e and t_o are given, find the observed redshift z.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of c, t_o, and z (i.e., without t_e appearing).

(d) At an arbitrary time t in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of c, t, and t_o.

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? (2020) *(30 points)*

(a) *(5 points)* In what way do spectral lines help us understand the properties of nearby stars? Consider the following options:

(A) We can determine their chemical composition.

(B) We can determine their speed relative to us.

(C) We can determine their acceleration relative to us.

(D) We can determine their mass.

(E) We can determine their volume.

Which one of the following combinations is correct:

(i) $\boxed{\text{(A) and (B)}}$

(ii) (A) and (C)

(iii) (B) and (C)

(iv) (D) and (E)

(v) (A), (B), (C), (D), and (E)

[*Comment:* The redshift of the spectral lines determines the speed of the star relative to us, and the presence and strengths of lines tells us about the composition. Although this information can identify the type of star and thereby give some indirect information about the mass and volume, the word "determine" does not really apply to the mass or volume.]

(b) *(5 points)* What is the cosmological principle?

(i) We live in the center of the universe.

(ii) $\boxed{\text{The universe is homogeneous and isotropic on large distance scales.}}$

(iii) The universe is isotropic.

(iv) All galaxies are evenly spaced throughout the universe.

(v) Newtonian gravity applies on large distance scales.

(c) *(5 points)* Roughly, what is the size of the Milky Way? And, at what length scale does the cosmological principle become valid?

(i) 10^7 light years and 100 Mpc

(ii) 10^5 light years and 1 Mpc

(iii) $\boxed{10^5 \text{ light years and } 100 \text{ Mpc}}$

(iv) 10^3 light years and 1 Mpc

(v) 10^3 light years and 100 Mpc

(d) *(5 points)* Which one of the following statements is a valid description of the history of what is now known as the "Cosmic Microwave Background"?

(i) The CMB radiation initially had a temperature less than 2.7 degrees Kelvin, but it increased gradually to its current value as it absorbed the energy of the light from stars.

(ii) Plasma instabilities in the early universe generated large fluctuations in electromagnetic radiation, which we now see as a "background" to astronomical observations.

(iii) The universe was originally filled with microwave radiation, and it persists until today.

(iv) It is a primordial kind of radiation with a blackbody spectrum of 2.7 degrees Kelvin at the time it was generated.

(v) At some point in the history of our universe, photons, which were once in thermal equilibrium with other particles, stopped interacting because electrons and atomic nuclei combined into atoms. The photons then free-streamed throughout the universe, from then until now.

[*Comment:* Choices (i), (iii), and (iv) are all contradicted by the fact that the temperature of the CMB has been cooling as the universe expands. 2.7 degrees is the current temperature; in the past it was much hotter, and hence not in the microwave band. The origin is not related to plasma instabilities, which are non-equilibrium phenomena which would not be capable of producing the thermal spectrum of the CMB.]

(e) *(5 points)* Why is the night sky not uniformly bright due to the universe being homogeneously populated with stars (or other bright objects)? Consider the following options:

(A) Space is not transparent.

(B) The universe is not infinitely large.

(C) The universe is not infinitely old.

(D) The absolute brightness of stars depends on their distance away from us.

(E) The cosmological redshift makes stars look dimmer and dimmer as they are further away from us.

Which one of the following combinations is correct?

(i) (A) and (C) — accepted

(ii) (B) and (D) — accepted

(iii) $\boxed{\text{(C) and (E)}}$ — preferred

(iv) (A), (C), and (E) — accepted

(v) (B), (C), and (D) — accepted

[*Comment:* When we (Alan and Bruno) wrote this problem we intended (iii) to be the correct answer, but after the quiz we decided that all the answers are arguably correct, so we decided to give full credit to everybody.

Despite the fact that this question is irrelevant to your grade, we hope you will find the following discussion interesting and informative.

The question is complicated by the fact that we do not fully agree with Ryden's statement that "the primary resolution to Olbers' paradox comes from the fact that the universe has a finite age," which corresponds to choice (C) in the problem statement. She also mentions that "However, in an expanding universe, the surface brightness of distant light sources is decreased relative to what you would see in a static universe," which is a way of describing choice (E). It's hard to know what "primary resolution" means, since these two explanations are both relevant and are somewhat intertwined. It is clear that a finite age is not essential to resolving Olber's paradox. In the steady-state theory (discussed by Ryden on pp. 17 and 18), which was considered very plausible before the discovery of the CMB, the age of the universe was infinite, but Olber's paradox was entirely avoided by the redshift effects.

Ryden was not quantitative when she spoke of the decrease in surface brightness, but in fact you have already done the calculations to show this. You calculated on Problem 6 of Problem Set 2 that the power received from a distant source is reduced by a factor of $(1+z)^2$, where one factor of $(1+z)$ came from the reduction in energy of each photon, and the other factor came from the reduction in the arrival rate of photons. You also found in Problem 5 that the angular size is enhanced by a factor of $(1+z)$, which means that the solid angle is enhanced by a factor of $(1+z)^2$. This means that the surface brightness, the amount of power that we receive per surface area on the sky, is actually reduced by a factor of $(1+z)^4$. So as we look to very large redshifts, the stars contribute very little to the brightness of the night sky.

The effect of the finite age means that we cannot see stars at distances larger than the horizon distance, which (as discussed in Lecture Notes 4) is the present physical distance of the furthest objects for which light has had time to reach us, since the beginning. The light we receive from such objects left them at $t = 0$, when $a(t) = 0$, so the redshift is infinite. Since the brightness of high z stars was suppressed by a factor of $1/(1+z)^4$, the fact that we see nothing beyond $z =$ infinity does not seem as relevant. But the finite age can still be considered relevant in the sense that if we considered a toy model of a universe that was static, so there would be no redshift, Olber's paradox could still be resolved if the universe were not infinitely old.

Option (A): the lack of spatial transparency is not usually mentioned in discussions of Olber's paradox. From observations of the CMB, we know that the universe is highly transparent, out to redshifts of about $z \approx 1000$. From the point of view of Olber's paradox, however, $z \approx 1000$ is very close. As Ryden explains, Olber's paradox involves thinking about distances of order 10^{18} Mpc, which is about 10^{14} times larger than the horizon distance. At $z \approx 1000$ we are looking so far back in time that we see the early plasma phase of the universe, when space was very opaque (non-transparent). This region is called the *surface of last scattering*, since most of the CMB photons were last scattered in this region. The surface of last scattering is fairly close to the horizon distance, so the lack of transparency in this region can be used as a variant to discussing the effects of the horizon.

Ryden dismisses transparency as an explanation by pointing out that if the universe is infinitely old, then eventually any light-absorbing material would be heated to the temperature of the stars, and would glow brightly. But if the universe is not infinitely old (as it does not appear to be), then this argument does not hold.

Option (B): the possibility that the universe has a finite size. Ryden points out that our universe might not be infinite — we don't know. If *our* universe has a finite size, however, it would not have much relevance to Olber's paradox, since the finite size would have to be about equal to or larger than the horizon distance, or else we would probably see evidence of it. In toy models of a universe, however, a finite size can serve as a way to avoid Olber's paradox.

Option (D): The validity of (D) as an explanation depends on the interpretation of the words. At a fixed cosmic time, we believe that the universe is on average homogeneous, so the absolute brightness of stars does not depend on their distance from us. However, if the statement is assumed to refer to the absolute brightness of the stars at the time that we see them, then of course we are seeing distant stars as they appeared long ago, and the absolute brightness can be different. If we look so far back that there are no stars, then the brightness would be zero. By this interpretation (D) can be valid, and is really just another way of discussing the finite age.]

(f) *(5 points)* Which of the following statements best describes Einstein's equivalence principle?

(i) Physical laws do not depend on the place of the universe where we make measurements.

(ii) General relativity is equivalent to special relativity when only massless particles are considered.

(iii) An observer on the surface of the Earth experiences the same external forces as a free-falling observer.

(iv) An observer on the surface of the Earth, without looking through a window, cannot tell whether they are actually on Earth or if they are on a rocket ship accelerating at $9.8\ \mathrm{m/s^2}$ in empty space.

(v) Any calculation done in the framework of General Relativity must be equivalent to another in Newtonian gravity in the appropriate limit.

PROBLEM 2: DID YOU DO THE READING (2018)? *(30 points)*

(a) *(5 points)* After telescopes became available, more and more extended objects in the sky, called nebulae, were discovered, but those were thought as members of our galaxy. Who is the person who first proposed that some of the nebulae are galaxies like our own located outside our galaxy?

(i) Isaac Newton

(ii) Immanuel Kant

(iii) Edwin Hubble

(iv) Albert Einstein

(b) *(5 points)* Before 1923, questions of the nature of the spiral and elliptical nebulae could not be settled without some reliable method of determining how far away they are. In 1923, Edwin Hubble was for the first time able to resolve the Andromeda Nebula (galaxy) into separate stars and estimated the distance to the Andromeda Nebula. What observational quantity did he measure to estimate the distance?

(i) the radial velocity of individual stars in the Adromeda Nebula

(ii) the radial velocity of the Andromeda Nebula itself

(iii) the periods of variation of a class of stars in the Andromeda Nebula

(iv) the parallax of bright stars in the Adromeda Nebula

[Comment: Hubble used Cepheid variables to estimate the distance to the Andromeda Nebula (galaxy) with a tight relation between the observed periods of variation of the

Cepheids and their absolute luminosities provided by Henrietta Swan Leavitt and Harlow Shapley.]

(c) *(5 points)* In 1917, a year after the completion of Einstein's general theory of relativity, __________ looked specifically for a solution that would be homogeneous, isotropic, and static, and thus was forced to mutilate the equations by introducing a term, the so-called cosmological constant. In the same year, another solution of the modified theory was found by the Dutch astronomer __________. Although this solution appeared to be static, it had the remarkable property of predicting a redshift proportional to the distance. In 1922, the general homogeneous and isotropic solution of the original Einstein equations was found by the Russian mathematician __________, which provides a mathematical background for the most modern cosmological theories. Which is the right answer to fill in the blanks in turn?

- (i) Friedmann — Einstein — de Sitter
- (ii) Friedmann — de Sitter — Einstein
- (iii) Einstein — Friedmann — de Sitter
- (iv) $\boxed{\text{Einstein — de Sitter — Friedmann}}$
- (v) de Sitter — Einstein — Friedmann
- (vi) de Sitter — Friedmann — Einstein

(d) *(5 points)* After radio noises with the equivalent temperature of about 3.5° K were detected, Penzias, Wilson, Dicke, Peebles, Roll, and Wilkinson decided to publish a pair of companion letters in the Astrophysical Journal, in which Penzias and Wilson would announce their observations, and Dicke, Peebles, Roll, and Wilkinson would explain the cosmological interpretation. What is the title of the paper written by Penzias and Wilson?

- (i) $\boxed{\text{“A Measurement of Excess Antenna Temperature at 4,080 Mc/s”}}$
- (ii) "Cosmic Black-Body Radiation"
- (iii) "Origin of the Microwave Radio Background"
- (iv) "Three Degrees Above Zero: Bell Labs in the Information Age"

[Comment: (ii) is the title of the companion letter by Dicke, Peebles, Roll, and Wilkinson; (iii) is the title of a paper written by Peebles and Dicke in 1966, in which they refuted a suggestion by Michele Kaufman that the background radiation was emitted by ionized intergalactic hydrogen; and (iv) is the title of a book written by Jeremy Bernstein.]

(e) *(5 points)* The universe contains different types of particles. Which of the following statements is NOT true?

- (i) A baryon is defined as a particle made of three quarks.

(ii) Electrons and neutrinos are leptons.

(iii) There are three types of neutrinos and they all have zero charge.

(iv) The component of the universe made of ions, atoms, and molecules is generally referred to as baryonic matter, since only the baryons (protons and neutrons) contribute significantly to the mass density.

(v) $\boxed{\text{About three-fourths of the baryonic matter in the universe is currently in the form of helium.}}$

[Comment: about three-fourths of the baryonic matter in the universe is currently in the form of hydrogen.]

(f) *(5 points)* If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1000 Mpc. (1 Mpc $= 10^6$ pc, 1 pc $= 3.086 \times 10^{16}$ m $= 3.262$ light-year).

(ii) $\boxed{\text{100 Mpc.}}$

(iii) 1 Mpc.

(iv) 100 kpc (1 kpc = 1000 pc).

(v) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

PROBLEM 3: DID YOU DO THE READING (2016)? *(35 points)*

(a) *(5 points)* The Milky Way has been known since ancient times as a band of light stretching across the sky. We now recognize the Milky Way as the galaxy of stars in which we live, with a large collection of stars, including our sun, arranged in a giant disk. Since the individual stars are mostly too small for our eyes to resolve, we observe the collective light from these stars, concentrated in the plane of the disk. The idea that the Milky Way is actually a disk of stars was proposed by

(i) Claudius Ptolemy, in the 2nd century AD.

(ii) Johannes Kepler, in 1610.

(iii) Isaac Newton, in 1695.

(iv) $\boxed{\text{Thomas Wright, in 1750.}}$

(v) Immanuel Kant, in 1755.

(vi) Edwin Hubble, in 1923.

(b) *(5 points)* Once it was recognized that we live in a galaxy, it was initially assumed that ours was the only galaxy. The suggestion that some of the patches of light known as nebulae might actually be other galaxies like our own was made by

(i) Claudius Ptolemy, in the 2nd century AD.

(ii) Johannes Kepler, in 1610.

(iii) Isaac Newton, in 1695.

(iv) Thomas Wright, in 1750.

(v) $\boxed{\text{Immanuel Kant, in 1755.}}$

(vi) Edwin Hubble, in 1923.

(c) *(5 points)* The first firm evidence that there is more than one galaxy stemmed from the ability to observe the Andromeda Nebula with high enough resolution to distinguish its individual stars. In particular, the observation of Cepheid variable stars in Andromeda allowed a distance estimate that placed it well outside the Milky Way. The observation of Cepheid variable stars in Andromeda was first made by

(i) Johannes Kepler, in 1610.

(ii) Isaac Newton, in 1695.

(iii Thomas Wright, in 1750.

(iv) Immanuel Kant, in 1755.

(v) Henrietta Swan Leavitt and Harlow Shapley in 1915.

(vi) $\boxed{\text{Edwin Hubble, in 1923.}}$

(d) *(5 points)* The first hint that the universe is filled with radiation with an effective temperature near 3 K, although not recognized at the time, was an observation of absorption lines in cyanogen (CN) by Adams and McKellar in 1941. They observed dark spectral lines which they interpreted as absorption by the cyanogen of light coming from the star behind the gas cloud. Explain in a few sentences how these absorption lines can be used to make inferences about the cosmic background radiation bathing the cyanogen gas cloud.

Answer:

When an atom absorbs a photon, it is excited from its initial state to some final state, and the energy of the photon must match the energy difference betwen the two states. One of the observed cyanogen lines was associated with a transition starting in the ground state, and two other observed lines were associated with transitions starting from an excited state. By comparing the intensities of these absorption lines, the astronomers could infer the relative abundance of ground state and excited cyanogen molecules, which in turn allowed them to infer the temperature of the gas cloud. They found a temperature of 2.2 K.

(e) *(5 points)* As the universe expands, the temperature of the cosmic microwave background

(i) goes up in proportion to the scale factor $a(t)$.

(ii) stays constant.

(iii) $\boxed{\text{goes down in proportion to } 1/a(t).}$

(iv) goes down in proportion to $1/a^2(t)$.

(f) *(5 points)* When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, $\boxed{\text{2 billion years,}}$ 10 billion years, or 20 billion years?

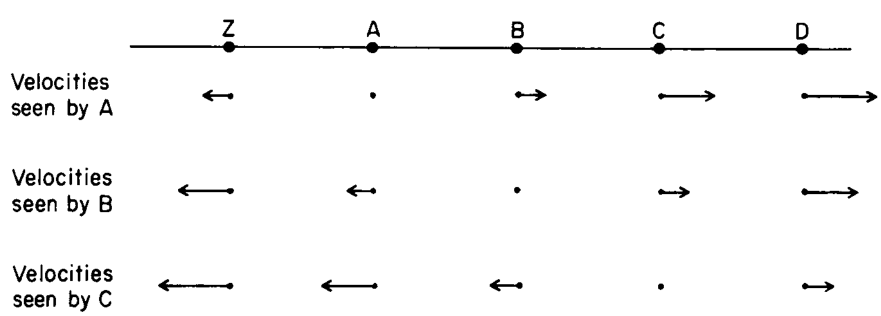
(g) *(5 points)* Explain in a few sentences what is meant by the equivalence principle?

Answer:

Ryden states that the equivalence principle is the fact that the gravitational mass of any object is equal to its inertial mass. It would also be correct to say that the graviational mass is proportional to the inertial mass. (If they are proportional, there is always a value of G which makes them equal.) The equivalence principle can also be described more generally by saying that gravity is equivalent to acceleration, so that within a small volume the effects of gravity can be removed by describing the system in an accelerating coordinate system.

PROBLEM 4: DID YOU DO THE READING? (2013) *(25 points)*

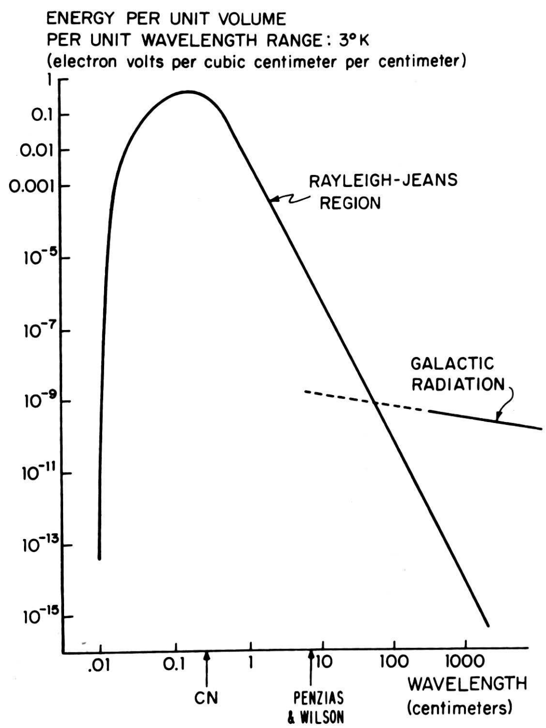
(a) *(5 points)*



The diagram was used to explain how Hubble's Law is consistent with homogeneity. Naively Hubble's law sounds like it makes us the center of the universe, since all distant galaxies are receding from us. The diagram shows, however, that if observers on Galaxy A see a Hubble expansion pattern, as shown on the top line, then observers on Galaxies B and C would also see a Hubble expansion pattern, centered on themselves.

(b) *(5 points)* The intensity at short wavelengths is suppressed by the effect of quantum theory, which implies that the electomagnetic field is composed of quanta, or photons, each with an energy of Planck's constant h times the frequency $\nu = c/\lambda$, where λ is the wavelength. When the wavelength is so small that the photon energy is large compared to kT, the typical thermal energy, then it is rare for these high-energy photons to exist.

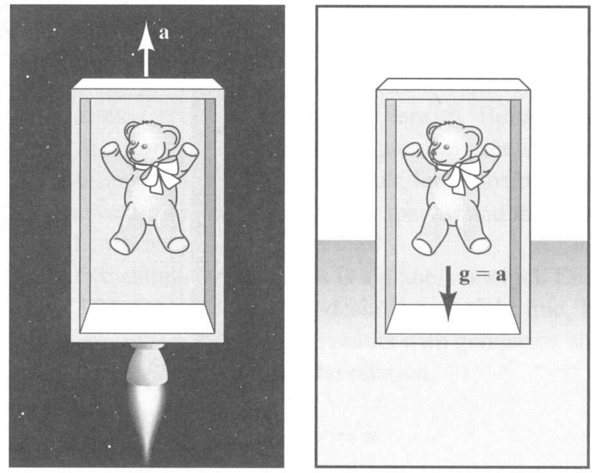
(c) *(5 points)* By observing multiple closely spaced absorption lines of the cyanogen radicals, astronomers can infer the relative population of the ground state and an excited state.



This allows astronomers to measure the effective temperature of interstellar gas clouds that are in equilibrium with the CMB.

(d) *(5 points)* Arno Penzias and Robert Wilson discovered the cosmic microwave background *(3 points)*, while employed at Bell Telephone Laboratories *(2 points)*.

(e) *(5 points)* Ryden used the teddy-bear diagram at the right to illustrate the equivalence principle, a key assumption of general relativity. The principle implies that they teddy bear has no way of distinguishing whether she is experiencing the gravity of the Earth, or whether she is in an acclerating rocket in otherwise empty space.



PROBLEM 5: DID YOU DO THE READING (2011)? *(25 points)*†

†Solution written by Daniele Bertolini.

(a) *(10 points)* To determine the distance of the galaxies he was observing Hubble used so called *standard candles*. Standard candles are astronomical objects whose intrinsic luminosity is known and whose distance is inferred by measuring their apparent luminosity. First, he used as standard candles variable stars, whose intrinsic luminosity can be related to the period of variation. Quoting Weinberg's *The First Three Minutes*, chapter 2, pages 19-20:

In 1923 Edwin Hubble was for the first time able to resolve the Andromeda Nebula into separate stars. He found that its spiral arms included a few bright variable stars, with the same sort of periodic variation of luminosity as was already familiar for a class of stars in our galaxy known as Cepheid variables. The reason this was so important was that in the preceding decade the work of Henrietta Swan Leavitt and Harlow Shapley of the Harvard College Observatory had provided a tight relation between the observed periods of variation of the Cepheids and their absolute luminosities. (Absolute luminosity is the total radiant power emitted by an astronomical object in all directions. Apparent luminosity is the radiant power received by us in each square centimeter of our telescope mirror. It is the apparent rather than the absolute luminosity that determines the subjective degree of brightness of astronomical objects. Of course, the apparent luminosity depends not only on the absolute luminosity, but also on the distance; thus, knowing both the absolute and the apparent luminosities of an astronomical body, we can infer its distance.) Hubble, observing the apparent luminosity of the Cepheids in the Andromeda Nebula, and estimating their absolute luminosity from their periods, could immediately calculate their distance, and hence the distance of the Andromeda Nebula, using the simple rule that apparent luminosity is proportional to the absolute luminosity and inversely proportional to the square of the distance.

He also used particularly bright stars as standard candles, as we deduce from page 25:

Returning now to 1929: Hubble estimated the distance to 18 galaxies from the apparent luminosity of their brighest stars, and compared these distances with the galaxies' respective velocities, determined spectroscopically from their Doppler shifts.

Note: since from reading just the first part of Weinberg's discussion one could be induced to think that Hubble used just Cepheids as standard candles, students who mentioned only Cepheids got 9 points out of 10. In fact, however, Hubble was able to identify Cepheid variables in only a few galaxies. The Cepheids were crucial, because they served as a calibration for the larger distances, but they were not in themselves sufficient.

(b) *(5 points)* Quoting Weinberg's *The First Three Minutes*, chapter 2, page 21:

We would expect intuitively that at any given time the universe ought to look the same to observers in all typical galaxies, and in whatever directions they look. (Here, and below, I will use the label "typical" to indicate galaxies that do not have any large peculiar motion of their own, but are simply carried along with the general cosmic flow of galaxies.) This hypothesis is so natural (at least since Copernicus) that it has been called **the** *Cosmological Principle by the English astrophysicist Edward Arthur Milne.*

So the Cosmological principle basically states that the universe appears as homogeneous and isotropic (on scales of distance large enough) to any typical observer, where typical is referred to observers with small local motion compared to the expansion flow. Ryden gives a more general definition of Cosmological Principle, which is valid as well. Quoting Ryden's *Introduction to Cosmology*, chapter 2, page 11 or 14 (depending on which version):

However, modern cosmologists have adopted the **cosmological principle,** *which states: There is nothing special about our location in the universe. The cosmological principle holds true only on large scales (of 100 Mpc or more).*

(c) *(10 points)* Quoting again Ryden's *Introduction to Cosmology*, chapter 2, page 9 or 11:

Saying that the universe is **isotropic** *means that there are no preferred directions in the universe; it looks the same no matter which way you point your telescope. Saying that the universe is* **homogeneous** *means that there are no preferred locations in the universe; it looks the same no matter where you set up your telescope.*

(i) $\boxed{\text{False.}}$ If the universe is isotropic around one point it does not need to be homogeneous. A counter-example is a distribution of matter with spherical symmetry, that is, with a density which is only a function of the radius but does not depend on the direction: $\rho(r, \theta, \phi) \equiv \rho(r)$. In this case for an observer at the center of the distribution the universe looks isotropic but it is not homogeneous.

(ii) $\boxed{\text{True.}}$ For the case of Euclidean geometry isotropy around two or more distinct points does imply homogeneity. Weinberg shows this in chapter 2, page 24. Consider two observers, and two arbitrary points A and B which we would like to prove equivalent. Consider a circle through point A, centered on observer 1, and another circle through point B, centered on observer 2. If C is a point on the intersection of the two circles, then isotropy about the two observers implies that $A = C$ and $B = C$, and hence $A = B$. (This argument was good enough for Weinberg and hence good enough to deserve full credit, but it is actually incomplete: one can find points A and B for which the two circles will not intersect. On your next problem set you will have a chance to invent a better proof.)

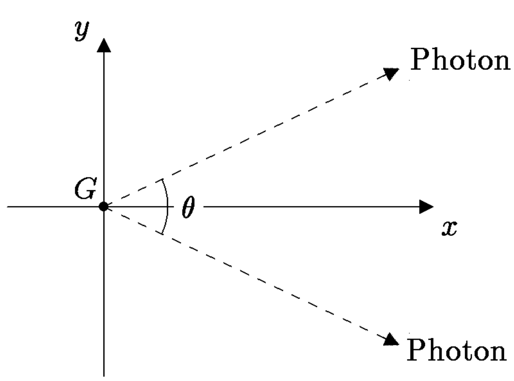
(d) *(2 points extra credit)* $\boxed{\text{False.}}$ If we relax the hypothesis of Euclidean geometry, then isotropy around two points does not necessarily imply homogeneity. A counter-example we mentioned in class is a two-dimensional universe consisting of the surface of a sphere. Think of the sphere in three Euclidean dimensions, but the model "universe" consists only of its two-dimensional surface. Imagine latitude and longitude lines to give coordinates to the surface, and imagine a matter distribution that depends only on latitude. This would not be homogeneous, but it would look isotropic to observers at both the north and south poles. While this example describes a two-dimensional universe, which therefore cannot be our universe, we will learn shortly how to construct a three-dimensional non-Euclidean universe with these same properties.

PROBLEM 6: LIGHT RAYS TRAVELING THROUGH A MATTER-DOMINATED FLAT UNIVERSE *(40 points)*

Consider a flat, matter-dominated universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. Now consider a galaxy G in this universe which at time t_1 emits two photons, with an angular separation θ between their paths, as shown in the diagram:



(a) *(10 points)* At cosmic time t (for $t > t_1$), what is the physical distance $\ell_{1,\text{phys}}(t)$ of each of these photons from the galaxy G?

Answer:

The coordinate speed of light is $c/a(t)$, so the coordinate distance traveled is

$$\begin{aligned}
\ell_{1,c}(t) &= \int_{t_1}^{t} \frac{c}{a(t')}\mathrm{d}t' \\
&= \left(\frac{3c}{b}\right) t^{1/3}\Big|_{t_1}^{t} \\
&= \left(\frac{3c}{b}\right) t^{1/3}\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .
\end{aligned}$$

The physical distance is then

$$\begin{aligned}
\ell_{1,\text{phys}}(t) &= a(t)\ell_{1,c}(t) = bt^{2/3}\ell_{1,c}(t) \\
&= \boxed{3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .}
\end{aligned}$$

(b) *(5 points)* If the frequency of the photons was ν_1 when they were emitted, what is their frequency $\nu(t)$ at cosmic time t (for $t > t_1$)? $\nu(t)$ should be the frequency as it would be measured by a comoving observer, i.e. an observer at rest with respect to the matter at the same location.

Answer:

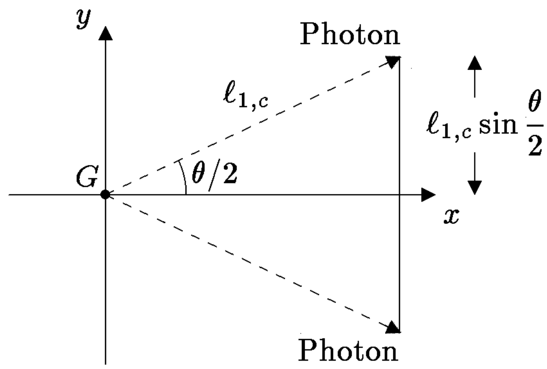
The wavelength of a photon is stretched in proportion to the scale factor, so the frequency is inversely proportional to the scale factor. So

$$\boxed{\nu(t) = \frac{a(t_1)}{a(t)}\nu_1 = \left(\frac{t_1}{t}\right)^{2/3} \nu_1 .}$$

(c) *(10 points)* What is the physical distance $\ell_{2,\text{phys}}(t)$ between the two photons at time t (for $t > t_1$)?

Answer:

Since the universe is flat, we can use ordinary Euclidean geometry, as shown in the diagram:



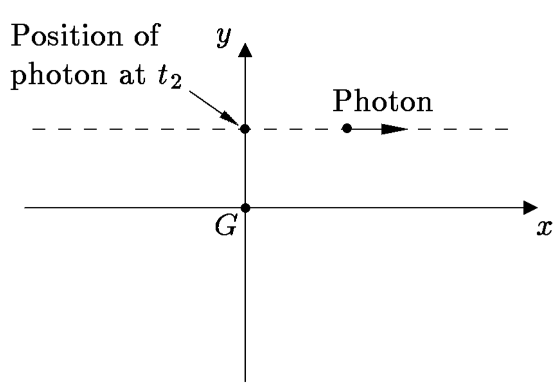
The coordinate distance between the two photons is then given by

$$\ell_{2,c} = 2\ell_{1,c} \sin \frac{\theta}{2} \ .$$

The physical distance is then

$$\begin{aligned} \ell_{2,\text{phys}} &= 2a(t)\ell_{1,c} \sin \frac{\theta}{2} \\ &= 2bt^{2/3} \left(\frac{3c}{b}\right) t^{1/3} \left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] \sin \frac{\theta}{2} \\ &= \boxed{6ct \left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] \sin \frac{\theta}{2} \ .} \end{aligned}$$

Now consider a different situation, but in the same universe. This time we consider a photon that travels past the galaxy G, traveling in the x direction, in the x-y plane, as shown in the diagram below. We are told that the photon crosses the y axis at time t_2, and at that time the photon is a physical distance h from the galaxy.



(d) *(10 points)* What is the physical distance $\ell_{3,\mathrm{phys}}(t)$ between the photon and the galaxy G at arbitrary time t, which might be earlier or later than t_2?

Answer:

It is important to recognize here that the coordinates shown are comoving coordinates, or map coordinates, so that physical distances are obtained by multiplying by the scale factor. (If these coordinates represented physical distances from the origin, then the Hubble expansion would be driving all particles outward, and the photon trajectory would not be a straight line.) So, if the physical distance between the photon and the galaxy is equal to h at time t_2, then the y coordinate of the photon is equal to

$$y = \frac{h}{a(t_2)} .$$

The x coordinate is determined by the fact that it vanishes at time t_2, and then moves toward positive values at the coordinate speed of light,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} .$$

Thus,

$$x(t) = \int_{t_2}^{t} \frac{c}{bt'^{2/3}} \mathrm{d}t' = \frac{3c}{b} t^{1/3} \left[1 - \left(\frac{t_2}{t} \right)^{1/3} \right] .$$

The coordinate distance from the origin is then given by the Pythagorean theorem,

$$\ell_{3,c}(t) = \left[x^2(t) + y^2(t) \right]^{1/2} ,$$

so

$$\ell_{3,\mathrm{phys}}(t) = bt^{2/3} \left[x^2(t) + y^2(t) \right]^{1/2}$$

$$= \boxed{ \left\{ 9c^2 t^2 \left[1 - \left(\frac{t_2}{t} \right)^{1/3} \right]^2 + \left(\frac{t}{t_2} \right)^{4/3} h^2 \right\}^{1/2} } .$$

(e) *(5 points)* At time t_2, what is the recessional speed $d\ell_{3,\mathrm{phys}}(t)/dt$ of the photon from the galaxy. *Hint:* if you are clever, this can be done with very little calculation.

Answer:

Note, first of all, that one cannot blindly assume that the photon obeys Hubble's law, since Hubble's law applies only to the comoving matter in the model universe, which is undergoing uniform expansion. It does not apply to objects, such as photons, that are moving relative to the comoving matter. (Motion relative to the comoving matter is called proper motion.)

The answer can be obtained by simply differentiating the above expression for $\ell_{3,\text{phys}}(t)$ with respect to t, and then setting $t = t_2$, but there is a shorter way. If we go back to

$$\ell_{3,\text{phys}}(t) = a(t)\ell_{3,c}(t) ,$$

we note that, unlike the the description of uniform Hubble expansion, in this case the coordinate distance $\ell_{3,c}$ depends on time. The coordinate distance between two pieces of comoving matter (i.e., matter expanding with the universe) does not change with time, but here we have the distance between a galaxy (at fixed coordinates) and a photon (which is traveling). However, we can easily see from the diagram that at time t_2, the coordinate distance $\ell_{3,c}(t)$ is at its minimum, and therefore its time derivative at t_2 must be zero. Therefore,

$$\begin{aligned}\left.\frac{\mathrm{d}\ell_{3,\text{phys}}}{\mathrm{d}t}\right|_{t_2} &= \dot{a}(t_2)\ell_{3,c}(t_2) \\ &= \left(\frac{\dot{a}}{a}\right)[a\ell_{3,c}(t_2)] = H(t_2)\ell_{3,\text{phys}}(t_2) \\ &= \boxed{\left(\frac{2}{3t_2}\right) h \ .}\end{aligned}$$

The average grade on this problem was only 2.8/5, or 55%, which was the lowest for any problem on the quiz. Many students assumed that Hubble's law applied to the photon. This assumption leads to the correct answer at time t_2, when the photon proper velocity is perpendicular to the direction from the photon to the galaxy, but not at other times. Students who gave the correct answer, but attributed it to Hubble's law, were given 4 points out of 5. Another common error was to assert that the speed of light is always measured as c, so $\mathrm{d}\ell_{3,\text{phys}}(t)/\mathrm{d}t = c$. The correct description of the invariance of the speed of light is to say that any inertial observer (which includes all comoving observers) will measure the speed of a photon that passes him as being equal to c. But if the photon is at a different location, then one has to take into account the expansion of the universe, which is done by basing all calculations on the principle that the coordinate speed of light is always equal to $c/a(t)$.

PROBLEM 7: THE STEADY-STATE UNIVERSE THEORY *(30 points)*

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties were confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density does not fall. Like the conventional theory, the steady-state theory describes a homogeneous, isotropic, expanding universe, so the same comoving coordinate formulation can be used.

a) *(15 points)* The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $a(t)$ which is consistent with this hypothesis.

Answer:

The Hubble expansion rate is related to $a(t)$ by

$$H(t) = \frac{1}{a(t)} \frac{da}{dt} ,$$

so in this case

$$\frac{1}{a(t)} \frac{da}{dt} = H_0 ,$$

which can be rewritten as

$$\frac{da}{a} = H_0 \, dt .$$

Integrating,

$$\ln a = H_0 \, t + c ,$$

where c is a constant of integration. Exponentiating,

$$\boxed{a = b e^{H_0 \, t} ,}$$

where $b = e^c$ is an arbitrary constant.

b) *(15 points)* Suppose that the mass density of the universe is ρ_0, which of course does not change with time. In terms of the general form for $a(t)$ that you found in part (a), calculate the rate at which new matter must be created for ρ_0 to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. *[If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $a(t)$.]*

Answer:

Consider a cube of side ℓ_c drawn on the comoving coordinate system diagram. The physical length of each side is then $a(t)\,\ell_c$, so the physical volume is

$$V(t) = a^3(t)\,\ell_c^3\ .$$

Since the mass density is fixed at $\rho = \rho_0$, the total mass inside this cube at any given time is given by

$$M(t) = a^3(t)\,\ell_c^3\,\rho_0\ .$$

In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

$$\begin{aligned}
\text{Rate} &= \frac{1}{V(t)}\frac{dM}{dt} \\
&= \frac{1}{a^3(t)\,\ell_c^3}\,3a^2(t)\frac{da}{dt}\,\ell_c^3\,\rho_0 \\
&= \frac{3}{a}\frac{da}{dt}\,\rho_0 \\
&= \boxed{3H_0\,\rho_0\ .}
\end{aligned}$$

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose we take $H_0 = 70$ km-sec^{-1}-Mpc^{-1}, and take ρ_0 to be the critical density, $\rho_c = 3H_0^2/8\pi G$. Then

$$\begin{aligned}
\text{Rate} &= \frac{9H_0^3}{8\pi G} \\
&= \frac{9\times(70\ \text{km-s}^{-1}\text{-Mpc}^{-1})^3}{8\pi\times 6.67260\times10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\
&= \frac{9\times(70\ \cancel{\text{km}}\text{-s}^{-1}\text{-}\cancel{\text{Mpc}}^{-1})^3}{8\pi\times 6.67260\times10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\
&\qquad\times\left(\frac{1\ \cancel{\text{Mpc}}}{3.086\times10^{22}\ \cancel{\text{m}}}\right)^3\times\left(\frac{10^3\ \cancel{\text{m}}}{\cancel{\text{km}}}\right)^3 \\
&= 6.26\times10^{-44}\ \text{kg-m}^{-3}\text{-s}^{-1}\ .
\end{aligned}$$

To put this number into more meaningful terms, note that the mass of a hydrogen atom is 1.67×10^{-27} kg, and that 1 year $= 3.156 \times 10^7$ s. The rate of matter production required for the steady-state universe theory can then be expressed as roughly one hydrogen atom per cubic meter per billion years! Needless to say, such a rate of matter production is totally undetectable, so the steady-state theory cannot be ruled out by the failure to detect matter production.

PROBLEM 8: OBSERVING A DISTANT GALAXY IN A MATTER-DOMINATED FLAT UNIVERSE *(40 points)*

Suppose that we are living in a matter-dominated flat universe, with a scale factor given by

$$a(t) = bt^{2/3} ,$$

where b is a constant. The present time is denoted by t_0.

(a) *(5 points)* If we measure time in seconds, distance in meters, and coordinate distances in notches, what are the units of b?

Answer:

$a(t)$ would be measured in meters/notch, and t would be measured in seconds. So

$$[b] = \frac{[a(t)]}{[t]^{2/3}} = \boxed{\frac{\text{m}}{\text{notch-s}^{2/3}}\ .}$$

(b) *(5 points)* Suppose that we observe a distant galaxy which is one half of a "Hubble length" away, which means that the physical distance today is $\ell_p = \frac{1}{2}cH_0^{-1}$, where c is the speed of light and H_0 is the present value of the Hubble expansion rate. What is the proper velocity $v_p \equiv \frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t}$ of this galaxy relative to us?

Answer:

By Hubble's law, the velocity of recession is equal to H_0 times the physical distance, so

$$v_p = H_0 \left[\frac{1}{2}cH_0^{-1}\right] = \boxed{\frac{1}{2}c\ .}$$

A common error in this part was to use

$$H_0 = \frac{\dot{a}}{a} = \frac{2}{3}\frac{bt_0^{-1/3}}{bt_0^{2/3}} = \frac{2}{3t_0}$$

to write

$$\ell_p = \frac{3}{4} c t_0 \ ,$$

and then to differentiate this expression with respect to t_0, finding $v_p = 3c/4$. The problem with this approach is that it assumes that the relation $\ell_p = \frac{1}{2} c H^{-1}$ holds for all t, so that one can differentiate it to find the velocity. But an object that is at distance $\frac{1}{2} c H^{-1}$ does not remain at a distance $\frac{1}{2} c H^{-1}$ as time progresses. It is the coordinate distance ℓ_c, and not the physical distance measured in Hubble lengths, that remains constant as the universe expands.

(c) *(5 points)* What is the coordinate distance ℓ_c between us and the distant galaxy? Express your answer in terms of b, t_0, and c (but not H_0).

Answer:

We know that $\ell_p(t) = a(t)\ell_c$, so

$$\ell_c = \frac{\ell_p(t_0)}{a(t_0)} = \frac{c}{2bH_0 t_0^{2/3}} \ .$$

To eliminate H_0, which is not allowed in the answer, we can use

$$H_0 = \frac{1}{a(t_0)} \frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0} = \frac{1}{bt_0^{2/3}} \left[\frac{2}{3} b t_0^{-1/3} \right] = \frac{2}{3t_0} \ .$$

Inserting the result into the line above,

$$\boxed{\ell_c = \frac{3}{4} \frac{ct_0^{1/3}}{b} \ .}$$

If you did not answer the previous part, you may still continue with the following parts, using the symbol ℓ_c for the coordinate distance to the galaxy.

(d) *(5 points)* At what time t_e was the light that we are now receiving from the galaxy emitted?

Answer:

We know that the coordinate velocity of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} = \frac{c}{bt^{2/3}} .$$

We can find t_e by the requirement that the coordinate distance that light travels between t_e and t_0 must be equal to ℓ_c found in part (c):

$$\int_{t_e}^{t_0} \frac{c}{bt'^{2/3}} \, \mathrm{d}t' = \frac{3}{4} \frac{ct_0^{1/3}}{b} \ .$$

Integrating,

$$\frac{3c}{b} \left[t_0^{1/3} - t_e^{1/3} \right] = \frac{3}{4} \frac{ct_0^{1/3}}{b} \ .$$

With a little algebra we see

$$t_e^{1/3} = \frac{3}{4} t_0^{1/3} \quad \Longrightarrow \quad \boxed{t_e = \frac{27}{64} t_0 \ .}$$

(e) *(5 points)* What is the redshift z of the light that we are now receiving from the distant galaxy?

Answer:

The redshift is related to the scale factor by

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/3} = \left(\frac{64}{27}\right)^{2/3} = \frac{16}{9} \; ,$$

so

$$\boxed{z = \frac{7}{9} \; .}$$

(f) *(10 points)* Consider a light pulse that leaves the distant galaxy at time t_e, as calculated in part (d), and arrives here at the present time, t_0. Calculate the physical distance $r_p(t)$ between the light pulse and us. Find $r_p(t)$ as a function of t for all t between t_e and t_0.

Answer:

We first calculate the coordinate separation $r_c(t)$ between the light pulse and us, as a function of t. At time t_e it is equal to the value of ℓ_c found in part (c), and from that time onward it is reduced by the coordinate distance that light can travel between times t_e and t. Therefore,

$$\begin{aligned} r_c(t) &= \frac{3}{4}\frac{ct_0^{1/3}}{b} - \int_{t_e}^{t} \frac{c}{b\, t'^{2/3}}\, \mathrm{d}t' \\ &= \frac{3}{4}\frac{ct_0^{1/3}}{b} - \frac{3c}{b}\left[t^{1/3} - t_e^{1/3}\right] \\ &= \frac{3}{4}\frac{ct_0^{1/3}}{b} - \frac{3c}{b}\left[t^{1/3} - \frac{3}{4}t_0^{1/3}\right] \\ &= \frac{3c}{b}\left[t_0^{1/3} - t^{1/3}\right] \; . \end{aligned}$$

The physical distance is then

$$r_p(t) = bt^{2/3}\, r_c(t) = 3c\left[t_0^{1/3} t^{2/3} - t\right] = \boxed{3ct\left[\left(\frac{t_0}{t}\right)^{1/3} - 1\right] \; .}$$

(g) *(5 points)* If we send a radio message now to the distant galaxy, at what time t_r will it be received?

Answer:

We calculate the time t_r by which a light ray, starting at t_0, can travel a coordinate distance equal to the value we found in part (c):

$$\int_{t_0}^{t_r} \frac{c}{b\,t'^{2/3}}\,\mathrm{d}t' = \ell_c = \frac{3}{4}\frac{ct_0^{1/3}}{b} \ .$$

Integrating,

$$\frac{3c}{b}\left[t_r^{1/3} - t_0^{1/3}\right] = \frac{3}{4}\frac{ct_0^{1/3}}{b} \ ,$$

from which we find

$$t_r^{1/3} = \frac{5}{4}t_0^{1/3} \qquad \Longrightarrow \qquad \boxed{t_r = \frac{125}{64}\,t_0 \ .}$$

PROBLEM 9: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that $k = 0$, so

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

Since

$$\rho(t) \propto \frac{1}{a^4(t)} \ ,$$

it follows that

$$\frac{da}{dt} = \frac{\text{const}}{a} \ .$$

Rewriting this as

$$a\,da = \text{const}\,dt \ ,$$

the indefinite integral becomes

$$\frac{1}{2}a^2 = (\text{const})t + c' \ ,$$

where c' is a constant of integration. Different choices for c' correspond to different choices for the definition of $t = 0$. We will follow the standard convention of choosing $c' = 0$, which sets $t = 0$ to be the time when $a = 0$. Thus the above equation implies that $a^2 \propto t$, and therefore

$$\boxed{a(t) \propto t^{1/2}}$$

for a photon-dominated flat universe.

PROBLEM 10: THE TRAJECTORY OF A PHOTON ORIGINATING AT THE HORIZON *(25 points)*

(a) They key idea is that the coordinate speed of light is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} ,$$

so the coordinate distance (in notches) that light can travel between $t = 0$ and now ($t = t_0$) is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} .$$

The corresponding physical distance is the horizon distance:

$$\ell_{p,\text{horizon}}(t_0) = a(t_0) \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} .$$

Evaluating,

$$\ell_{p,\text{horizon}}(t_0) = bt_0^{2/3} \int_0^{t_0} \frac{c\,\mathrm{d}t}{bt^{2/3}} = t_0^{2/3}\left[3ct_0^{1/3}\right] = \boxed{3ct_0 .}$$

(b) As stated in part (a), the coordinate distance that light can travel between $t = 0$ and $t = t_0$ is given by

$$\ell_c = \int_0^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{3ct_0^{1/3}}{b} .$$

Thus, if we are at the origin, at $t = 0$ the photon must have been at

$$\boxed{x_0 = \frac{3ct_0^{1/3}}{b} .}$$

(c) The photon starts at $x = x_0$ at $t = 0$, and then travels in the negative x-direction at speed $c/a(t)$. Thus, it's position at time t is given by

$$x(t) = x_0 - \int_0^{t} \frac{c\,\mathrm{d}t'}{a(t')} = \frac{3ct_0^{1/3}}{b} - \frac{3ct^{1/3}}{b} = \boxed{\frac{3c}{b}\left(t_0^{1/3} - t^{1/3}\right) .}$$

(d) Since the coordinate distance between us and the photon is $x(t)$, measured in notches, the physical distance (in, for example, meters) is just $a(t)$ times $x(t)$. Thus.

$$\ell_p(t) = a(t)x(t) = \boxed{3ct^{2/3}\left(t_0^{1/3} - t^{1/3}\right)\ .}$$

(e) To find the maximum of $\ell_p(t)$, we set the derivative equal to zero:

$$\frac{\mathrm{d}\ell_p(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[3c\left(t^{2/3}t_0^{1/3} - t\right)\right] = 3c\left[\frac{2}{3}\left(\frac{t_0}{t}\right)^{1/3} - 1\right] = 0\ ,$$

so

$$\left(\frac{t_0}{t_{\text{max}}}\right)^{1/3} = \frac{3}{2} \quad\Longrightarrow\quad \boxed{t_{\text{max}} = \left(\frac{2}{3}\right)^3 t_0 = \frac{8}{27}\,t_0\ .}$$

The maximum distance is then

$$\ell_{p,\text{max}} = \ell_p(t_{\text{max}}) = 3c\left(\frac{2}{3}\right)^2 t_0^{2/3}\left[t_0^{1/3} - \left(\frac{2}{3}\right)t_0^{1/3}\right] = 3c\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)t_0$$

$$= \boxed{\frac{4}{9}ct_0\ .}$$

PROBLEM 11: SIGNAL PROPAGATION IN A FLAT MATTER-DOMINATED UNIVERSE *(55 points)*

(a)-(i) If we let $\ell_c(t)$ denote the coordinate distance of the light signal from A, then we can make use of Eq. (3.8) from the lecture notes for the coordinate velocity of light:

$$\frac{\mathrm{d}\ell_c}{\mathrm{d}t} = \frac{c}{a(t)}\ . \tag{11.1}$$

Integrating the velocity,

$$\begin{aligned}\ell_c(t) &= \int_{t_1}^{t}\frac{c\,\mathrm{d}t'}{a(t')} = \frac{c}{b}\int_{t_1}^{t}\frac{\mathrm{d}t'}{t'^{2/3}} \\ &= \frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right]\ .\end{aligned} \tag{11.2}$$

The physical distance is then

$$\begin{aligned}
\ell_{p,sA}(t) = a(t)\ell_c(t) &= bt^{2/3}\frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right] \\
&= 3c\left(t - t^{2/3}t_1^{1/3}\right) \\
&= 3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .
\end{aligned} \tag{11.3}$$

We now need to differentiate, which is done most easily with the middle line of the above equation:

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} = c\left[3 - 2\left(\frac{t_1}{t}\right)^{1/3}\right] .} \tag{11.4}$$

(ii) At $t = t_1$, the time of emission, the above formula gives

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} = c .} \tag{11.5}$$

This is what should be expected, since the speed of separation of the light signal at the time of emission is really just a local measurement of the speed of light, which should always give the standard value c.

(iii) At arbitrarily late times, the second term in brackets in Eq. (11.4) becomes negligible, so

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} \to 3c .} \tag{11.6}$$

Although this answer is larger than c, it does not violate relativity. Once the signal is far from its origin it is carried by the expansion of the universe, and relativity places no speed limit on the expansion of the universe.

(b) This part of the problem involves $H(t_1)$, so we can start by evaluating it:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(bt^{2/3})}{bt^{2/3}} = \frac{2}{3t} . \tag{11.7}$$

Thus, the physical distance from A to B at time t_1 is

$$\ell_{p,BA} = \frac{3}{2} c t_1 \; . \tag{11.8}$$

The coordinate distance is the physical distance divided by the scale factor, so

$$\ell_{c,BA} = \frac{cH^{-1}(t_1)}{a(t_1)} = \frac{\frac{3}{2}ct_1}{bt_1^{2/3}} = \frac{3c}{2b}\, t_1^{1/3} \; . \tag{11.9}$$

Since light travels at a coordinate speed $c/a(t)$, the light signal will reach galaxy B at time t_2 if

$$\begin{aligned} \ell_{c,BA} &= \int_{t_1}^{t_2} \frac{c}{bt'^{2/3}} \, \mathrm{d}t' \\ &= \frac{3c}{b} \left[t_2^{1/3} - t_1^{1/3} \right] \; . \end{aligned} \tag{11.10}$$

Setting the expressions (11.9) and (11.10) for $\ell_{c,BA}$ equal to each other, one finds

$$\frac{1}{2} t_1^{1/3} = t_2^{1/3} - t_1^{1/3} \quad \Longrightarrow \quad t_2^{1/3} = \frac{3}{2} t_1^{1/3} \quad \Longrightarrow \quad \boxed{t_2 = \frac{27}{8} t_1 \; .} \tag{11.11}$$

(c)-(i) Physical distances are additive, so if one adds the distance from A and the light signal to the distance from the light signal to B, one gets the distance from A to B:

$$\ell_{p,sA} + \ell_{p,sB} = \ell_{p,BA} \; . \tag{11.12}$$

But $\ell_{p,BA}(t)$ is just the scale factor times the coordinate separation, $a(t)\ell_{c,BA}$. Using the previous relations (11.3) and (11.9) for $\ell_{p,sA}(t)$ and $\ell_{c,BA}$, we find

$$3ct \left[1 - \left(\frac{t_1}{t} \right)^{1/3} \right] + \ell_{p,sB}(t) = \frac{3}{2} c t_1^{1/3} t^{2/3} \; , \tag{11.13}$$

so

$$\ell_{p,sB}(t) = \frac{9}{2} c t_1^{1/3} t^{2/3} - 3ct = 3ct \left[\frac{3}{2} \left(\frac{t_1}{t} \right)^{1/3} - 1 \right] \; . \tag{11.14}$$

As a check, one can verify that this expression vanishes for $t = t_2 = (27/8)\,t_1$, and that it equals $(3/2)ct_1$ at $t = t_1$. But we are asked to find the speed of approach, the negative of the derivative of Eq. (11.14):

$$\begin{aligned}\text{Speed of approach} &= -\frac{\mathrm{d}\ell_{p,sB}}{\mathrm{d}t} \\ &= -3ct_1^{1/3}t^{-1/3} + 3c \\ &= \boxed{3c\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right]} .\end{aligned} \tag{11.15}$$

(ii) At the time of emission, $t = t_1$, Eq. (11.15) gives

$$\boxed{\text{Speed of approach} = 0\ .} \tag{11.16}$$

This makes sense, since at $t = t_1$ galaxy B is one Hubble length from galaxy A, which means that its recession velocity is exactly c. The recession velocity of the light signal leaving A is also c, so the rate of change of the distance from the light signal to B is initially zero.

(iii) At the time of reception, $t = t_2 = (27/8)\,t_1$, Eq. (11.15) gives

$$\boxed{\text{Speed of approach} = c\ ,} \tag{11.17}$$

which is exactly what is expected. As in part (a)-(ii), this is a local measurement of the speed of light.

(d) To find the redshift, we first find the time t_{BA} at which a light pulse must be emitted from galaxy B so that it arrives at galaxy A at time t_1. Using the coordinate distance given by Eq. (11.9), the time of emission must satisfy

$$\frac{3c}{2b}\,t_1^{1/3} = \int_{t_{BA}}^{t_1} \frac{c}{bt'^{2/3}}\,\mathrm{d}t' = \frac{3c}{b}\left(t_1^{1/3} - t_{BA}^{1/3}\right)\ , \tag{11.18}$$

which can be solved to give

$$t_{BA} = \frac{1}{8}\,t_1\ . \tag{11.19}$$

The redshift is given by

$$1 + z_{BA} = \frac{a(t_1)}{a(t_{BA})} = \left(\frac{t_1}{t_{BA}}\right)^{2/3} = 4 \ . \tag{11.20}$$

Thus,

$$\boxed{z_{BA} = 3 \ .} \tag{11.21}$$

(e) Applying Euclidean geometry to the triangle C-A-B shows that the physical distance from C to B, at time t_1, is $\sqrt{2}cH^{-1}$. The coordinate distance is also larger than the A-B separation by a factor of $\sqrt{2}$. Thus,

$$\ell_{c,BC} = \frac{3\sqrt{2}c}{2b}\, t_1^{1/3} \ . \tag{11.22}$$

If we let t_{BC} be the time at which a light pulse must be emitted from galaxy B so that it arrives at galaxy C at time t_1, we find

$$\frac{3\sqrt{2}c}{2b}\, t_1^{1/3} = \int_{t_{BC}}^{t_1} \frac{c}{bt'^{2/3}}\, \mathrm{d}t' = \frac{3c}{b}\left(t_1^{1/3} - t_{BC}^{1/3}\right) \ , \tag{11.23}$$

which can be solved to find

$$t_{BC} = \left(1 - \frac{\sqrt{2}}{2}\right)^3 t_1 \ . \tag{11.24}$$

Then

$$1 + z_{BC} = \frac{a(t_1)}{a(t_{BC})} = \left(\frac{t_1}{t_{BC}}\right)^{2/3} = \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} \ , \tag{11.25}$$

and

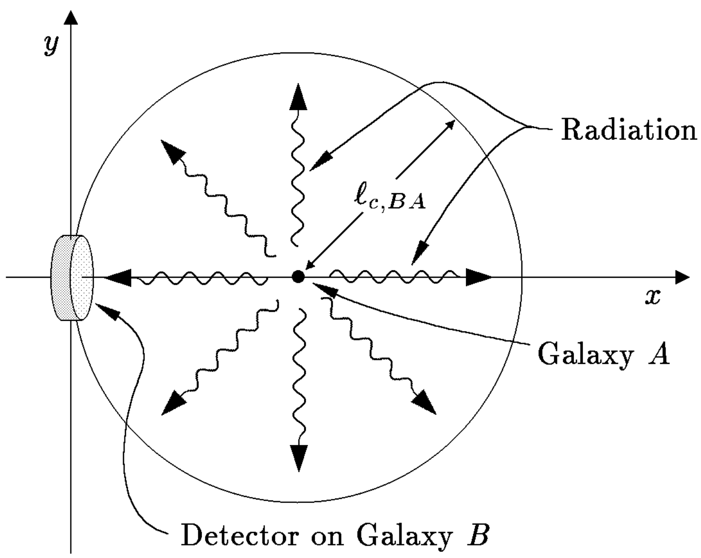
$$\boxed{z_{BC} = \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} - 1 \ .} \tag{11.26}$$

Full credit will be given for the answer in the form above, but it can be simplified by rationalizing the fraction:

$$\begin{aligned} z_{BC} &= \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} \frac{\left(1 + \frac{\sqrt{2}}{2}\right)^2}{\left(1 + \frac{\sqrt{2}}{2}\right)^2} - 1 \\ &= \frac{1 + \sqrt{2} + \frac{1}{2}}{\frac{1}{4}} - 1 \\ &= \boxed{5 + 4\sqrt{2} \ .} \end{aligned} \tag{11.27}$$

Numerically, $z_{BC} = 10.657$.

(f) Following the solution to Problem 6 of Problem Set 2, we draw a diagram in comoving coordinates, putting the source at the center of a sphere:



The energy from galaxy A will radiate uniformly over the sphere. If the detector has physical area A_D, then in the comoving coordinate picture it has coordinate area $A_D/a^2(t_2)$, since the detection occurs at time t_2 The full coordinate area of the sphere is $4\pi\ell_{c,BA}^2$, so the fraction of photons that hit the detector is

$$\text{fraction} = \frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2} \ . \tag{11.28}$$

As in Problem 6, the power hitting the detector is reduced by two factors of $(1+z)$: one factor because the energy of each photon is proportional to the frequency, and hence is reduced by the redshift, and one more factor because the rate of arrival of photons is also reduced by the redshift factor $(1+z)$. Thus,

$$\begin{aligned}\text{Power hitting detector} &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\frac{1}{(1+z)^2} \\ &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\left[\frac{a(t_1)}{a(t_2)}\right]^2 \\ &= P\frac{A}{4\pi\ell_{c,BA}^2}\frac{a^2(t_1)}{a^4(t_2)} \ .\end{aligned} \tag{11.29}$$

The energy flux is given by

$$J = \frac{\text{Power hitting detector}}{A} \ , \tag{11.30}$$

so

$$J = \frac{P}{4\pi \ell_{c,BA}^2} \frac{a^2(t_1)}{a^4(t_2)} \ . \tag{11.31}$$

From here it is just algebra, using Eqs. (11.9) and (11.11), and $a(t) = bt^{2/3}$:

$$\begin{aligned} J &= \frac{P}{4\pi \left[\frac{3c}{2b} t_1^{1/3}\right]^2} \frac{b^2 t_1^{4/3}}{b^4 t_2^{8/3}} \\ &= \frac{P}{4\pi \left[\frac{3c}{2b} t_1^{1/3}\right]^2} \frac{b^2 t_1^{4/3}}{\left(\frac{27}{8}\right)^{8/3} b^4 t_1^{8/3}} \\ &= \frac{P}{4\pi \left[\frac{3c}{2} t_1^{1/3}\right]^2} \frac{t_1^{4/3}}{\left(\frac{3}{2}\right)^8 t_1^{8/3}} \\ &= \boxed{\frac{2^8}{3^{10}\pi} \frac{P}{c^2 t_1^2}} \\ &= \boxed{\frac{256}{59,049\pi} \frac{P}{c^2 t_1^2} \ .} \end{aligned} \tag{11.32}$$

It is debatable which of the last two expressions is the simplest, so I have boxed both of them. One could also write

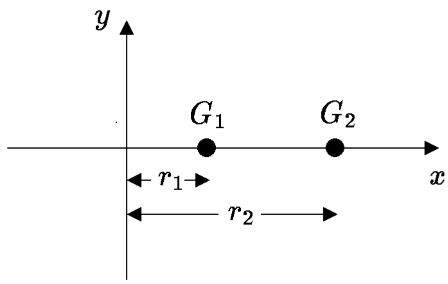
$$J = 1.380 \times 10^{-3} \frac{P}{c^2 t_1^2} \ . \tag{11.33}$$

PROBLEM 12: INTERGALACTIC SIGNALING AND TRAVEL *(30 points)*

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{2/3} \ ,$$

where b is a constant. Suppose that galaxy G_1 is located at comoving coordinates $(x, y, z) = (r_1, 0, 0)$ and galaxy G_2 is located at $(x, y, z) = (r_2, 0, 0)$, as shown in the following diagram:



(a) *(10 points)* At cosmic time $t = t_1$, a light signal is sent from G_1, in the direction of G_2. At what time t_2 does the light signal reach G_2? Your answer should be expressed in terms of some or all of the given variables t_1, b, r_1, r_2, and the speed of light c.

Answer: The coordinate distance that must be traveled is $r_2 - r_1$, and the coordinate speed of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} \ .$$

The integral of the coordinate speed over time is the coordinate distance traveled, so

$$\int_{t_1}^{t_2} \frac{c\,\mathrm{d}t'}{b\,t'^{2/3}} = r_2 - r_1 \ . \tag{12.1}$$

Carrying out the integration,

$$\int_{t_1}^{t_2} \frac{c\,\mathrm{d}t'}{b\,t'^{2/3}} = 3\frac{c}{b}\left[t_2^{1/3} - t_1^{1/3}\right] = r_2 - r_1 \ . \tag{12.2}$$

Solving for t_2, one finds

$$\boxed{t_2 = \left[t_1^{1/3} + \frac{b}{3c}(r_2 - r_1)\right]^3 \ .} \tag{12.3}$$

(b) *(6 points)* When the light signal arrives at G_2, what is its redshift z? Your answer can be expressed in terms of any or all of the variables mentioned in part (a), and also t_2, the answer to part (a).

Answer: This is an example of the cosmological redshift, given by

$$1 + z = \frac{a(t_{\text{observer}})}{a(t_{\text{source}})} = \frac{a(t_2)}{a(t_1)} \ , \tag{12.4}$$

so

$$\boxed{z = \frac{a(t_2)}{a(t_1)} - 1 = \left(\frac{t_2}{t_1}\right)^{2/3} - 1 \ .} \tag{12.5}$$

The equation above is an acceptable answer, but one could also insert Eq. (12.3) for t_2:

$$\boxed{z = \left[1 + \frac{b}{3ct_1^{1/3}}(r_2 - r_1)\right]^2 - 1 \ .} \tag{12.6}$$

(c) *(7 points)* Now suppose that at time t_1, the galaxy G_1 also launches a starship towards G_2. This is a science-fiction starship, which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity, and no time to decelerate at the end of its journey. At what time t_3 does the starship arrive at G_2? Your answer may depend on any of the allowed variables in part (b).

Answer: This part is analogous to part (a), except that the trajectory has a physical speed $\frac{1}{2}c$ instead of c, and hence a coordinate speed

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{2a(t)} \ . \tag{12.7}$$

The answer is then identical to Eq. (12.3), except that c is replaced by $\frac{1}{2}c$:

$$\boxed{t_3 = \left[t_1^{1/3} + \frac{2b}{3c}(r_2 - r_1)\right]^3 \ .} \tag{12.8}$$

(d) *(7 points)* We have not discussed how the clocks on such a starship would behave, but general relativity implies that, as the starship passes any co-moving observer, the comoving observer would measure the clocks on the starship to be time-dilated exactly as they would be in special relativity, for the same relative velocity. Given this fact, what time will the clocks on the starship read when the ship arrives at G_2? Your answer may include any of the allowed variables in part (c), and also t_3, the answer to part (c).

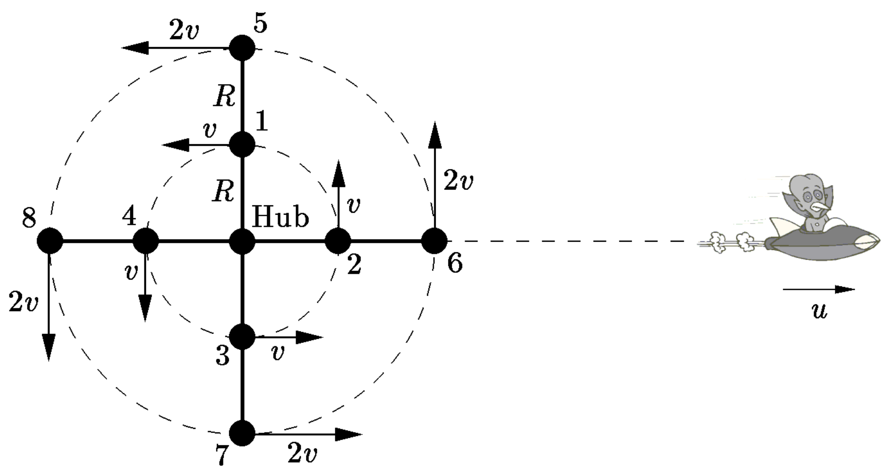
Answer: The time dilation factor for $v = \frac{1}{2}c$ is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2}{\sqrt{3}} \ . \tag{12.9}$$

When the starship clocks are passing the clocks of the comoving observers, they will appear to be running slowly by this factor. During the starship voyage the comoving clocks will advance from time t_1 to t_3, so the clocks on the starship will advance from t_1 to

$$\boxed{t_{\text{final,starship}} = t_1 + \frac{\sqrt{3}}{2}(t_3 - t_1) \ .} \tag{12.10}$$

PROBLEM 13: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND *(15 points)*



(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed v, and the outer cars are moving at speed $2v$. In the laboratory frame, the time interval between the wave crests emitted by the source $\Delta t_S^{\rm Lab}$ will be exactly equal to the time interval $\Delta t_O^{\rm Lab}$ between two crests reaching the observer:

$$\Delta t_O^{\rm Lab} = \Delta t_S^{\rm Lab} \ .$$

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slowly by the factor

$$\gamma_1 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

for the inner cars, and by the factor

$$\gamma_2 = \frac{1}{\sqrt{1 - 4v^2/c^2}}$$

for the outer cars. Thus, if we let Δt_S denote the time between crests as measured by a clock on the source, and Δt_O as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma_2 \Delta t_S = \Delta t_S^{\rm Lab} \quad \text{and} \quad \gamma_1 \Delta t_O = \Delta t_O^{\rm Lab} \ .$$

To make sure that the γ-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that

$$\Delta t_O = \frac{\gamma_2}{\gamma_1} \Delta t_S \ .$$

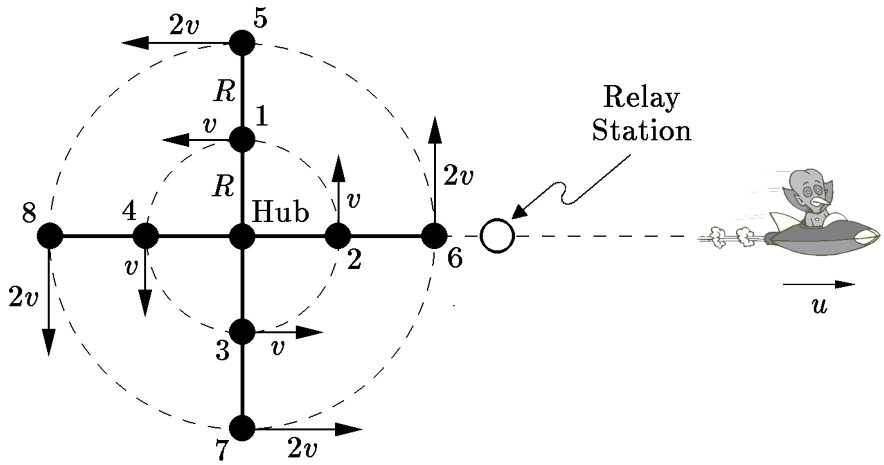
The redshift z is defined by

$$\Delta t_O \equiv (1+z)\, \Delta t_S \ ,$$

so

$$\boxed{z = \frac{\gamma_2}{\gamma_1} - 1 = \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 - \frac{4v^2}{c^2}}} - 1 \ .}$$

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.



The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and a change in path lengths between successive pulses, but we do not need to do it. It is the standard situation of a source and observer moving directly away from each other, as discussed at the end of Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed u, it becomes

$$(1+z)\big|_{\text{relay}} = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} \ .$$

If we again use the symbol Δt_S for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is

$$\Delta t_R = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\ \Delta t_S \ .$$

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in Δt, so it can be ignored. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station:

$$\Delta t_O^{\rm Lab} = \Delta t_R \ .$$

As in part (a), the time dilation implies that

$$\gamma_2 \Delta t_O = \Delta t_O^{\rm Lab} \ .$$

Combining the formulas above,

$$\Delta_O = \frac{1}{\gamma_2}\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\,\Delta t_S \ .$$

Again $\Delta t_O \equiv (1+z)\,\Delta t_S$, so

$$\boxed{z = \frac{1}{\gamma_2}\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} - 1 = \sqrt{\frac{\left(1-\frac{4v^2}{c^2}\right)\left(1+\frac{u}{c}\right)}{1-\frac{u}{c}}} - 1 \ .}$$

PROBLEM 14: AN EXPONENTIALLY EXPANDING FLAT UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{a}/a \ .$$

So

$$H = \frac{\chi a_0 e^{\chi t}}{a_0 e^{\chi t}} = \boxed{\ \chi\ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{a(t)} = \frac{c}{a_0}e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{a_0}\int_0^t e^{-\chi t'}dt' \\ &= \frac{c}{a_0}\left[-\frac{1}{\chi}e^{-\chi t'}\right]_0^t \\ &= \boxed{\frac{c}{\chi a_0}\left[1-e^{-\chi t}\right] \ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1 + z = \frac{a(t_r)}{a(t_e)} \; .$$

Here $t_e = 0$, so

$$\begin{aligned} 1 + z &= \frac{a_0 e^{\chi t_r}}{a_0} \\ &\Longrightarrow \quad e^{\chi t_r} = 1 + z \\ &\Longrightarrow \quad \boxed{t_r = \frac{1}{\chi} \ln(1 + z) \; .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1 + z \; ,$$

and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi a_0} \left[1 - e^{-\chi t_r} \right] \\ &= \frac{c}{\chi a_0} \left[1 - \frac{1}{1+z} \right] \\ &= \frac{cZ}{\chi a_0 (1 + z) \; .} \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = a(t_r) x(t_r) = \frac{cz e^{\chi t_r}}{\chi(1+z)} = \boxed{\frac{cz}{\chi} \; .}$$

PROBLEM 15: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE

(a) The physical horizon distance is given in general by

$$\ell_{p,\text{horizon}} = a(t) \int_0^{t_f} \frac{c}{a(t)} \, dt \; ,$$

so in this case

$$\ell_{p,\text{horizon}} = bt^{1/2} \int_0^{t_f} \frac{c}{bt^{1/2}}\, dt = \boxed{\;2ct_f\;.\;}$$

(b) If the source is at the horizon distance, it means that a photon leaving the source at $t = 0$ would just be reaching the origin at t_f. So, $\boxed{t_e = 0\;.}$

(c) The coordinate distance between the source and the origin is the coordinate horizon distance, given by

$$\ell_{c,\text{horizon}} = \int_0^{t_f} \frac{c}{bt^{1/2}}\, dt = \boxed{\;\frac{2ct_f^{1/2}}{b}\;.\;}$$

(d) The photon starts at coordinate distance $2c\sqrt{t_f}/b$, and by time t it will have traveled a coordinate distance

$$\int_0^t \frac{c}{bt'^{1/2}}\, dt' = \frac{2c\sqrt{t}}{b}$$

toward the origin. Thus the photon will be at coordinate distance

$$\ell_c = \frac{2c}{b}\left(\sqrt{t_f} - \sqrt{t}\right)$$

from the origin, and hence a physical distance

$$\ell_p(t) = a(t)\ell_c = \boxed{\;2c\left(\sqrt{t\,t_f} - t\right)\;.\;}$$

(e) To find the maximum of $\ell_p(t)$, we differentiate it and set the derivative to zero:

$$\frac{d\ell_p}{dt} = \left(\sqrt{\frac{t_f}{t}} - 2\right) c\;,$$

so the maximum occurs when

$$\sqrt{\frac{t_f}{t_{\max}}} = 2\;,$$

or

$$\boxed{\;t_{\max} = \frac{1}{4} t_f\;.\;}$$

PROBLEM 16: TRANSVERSE DOPPLER SHIFTS

(a) Describing the events in the coordinate system shown, the Xanthu is at rest, so its clocks run at the same speed as the coordinate system time variable, t. The emission of the wavecrests of the radio signal are therefore separated by a time interval equal to the time interval as measured by the source, the Xanthu:

$$\Delta t = \Delta t_s \ .$$

Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wavecrests will travel the same distance, as long as $c\Delta t \ll a$. Since the wavecrests travel the same distance, the time separation of their arrival at the Emmerac is Δt, the same as the time separation of their emission. The clocks on the Emmerac, however, and running slowly by a factor of

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ .$$

The time interval between wave crests as measured by the receiver, on the Emmerac, is therefore smaller by a factor of γ,

$$\Delta t_r = \frac{\Delta t_s}{\gamma} \ .$$

Thus, there is a $\boxed{\text{blueshift.}}$ The redshift parameter z is defined by

$$\frac{\Delta t_r}{\Delta t_s} = 1 + z \ ,$$

so

$$\frac{1}{\gamma} = 1 + z \ ,$$

or

$$\boxed{z = \frac{1 - \gamma}{\gamma} \ .}$$

Recall that $\gamma > 1$, so z is negative.

(b) Describing this situation in the coordinate system shown, this time the source on the Xanthu is moving, so the clocks at the source are running slowly. The time between wavecrests, measured in coordinate time t, is therefore larger by a factor of γ than Δt_s, the time as measured by the clock on the source:

$$\Delta t = \gamma\, \Delta t_s \ .$$

Since the radio signal is emitted when the Xanthu is at its minimum separation from the Emmerac, the rate of change of the separation is zero, so each wavecrest travels the same distance (again assuming that $c\Delta t \ll a$). Since the Emmerac is at rest, its clocks run at the same speed as the coordinate time t, and hence the time interval between crests, as measured by the receiver, is

$$\Delta t_r = \Delta t = \gamma\, \Delta t_s \ .$$

Thus the time interval as measured by the receiver is longer than that measured by the source, and hence it is a $\boxed{\text{redshift.}}$ The redshift parameter z is given by

$$1 + z = \frac{\Delta t_r}{\Delta t_s} = \gamma \ ,$$

so

$$\boxed{z = \gamma - 1 \ .}$$

(c) The events described in (a) can be made to look a lot like the events described in (b) by transforming to a frame of reference that is moving to the right at speed v_0 — i.e., by transforming to the rest frame of the Emmerac. In this frame the Emmerac is of course at rest, and the Xanthu is traveling on the trajectory

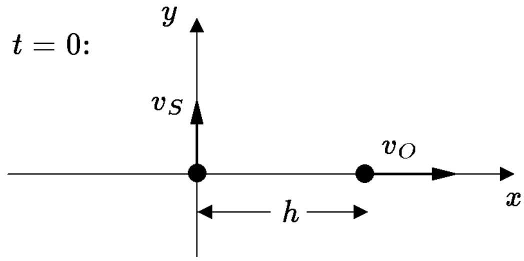
$$(x{=}-v_0 t, y{=}a, z{=}0) \ ,$$

as in part (b). However, just as the transformation causes the x-component of the velocity of the Xanthu to change from zero to a negative value, so the x-component of the velocity of the radio signal will be transformed from zero to a negative value. Thus in this frame the radio signal will not be traveling along the y-axis, so the events will not match those described in (b). The situations described in (a) and (b) are therefore physically distinct (which they must be if the redshifts are different, as we calculated above).

PROBLEM 17: THE NONRELATIVISTIC TRANSVERSE DOPPLER SHIFT *(10 points)*

At time $t = 0$, a source of sound waves is located at the origin of a coordinate system, and an observer is located at a distance h along the positive x axis, as shown in the diagram below. The source is moving at constant speed v_S along the positive y axis, and the observer is moving at constant speed v_O along the positive x axis. Denote the speed of sound by u.

At $t = 0$:



(a) *(6 points)* At $t = 0$, the sound source emits a wave crest. At what time t_1 does it arrive at the observer?

Answer: The wave crest follows the trajectory

$$x_c = ut \; .$$

The motion of the observer is described by

$$x_O = h + v_O t \; .$$

The intersection of these equations determines when the wave crest intersects the observer:

$$x_c = x_O \quad \Longrightarrow \quad ut_1 = h + v_O t_1 \quad \Longrightarrow$$

$$\boxed{t_1 = \frac{h}{u - v_O} \; .}$$

(b) *(4 points)* Under the usual assumption that the wavelength of the wave is very short compared to any other distance, what is the Doppler shift of the sound wave, as received by the observer?

Answer: Since this is a nonrelativistic problem, there is no time dilation. Thus, the Doppler shift is caused entirely by the rate of change of the path length $\ell(t)$, the length of the path of wave crests emitted at time t. Since the motion of the source is perpendicular to the direction of the observer, it does not contribute to $\mathrm{d}\ell/\mathrm{d}t$. Thus, the Doppler shift is given by the formula for the nonrelativistic Doppler shift when the observer is moving along a line, away from the source, which is given in the formula sheet:

$$\boxed{z = \frac{v_O/u}{1 - v_O/u} \; .}$$

This was derived in Lecture Notes 1, as Eq. (1.8).

PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT *(20 points)*

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler shift formula, which was given on the front of the exam:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \ , \tag{18.1}$$

where $\beta = v/c$. Remembering that the wavelength is stretched by a factor $1+z$, we find immediately that the wavelength of the radio wave received at Alpha-7 is given by

$$\lambda_{\text{Alpha}-7} = \sqrt{\frac{1+v_s/c}{1-v_s/c}}\, \lambda_{\text{emitted}} \ . \tag{18.2}$$

The photons that are received by the observer are in fact never received by Alpha-7, but the wavelength found by the observer will be the same as if Alpha-7 acted as a relay station, receiving the photons and retransmitting them at the received wavelength. So, applying Eq. (18.1) again, the wavelength seen by the observer can be written as

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\, \lambda_{\text{Alpha}-7} \ . \tag{18.3}$$

Combining Eqs. (18.2) and (18.3),

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}}\, \lambda_{\text{emitted}} \ , \tag{18.4}$$

so finally

$$\boxed{z = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \ .} \tag{18.5}$$

(b) Although we used the presence of Alpha-7 in determining the redshift z of Eq. (18.5), the redshift is not actually affected by the space station. So the special-relativity Doppler shift formula, Eq. (18.1), must directly describe the redshift resulting from the relative motion of the source and the observer. Thus

$$\sqrt{\frac{1+v_{\text{tot}}/c}{1-v_{\text{tot}}/c}} - 1 = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \ . \tag{18.6}$$

The equation above determines $v_{\rm tot}$ in terms of v_o and v_s, so the rest is just algebra. To simplify the notation, let $\beta_{\rm tot} \equiv v_{\rm tot}/c$, $\beta_o \equiv v_o/c$, and $\beta_s \equiv v_s/c$. Then

$$1 + \beta_{\rm tot} = \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s}\,(1 - \beta_{\rm tot})$$

$$\beta_{\rm tot}\left[1 + \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s}\right] = \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s} - 1$$

$$\beta_{\rm tot}\left[\frac{(1-\beta_o-\beta_s+\beta_o\beta_s)+(1+\beta_o+\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}\right] =$$

$$\frac{(1+\beta_o+\beta_s+\beta_o\beta_s)-(1-\beta_o-\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}$$

$$\beta_{\rm tot}[2(1+\beta_o\beta_s)] = 2(\beta_o+\beta_s)$$

$$\beta_{\rm tot} = \frac{\beta_o+\beta_s}{1+\beta_o\beta_s}$$

$$\boxed{v_{\rm tot} = \frac{v_o+v_s}{1+\dfrac{v_o v_s}{c^2}}\ .} \tag{18.7}$$

The final formula is the relativistic expression for the addition of velocities. Note that it guarantees that $|v_{\rm tot}| \le c$ as long as $|v_o| \le c$ and $|v_s| \le c$.

PROBLEM 19: A FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

a) *(5 points)* The cosmological redshift is given by the usual form,

$$1 + z = \frac{a(t_0)}{a(t_e)}\quad .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma}\quad .$$

So,

$$\boxed{z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1\quad .}$$

b) *(5 points)* The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0 \left(1 - \frac{t_e}{t_0}\right) \quad .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1+z)^{-1/\gamma} \quad .$$

Therefore,

$$\boxed{t_0 - t_e = t_0 \left[1 - (1+z)^{-1/\gamma}\right] \quad .}$$

c) *(10 points)* The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = a(t_0) \int_{t_e}^{t_0} \frac{c}{a(t)} dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{c t_0^\gamma}{1-\gamma} \left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Factoring $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0) = \frac{c t_0}{1-\gamma} \left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

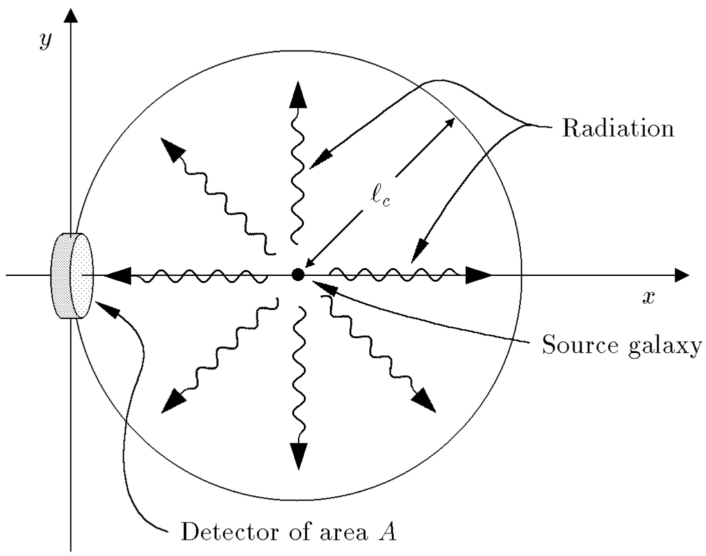
$$H_0 = \frac{\dot a(t_0)}{a(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = c H_0^{-1} \frac{\gamma}{1-\gamma} \left[1 - (1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) *(10 points)* A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/a(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/a(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/a(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \quad .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = a(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \quad .}$$

e) *(10 points)* Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \times \Delta t_{\text{source}} \ .$$

Note that I have used the formula from the front of the exam, but I have changed the size of v, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1+z)\Delta t_{\text{relay station}} \ .$$

Thus,

$$\begin{aligned} 1 + z_J \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} &= \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{relay station}}} \frac{\Delta t_{\text{relay station}}}{\Delta t_{\text{source}}} \\ &= (1+z)\big|_{\text{cosmological}} \times (1+z)\big|_{\text{special relativity}} \\ &= (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \ . \end{aligned}$$

Thus,

$$\boxed{z_J = (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} - 1 \ .}$$

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler shift formula to convert the redshift z of the galaxy to a velocity of recession, then subtracted from this the speed v of the jet, and then again used the special relativity Doppler shift formula to find the Doppler shift corresponding to this composite velocity. However, as discussed at the end of Lecture Notes 3, the cosmological Doppler shift is given by

$$1 + z \equiv \frac{\Delta t_o}{\Delta t_e} = \frac{a(t_o)}{a(t_e)} \; , \tag{3.11}$$

and is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating z to v does not apply.

PROBLEM 20: ANOTHER FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The key to this problem is to work in comoving coordinates.

[Some students have asked me why one cannot use "physical" coordinates, for which the coordinates really measure the physical distances. In principle one can use any coordinate system on likes, but the comoving coordinates are the simplest. In any other system it is difficult to write down the trajectory of either a particle or a light-beam. In comoving coordinates it is easy to write the trajectory of either a light beam, or a particle which is moving with the expansion of the universe (and hence standing still in the comoving coordinates). Note, by the way, that when one says that a particle is standing still in comoving coordinates, one has not really said very much about it's trajectory. One has said that it is moving with the matter which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor, $a(t)$.]

(a) The physical separation at t_o is given by the scale factor times the coordinate distance. The coordinate distance is found by integrating the coordinate velocity, so

$$\ell_p(t_o) = a(t_o) \int_{t_e}^{t_o} \frac{c\,dt'}{a(t')} = b t_o^{1/3} \int_{t_e}^{t_o} \frac{c\,dt'}{b t'^{1/3}} = \frac{3}{2} c t_o^{1/3} \left[t_o^{2/3} - t_e^{2/3} \right]$$

$$\boxed{= \frac{3}{2} c t_o \left[1 - (t_e/t_o)^{2/3} \right] \; .}$$

(b) From the front of the exam,

$$1 + z = \frac{a(t_o)}{a(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/3}$$

$$\Longrightarrow \qquad \boxed{z = \left(\frac{t_o}{t_e}\right)^{1/3} - 1\ .}$$

(c) By combining the answers to (a) and (b), one has

$$\boxed{\ell_p(t_o) = \frac{3}{2} c t_o \left[1 - \frac{1}{(1+z)^2}\right]\ .}$$

(d) The physical distance of the light pulse at time t is equal to $a(t)$ times the coordinate distance. The coordinate distance at time t is equal to the starting coordinate distance, $\ell_c(t_e)$, minus the coordinate distance that the light pulse travels between time t_e and time t. Thus,

$$\begin{aligned}
\ell_p(t) &= a(t)\left[\ell_c(t_e) - \int_{t_e}^{t} \frac{c\,dt'}{a(t')}\right] \\
&= a(t)\left[\int_{t_e}^{t_o} \frac{c\,dt'}{a(t')} - \int_{t_e}^{t} \frac{c\,dt'}{a(t')}\right] \\
&= a(t)\int_{t}^{t_o} \frac{c\,dt'}{a(t')} \\
&= bt^{1/3}\int_{t}^{t_o} \frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2} c t^{1/3}\left[t_o^{2/3} - t^{2/3}\right] \\
&= \boxed{\frac{3}{2} c t \left[\left(\frac{t_o}{t}\right)^{2/3} - 1\right]\ .}
\end{aligned}$$

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 5, 2022
Prof. Alan Guth

QUIZ 1

Problem	**Maximum**	**Score**
1	20	
2	50	
3	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(5 points)* Why are there dark regions when we look toward the center of our galaxy using visible wavelength, or at another galaxy edge on?

(i) There are few stars in those areas.

(ii) The stars in those areas have a lower brightness than the stars in the halo.

(iii) Clouds of dust absorb the light of the stars located behind.

(iv) The gravitational field of the galaxy binds the path of photons away from the edge.

(b) *(5 points)* Using a radio antenna, Penzias and Wilson took about a year to determine that the signal they were receiving was truly of astronomical origin. What made it hard to establish that the signal they received was not just noise?

(i) It was hard to distinguish it from the electronic noise of the instruments.

(ii) It was hard to distinguish it from a microwave signal coming from stars or distant galaxies.

(iii) It was hard to distinguish it from radio communications on Earth.

(c) *(5 points)* The universe contains various types of particles; leptons, such as electrons and neutrinos, and baryons, such as neutrons and protons. Why do we refer to the matter of the universe solely as baryonic?

(i) There are much more baryons than leptons in the universe.

(ii) Baryons are heavier than leptons, so they dominate the mass density.

(iii) Leptons are a subcategory of baryons.

(iv) Leptons bind together to form baryons.

— Problem 1 continues on next page. —

(d) *(5 points)* State whether each of the following statements is true (T) or false (F):

(i) Before the 1840s it was a mystery why stars had visibly different colors, but in 1842 Johann Christian Doppler correctly explained this phenomenon as the consequence of the different velocities of stars. **T** or **F** .

(ii) Frauenhofer let light from the Sun pass through a slit and then through a glass prism. The resulting spectrum showed bright lines that could be identified with known elements. **T** or **F** .

(iii) In black-body radiation, the number of photons per unit volume is directly proportional to the cube of the temperature. **T** or **F** .

(iv) The model of the universe proposed by de Sitter in 1917 predicts a redshift proportional to the distance. **T** or **F** .

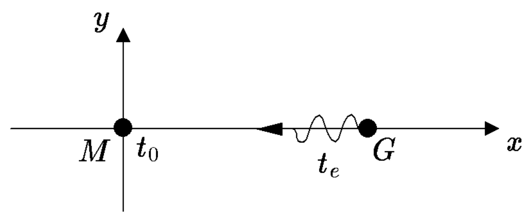
(v) The horizon distance bears its name because it is the edge of the universe; there are no stars or galaxies beyond it. **T** or **F** .

PROBLEM 2: INTERGALACTIC SIGNALING AND MARKSMANSHIP *(50 points)*

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{3/4} ,$$

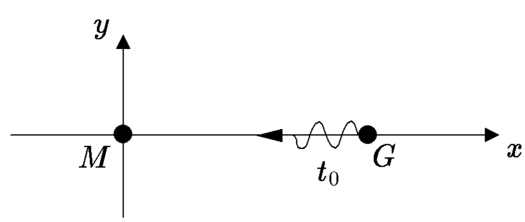
where b is a constant. Suppose that we live at cosmic time t_0 on galaxy M (maybe for Milky Way), located at the origin of a comoving coordinate system $(x, y, z) = (0, 0, 0)$. Galaxy G is located somewhere along the positive x axis, as shown in the diagram below:



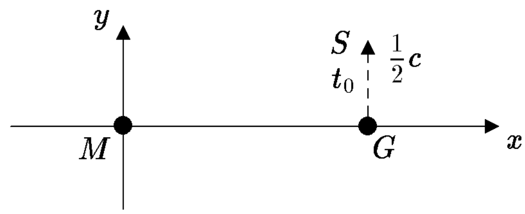
In all of the following questions, you can express your answer in terms of any or all of the given variables (i.e., any variable that has previously been assigned a symbol) and also any of the symbols that represent answers to the previous parts, whether or not you answered those parts.

(a) *(5 points)* What is the Hubble expansion rate H_0 that we would measure?

(b) *(7 points)* Suppose that we know that the light that we are receiving now (at time t_0) from G was emitted at time t_e. Given t_e and t_0, and the speed of light c, what is the physical distance $\ell_p(t_0)$ today between us and the galaxy G? What was the physical distance $\ell_p(t_e)$ at the time of emission t_e?



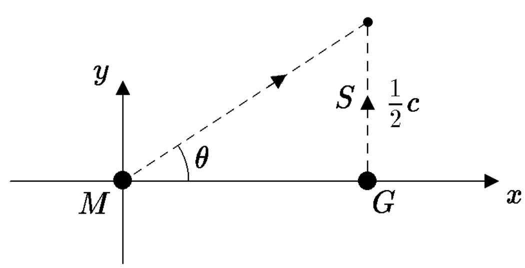
(c) *(8 points)* At the present cosmic time t_0, a light signal is emitted from G toward us, as shown in the diagram above. When we receive this light signal at some point in the future, what will be its redshift z_G?



(d) *(7 points)* Also at the present cosmic time t_0, a spaceship S is launched from the galaxy G in the y direction, as indicated in the diagram above. This is a science-fiction spaceship (just like one that you saw on the Quiz 1 Review Problems) which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity. Suppose, immediately after its launch, the spaceship emits a light signal directed toward us. When we eventually receive this signal (at M), what will be its redshift z_S? [*Hint:* it may help to imagine an observer at rest on G, who observes this light signal as it leaves the spaceship. You may assume that the redshift observed by this observer will be governed by special relativity. You can think of this observer as a relay station, since the signal will be unchanged if the observer absorbs it and retransmits it.]

(e) *(7 points)* Find the comoving coordinates $x_c(t)$ and $y_c(t)$ for the spaceship S at an arbitrary time $t > t_0$.

(f) *(7 points)* Find the physical distance $\ell_{p,S}(t)$ from S to M at an arbitrary time $t > t_0$.

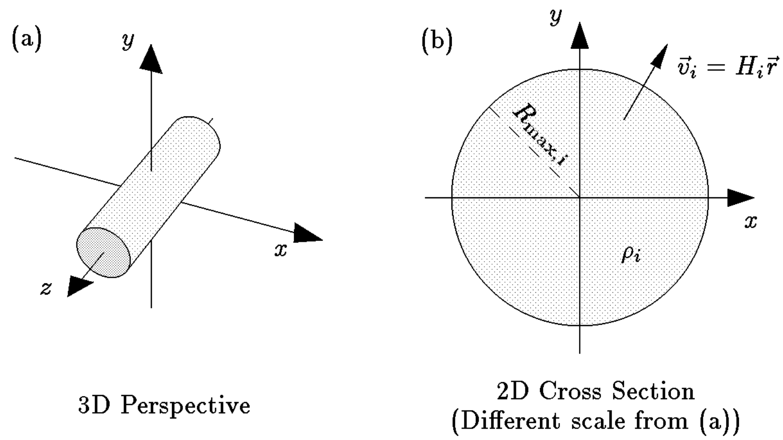


(g) *(9 points)* Now suppose that we, at the present time t_0, shoot a directed pulse of light at an angle θ with respect to the x axis, as shown in the diagram above. Our goal is to choose θ so that the light pulse intercepts the spaceship S. Write an equation that would determine the right value of θ to accomplish this interception. Your equation should involve only θ, given quantities, and answers to previous parts. You do *NOT* need to solve this equation (although you may find it easy to solve).

PROBLEM 3: A CYLINDRICAL UNIVERSE *(30 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points. This year it was Problem 1 of Problem Set 3.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \; .$$

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \; ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

— *Problem 2 continues on next page.* —

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

(c) *(7 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

(e) *(8 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a) .$$

What is $V(a)$? Will this universe expand forever, or will it collapse?

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 21, 2022
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 5, 2022

Problem	**Maximum**	**Score**
1	20	
2	50	
3	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(5 points)* Why are there dark regions when we look toward the center of our galaxy using visible wavelength, or at another galaxy edge on?

(i) There are few stars in those areas.

(ii) The stars in those areas have a lower brightness than the stars in the halo.

(iii) $\boxed{\text{Clouds of dust absorb the light of the stars located behind.}}$

(iv) The gravitational field of the galaxy binds the path of photons away from the edge.

[*Comment:* Absorption by dust clouds is discussed by Weinberg in the captions to photographs in Chapter 2.]

(b) *(5 points)* Using a radio antenna, Penzias and Wilson took about a year to determine that the signal they were receiving was truly of astronomical origin. What made it hard to establish that the signal they received was not just noise?

(i) $\boxed{\text{It was hard to distinguish it from the electronic noise of the instruments.}}$

(ii) It was hard to distinguish it from a microwave signal coming from stars or distant galaxies.

(iii) It was hard to distinguish it from radio communications on Earth.

[*Comment:* The problems of instrument noise are discussed by Weinberg at the start of Chapter 3.]

(c) *(5 points)* The universe contains various types of particles; leptons, such as electrons and neutrinos, and baryons, such as neutrons and protons. Why do we refer to the matter of the universe solely as baryonic?

(i) There are much more baryons than leptons in the universe.

(ii) $\boxed{\text{Baryons are heavier than leptons, so they dominate the mass density.}}$

(iii) Leptons are a subcategory of baryons.

(iv) Leptons bind together to form baryons.

[*Comment:* Ryden discusses the particle content of the universe in Chapter 2.]

(d) *(5 points)* State whether each of the following statements is true (T) or false (F):

(i) Before the 1840s it was a mystery why stars had visibly different colors, but in 1842 Johann Christian Doppler correctly explained this phenomenon as the consequence of the different velocities of stars. **T** or $\boxed{\textbf{F}}$.

(ii) Frauenhofer let light from the Sun pass through a slit and then through a glass prism. The resulting spectrum showed bright lines that could be identified with known elements. **T** or $\boxed{\textbf{F}}$.

(iii) In black-body radiation, the number of photons per unit volume is directly proportional to the cube of the temperature. $\boxed{\textbf{T}}$ or **F** .

(iv) The model of the universe proposed by de Sitter in 1917 predicts a redshift proportional to the distance. $\boxed{\textbf{T}}$ or **F** .

(v) The horizon distance bears its name because it is the edge of the universe; there are no stars or galaxies beyond it. **T** or $\boxed{\textbf{F}}$.

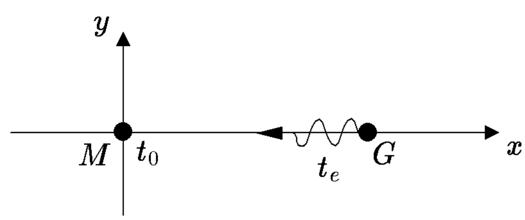
Comments: (i) In Chapter 2, Weinberg explains that the Doppler shift has essentially nothing to do with the color of stars. While the redshift may cause blue light to shift toward the red, at the same time the normally invisible ultraviolet light is shifted into the blue. The visible color differences are really caused by the surface temperature of stars. (ii) In Chapter 2, Weinberg also explains that the spectral lines in the Sun are dark, caused by the selective absorption of radiation by the cooler outer atmosphere of the Sun. (iii) Weinberg discusses this in Chapter 3, pointing out that it is equivalent to saying that the typical separation between photons in black-body radiation is roughly equal to the typical photon wavelength. (iv) Weinberg discusses the de Sitter model in Chapter 2. (v) Ryden explains in Chapter 2 that we cannot see beyond the horizon because light from such sources has not yet had time to reach us.

PROBLEM 2: INTERGALACTIC SIGNALING AND MARKSMANSHIP *(50 points)*

Consider a flat, matter-dominated universe, with a scale factor

$$a(t) = bt^{3/4} ,$$

where b is a constant. Suppose that we live at cosmic time t_0 on galaxy M (maybe for Milky Way), located at the origin of a comoving coordinate system $(x, y, z) = (0, 0, 0)$. Galaxy G is located somewhere along the positive x axis, as shown in the diagram below:



In all of the following questions, you can express your answer in terms of any or all of the given variables (i.e., any variable that has previously been assigned a symbol) and also any of the symbols that represent answers to the previous parts, whether or not you answered those parts.

(a) *(5 points)* What is the Hubble expansion rate H_0 that we would measure?

Answer: In general

$$H(t) = \frac{1}{a}\frac{da}{dt} ,$$

so in this case

$$\boxed{H_0 = H(t_0) = \frac{1}{bt_0^{3/4}}\left(\frac{3}{4}\right) bt_0^{-1/4} = \frac{3}{4t_0} .}$$

(b) *(7 points)* Suppose that we know that the light that we are receiving now (at time t_0) from G was emitted at time t_e. Given t_e and t_0, and the speed of light c, what is the physical distance $\ell_p(t_0)$ today between us and the galaxy G? What was the physical distance $\ell_p(t_e)$ at the time of emission t_e?

Answer: The coordinate speed of light is $c/a(t)$, so the coordinate distance that light travels between t_e and t_0 is given by

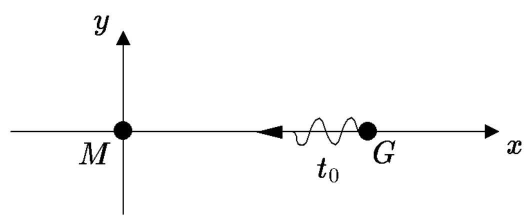
$$\ell_c = \int_{t_e}^{t_0} \frac{c}{bt^{3/4}}\, dt = \frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right) .$$

The physical distance $\ell_p(t_0)$ is obtained by multiplying by the scale factor $a(t_0)$, so

$$\boxed{\ell_p(t_0) = 4ct_0^{3/4}\left(t_0^{1/4} - t_e^{1/4}\right) = 4ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]} \; .$$

The physical distance at the time of emission $\ell_p(t_e)$ is found by multiplying by the scale factor at the time of emission, so

$$\boxed{\ell_p(t_e) = 4ct_e^{3/4}\left(t_0^{1/4} - t_e^{1/4}\right) = 4ct_e\left[\left(\frac{t_0}{t_e}\right)^{1/4} - 1\right] \; .}$$



(c) *(8 points)* At the present cosmic time t_0, a light signal is emitted from G toward us, as shown in the diagram above. When we receive this light signal at some point in the future, what will be its redshift z_G?

Answer: To find the redshift, we first find the time t_r at which we will receive the light signal. The coordinate distance that light travels between t_0 and t_r must be the same coordinate distance ℓ_c calculated in part (b), so

$$\frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right) = \int_{t_0}^{t_r} \frac{c}{a(t)}\, dt = \frac{4c}{b}\left(t_r^{1/4} - t_0^{1/4}\right) \; ,$$

so

$$t_r^{1/4} = 2t_0^{1/4} - t_e^{1/4} \; .$$

The redshift with which we will receive the light is

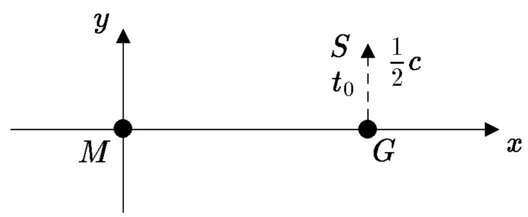
$$1 + z_G = \frac{a(t_r)}{a(t_0)} = \left(\frac{t_r}{t_0}\right)^{3/4} \; .$$

Using the formula above for $t_r^{1/4}$, this becomes

$$1 + z_G = \left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 \; ,$$

so

$$\boxed{z_G = \left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1\ .}$$



(d) *(7 points)* Also at the present cosmic time t_0, a spaceship S is launched from the galaxy G in the y direction, as indicated in the diagram above. This is a science-fiction spaceship (just like one that you saw on the Quiz 1 Review Problems) which moves at speed $\frac{1}{2}c$ relative to the comoving observers. That is, it moves at speed $\frac{1}{2}c$ relative to the observers who are at rest in the comoving coordinate system. For simplicity, we will assume that the starship takes no time to reach its cruising velocity. Suppose, immediately after its launch, the spaceship emits a light signal directed toward us. When we eventually receive this signal (at M), what will be its redshift z_S? [*Hint:* it may help to imagine an observer at rest on G, who observes this light signal as it leaves the spaceship. You may assume that the redshift observed by this observer will be governed by special relativity. You can think of this observer as a relay station, since the signal will be unchanged if the observer absorbs it and retransmits it.]

Answer: In the frame of the observer on G, the light signal emitted from the spaceship S is directed perpendicular to the velocity of S, so it is a transverse Doppler shift — the Doppler shift is due entirely to time dilation, and is given by

$$1 + z_{\text{relay}} = \frac{\lambda_{\text{relay}}}{\lambda_S} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2}{\sqrt{3}}\ ,$$

for $v/c = 1/2$. This redshift factor will simply multiply the redshift factor for transmission from G to M, so

$$1 + z_S = \frac{\lambda_M}{\lambda_S} = \frac{\lambda_M}{\lambda_{\text{relay}}}\frac{\lambda_{\text{relay}}}{\lambda_S} = (1 + z_G)\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3\ .$$

Thus,

$$\boxed{z_S = \frac{2}{\sqrt{3}}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1\ .}$$

Grading Error: The quizzes were initially graded under the incorrect assumption that the spaceship S was moving along the positive x axis, so the relevant redshift formula would be the one given on the Formula Sheet for motion along a line:

$$\frac{\lambda_{\rm relay}}{\lambda_S} = 1 + z = \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{3}\ ,$$

for $\beta \equiv v/c = 1/2$. In that case the final answer would be

$$z_S = \sqrt{3}\left[2 - \left(\frac{t_e}{t_0}\right)^{1/4}\right]^3 - 1\ .$$

Students who gave this answer were given full credit — those grades will not be revised. Students who gave the correct answer above were marked as wrong, but we expect to have these grading errors fixed at least within the next week. There is no need for you to do anything, since we have the scans and will make the corrections. Apologies for this confusion.

(e) *(7 points)* Find the comoving coordinates $x_c(t)$ and $y_c(t)$ for the spaceship S at an arbitrary time $t > t_0$.

Answer: Since S moves only in the y direction in the comoving coordinate system, $x_c(t)$ does not change. It is always given by

$$\boxed{x_c(t) = \ell_c = \frac{4c}{b}\left(t_0^{1/4} - t_e^{1/4}\right)\ .}$$

The spaceship S moves in the y direction at a physical speed $\frac{1}{2}c$ relative to the comoving coordinate system, so its speed in comoving coordinates is

$$\frac{dy}{dt} = \frac{c}{2a(t)}\ .$$

It starts at $y = 0$ at $t = t_0$, so

$$y_c(t) = \int_{t_0}^{t} \frac{c}{2bt'^{3/4}}\, dt' = \boxed{\frac{2c}{b}\left(t^{1/4} - t_0^{1/4}\right)\ .}$$

(f) *(7 points)* Find the physical distance $\ell_{p,S}(t)$ from S to M at an arbitrary time $t > t_0$.

Answer: Euclidean geometry is valid in a flat universe, so the coordinate distance from S to M at time t is given by the Pythagorean formula

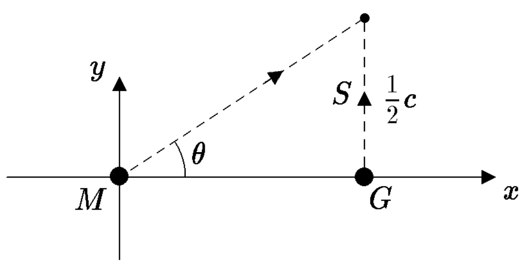
$$\ell_{c,S}(t) = \sqrt{x_c^2(t) + y_c^2(t)}\ ,$$

and then the physical distance is

$$\boxed{\ell_{p,S}(t) = bt^{3/4}\sqrt{x_c^2(t) + y_c^2(t)}\ .}$$

Since we were told that it is okay to answer in terms of symbols that represent the answers to previous parts, the answer above is completely acceptable. If one chooses to substitute for $x_c(t)$ and $y_c(t)$ from part (e), one should find

$$\ell_{p,S}(t) = 2ct^{3/4}t_0^{1/4}\left[5 - 8\left(\frac{t_e}{t_0}\right)^{1/4} + 4\left(\frac{t_e}{t_0}\right)^{1/2} - 2\left(\frac{t}{t_0}\right)^{1/4} + \left(\frac{t}{t_0}\right)^{1/2}\right]^{1/2}\ .$$



(g) *(9 points)* Now suppose that we, at the present time t_0, shoot a directed pulse of light at an angle θ with respect to the x axis, as shown in the diagram above. Our goal is to choose θ so that the light pulse intercepts the spaceship S. Write an equation that would determine the right value of θ to accomplish this interception. Your equation should involve only θ, given quantities, and answers to previous parts. You do *NOT* need to solve this equation (although you may find it easy to solve).

Answer: Probably the easiest way to solve this part is to recognize that both the light pulse and the spaceship start at $t = t_0$ to move at a constant velocity, starting from $y = 0$. When they collide they must also have the same y coordinate, so their y coordinate velocities must be equal. For the spaceship S, we noted in part (e) that

$$\frac{dy_S}{dt} = \frac{c}{2a(t)}\ .$$

For the light pulse, the coordinate speed is $c/a(t)$, and the component in the y direction is then

$$\frac{dy_\gamma}{dt} = \frac{c}{a(t)} \sin\theta \ .$$

Setting these expressions equal to each other, we have

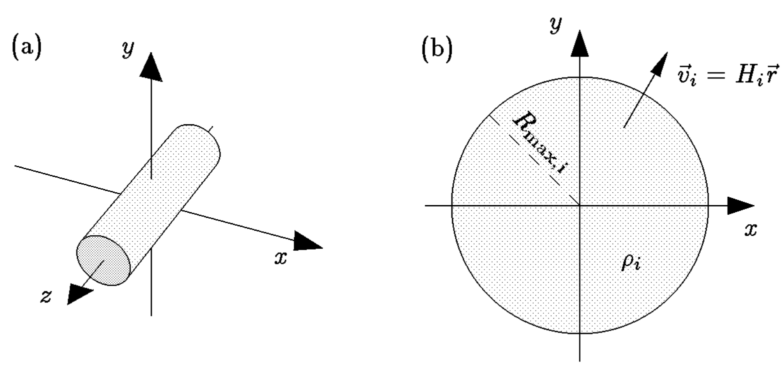
$$\boxed{\sin\theta = \frac{1}{2} \ .}$$

You were not asked to give the value of θ, but it is $\pi/6$, or $30°$.

PROBLEM 3: A CYLINDRICAL UNIVERSE *(30 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points. This year it was Problem 1 of Problem Set 3.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



3D Perspective

2D Cross Section
(Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \; .$$

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

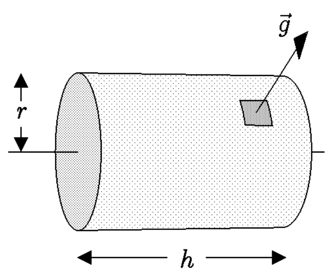
$$\vec{g} = -\frac{A\mu}{r}\hat{r} \; ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

Answer: Gauss's law of gravity states that

$$\oint \vec{g} \cdot d\vec{s} = -4\pi G M \; ,$$

where $\vec{g}$ is the acceleration of gravity, G is Newton's constant, and M is the total mass enclosed inside the volume. Apply this relation to the following slice of the infinite cylinder:



By symmetry $\vec{g}$ points radially outward, so the dot product $\vec{g} \cdot d\vec{s}$ vanishes for the disks that bound the cylindrical slice on the left and right. The only contribution comes from the curved surface of the cylinder, for which the cosine of the dot product is 1. Thus,

$$\oint \vec{g} \cdot d\vec{s} = 2\pi r h g_r \; ,$$

where g_r is the radial component of $\vec{g}$. The mass enclosed, M, is the length times the mass per length, or $h\mu$. (Recall that μ denotes the total mass within the radius r, so it is really a function of r and the mass density ρ.) Thus, Gauss's law gives

$$2\pi r h g_r = -4\pi G h \mu \; ,$$

so

$$g_r = -\frac{2G\mu}{r} \ .$$

Since the other components of $\vec{g}$ vanish by symmetry,

$$\boxed{\vec{g} = -\frac{2G\mu}{r}\,\hat{r}\ ,}$$

and we can read off the constant A,

$$\boxed{A = 2G}\ .$$

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

Answer: $\ddot{r}$ is the acceleration of r. The only force in the problem is gravity, so

$$\ddot{r} = g_r = -\frac{2G\mu}{r} \ ,$$

where I used the gravitational acceleration that was calculated in the previous part of the problem.

The evaluation of μ involves logic that is essentially identical to the spherical case discussed in Lecture Notes 3. If we label each particle by its initial radius r_i, then all the particles for a given r_i will form a cylindrical shell. As the motion proceeds it is conceivable that shells might cross each other, and then the evaluation of μ would become very complicated. However, the Hubble expansion incorporated into the initial conditions implies that initially any two shells are moving apart from each other. Since there are no infinite accelerations, we can count on the fact that there will be at least some nonzero time interval before any crossings can take place. Until such crossings take place, it is easy to find μ: the total mass contained within any shell of particles is independent of time, since the particles that lie at a smaller radius than any given shell at time t are exactly the same particles as those that were at smaller radius at time t_i. Then, even if shells start to cross at some time, the equations that we derive under the no-shell-crossing assumption will be valid until the time that shells cross. Thus, shells can cross only if our equations show the possibility that two shells starting at different values of r_i can come together. However, we will find below that the no-shell-crossing assumption leads to uniform expansion described by an overall scale factor $a(t)$. Since two shells never come together under this uniform expansion, we can conclude that no shell crossings will ever take place.

With no shell crossings, we can evaluate μ for a given shell r_i at the initial time. For definiteness we can consider a length h of the cylinder, so the volume of the cylinder of radius r_i is $\pi r_i^2 h$. The mass per length is then

$$\mu(r_i) = \frac{\pi r_i^2 h \rho_i}{h} = \pi r_i^2 \rho_i \ .$$

Thus,

$$\boxed{\ddot{r} = -\frac{2\pi G r_i^2 \rho_i}{r} \ .}$$

(c) *(7 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \ ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

Answer: The function $u(r_i, t)$ is determined by the differential equation that it obeys, combined with the initial conditions. Using the answer from part (b), the differential equation for u is

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{2\pi G r_i \rho_i}{r} \ ,$$

so

$$\ddot{u} = -\frac{2\pi G \rho_i}{u} \ ,$$

which does not depend on r_i. We still need to check that the initial conditions for u and $\dot{u}$ are independent of r_i. (There are two initial conditions, because the differential equation for u is second order.) Since $r(r_i, t_i) \equiv r_i$, the initial value of u is given by

$$u(r_i, t_i) = 1 \ .$$

Finally, since the initial velocities are set to agree with Hubble's law,

$$\dot{r}(r_i, t_i) = H_i r_i \ ,$$

it follows that

$$\dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \ .$$

Thus, neither the differential equation for $u(r_i, t)$ nor the initial conditions depend on r_i, so the solution will not depend on r_i. Thus, we can define $a(t) \equiv u(r_i, t)$.

Many students showed that the differential equation is independent of r_i, but did not recognize that the initial conditions are important. But they are crucial! If I throw a ball straight up into the air, and neglect air friction, then we know that it obeys the second order equation

$$\ddot{y} = -g \ ,$$

where g is the acceleration of gravity. Does this mean that the trajectory of the ball does not depend on the initial velocity, since the initial velocity does not appear in the differential equation? Of course not! If we specify $y(0)$ and $\dot{y}(0)$, then the solution will be uniquely determined.

(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

Answer: For clarity, we can consider a finite length h of the cylinder. The mass contained inside a cylinder of radius r_i at the initial time t_i is then

$$M(r_i, h) = \pi r_i^2 h \rho_i \ .$$

At time t, this same mass will be uniformly spread in a cylinder of radius $r(r_i, t)$ and length h. The density is therefore

$$\rho(t) = \frac{M(r_i, h)}{\pi r^2 h} = \boxed{\frac{\rho_i}{a^2(t)} \ .}$$

Using this result to replace ρ_i in the differential equation found in (c), we find

$$\boxed{\ddot{a} = -2\pi G \rho a \ .}$$

(e) *(8 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a) \ .$$

What is $V(a)$? Will this universe expand forever, or will it collapse?

Answer: Multiplying the differential equation by $\dot{a}$,

$$\dot{a}\left[\ddot{a} + \frac{2\pi G\rho_i}{a}\right] = 0 \ .$$

Note that I wrote the differential equation in terms of ρ_i rather than ρ, so that $a(t)$ is the only time-dependent quantity in the equation. This expression can be rewritten as

$$\frac{d}{dt}\left[\frac{1}{2}\dot{a}^2 + 2\pi G\rho_i \ln a\right] \equiv \frac{d}{dt}E = 0 \ .$$

In other words, the quantity in square brackets, which we have labeled E, must be constant:

$$E = \frac{1}{2}\dot{a}^2 + 2\pi G\rho_i \ln a \ = \text{const.}$$

We can now identify the rescaled potential energy as

$$\boxed{V(a) = 2\pi G\rho_i \ln a \ .}$$

The potential energy term $V(a)$ grows as $\ln a$ and is hence unbounded. No matter how large the initial value of $\dot{a}^2$, there can never be enough energy to allow the universe to grow to arbitrarily large a. Eventually the $V(a)$ term will grow to be as large as E, at which point $\dot{a}$ will vanish. At that point conservation of energy alone is not sufficient to control what happens next. As far as conservation of energy is concerned, $a(t)$ could remain constant after reaching the maximum value allowed by conservation of energy. (Similarly, if you threw a ball straight up in a uniform gravitational field, conservation of energy would allow you to calculate the speed at any height, but when the ball reaches its maximum height, conservation of energy would not rule out the possibility that the ball could just come to a stop.) To know what happens after $a(t)$ reaches its maximum value, we can look at the second order equation, $\ddot{a} = -2\pi G\rho a$. This equation implies that $\dot{a}$ cannot remain zero, but must become negative, and then conservation of energy would imply that $\dot{a}$ must become more negative as a descreases, so the universe collapses.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 3, 2022
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2

Revised Version*

QUIZ DATE: Wednesday, November 9, 2022.

COVERAGE: Lecture Notes 4, starting with *Evolution of a Closed Universe* on p. 5, Lecture Notes 5, and Lecture Notes 6 through *Relationship between a and T, p. 22.* Problem Sets 4–7. Weinberg, *The First Three Minutes,* Chapters 4 – 8 and *Afterword: Cosmology Since 1977.* In Ryden's *Introduction to Cosmology*, we have read Chapters 4 and Chapter 5 through Section 5.4.1 during this period These chapters, however, parallel what we have done or will be doing in lecture, so you should take them as an aid to learning the lecture material; there will be no questions explicitly based on these sections from Ryden. The quiz will include, however, Ryden's Chapters 7 (*Dark Matter*), Chapter 8 (*The Cosmic Microwave Background*) and Chapter 9 (*Nucleosynthesis and the Early Universe*), with the exceptions of Sections 8.3 (*The Physics of Recombination*), 9.2 (*Neutrons and Protons*), and 9.3 (*Deuterium Synthesis*). These three sections include detailed calculations of chemical equilibrium, which is a subject to which we will return. As I pointed out in the preamble to Problem Set 6, Eqs. (9.11) and (9.12) are not correct. For now, you should read over these three sections with enough understanding to be able to follow the sections that follow, but there will be no questions on the quiz that are based on these sections.

Chapters 4 and 5 of Weinberg's book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. We will not take off for the spelling of names, as long as they are vaguely recognizable. For dates before 1900, it will be sufficient for you to know when things happened to within 100 years. For dates after 1900, it will be sufficient if you can place events within 10 years.

You should expect one 20-point problem based on the readings, and several calculational problems. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the problem sets listed above (extra credit problems included), or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 6, 7, 11, 13, 15, 17, and 18.

QUIZ LOGISTICS: The logistics will be identical to Quiz 1. The quiz will take place during our regular class time, but in a different room, to be announced. The quiz will be closed book, no calculators, no internet, and 80 minutes long.

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* Revised 11/3/22 to fix the bookmarks of the solutions.

PURPOSE OF THE REVIEW PROBLEMS: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz. I thought that 20 review problems would be a good number, but that meant cutting a number of good problems. If by any chance you would like to see more review problems, let me know.

QUIZZES FROM PREVIOUS YEARS: In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, 2007, 2009, 2011, 2013, 2016, 2018, and 2020. The relevant problems from those quizzes have largely been incorporated into these review problems, but you still may be interested in looking at the quizzes, mainly to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match exactly the coverage from all previous years, but I believe that all these review problems would be fair problems for the upcoming quiz. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

REVIEW SESSION: To help you study for the quiz, Marianne Moore will hold a review session, at a time and place to be announced. Please fill out the Doodle Poll at https://doodle.com/meeting/participate/id/e919wYYe, to communicate your availability.

FUTURE QUIZ: Quiz 3 will be given on Wednesday, December 7, 2022.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" for your reference. For the second quiz, this useful information will be the following:

DOPPLER SHIFT (Definition:)

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} \ ,$$

where $\Delta t_{\text{observer}}$ and Δt_{source} are the period of the wave as measured by the observer and by the source, respectively, and $\lambda_{\text{observer}}$ and λ_{source} are the wavelength of the wave, as measured by the observer and by the source, respectively.

DOPPLER SHIFT (For motion along a line):

Nonrelativistic, u = wave speed, source moving at speed v away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed v away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity), $\beta \equiv v/c$, where c is the speed of light and v is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation. A clock that is moving at speed v relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor γ:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance ℓ_0 in their own frame, are moving together, in the direction of the line separating them, at speed v relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \;, \quad \vec{p} = \gamma m_0 \vec{v} \;, \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \;,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \;.$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,

where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,

where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned}\ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\ , \quad \ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a\ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\,\rho_r(t_i) \ \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right)\ , \quad \Omega \equiv \rho/\rho_c\ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G}\ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$\Omega = 1$.

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)\ , \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$

$\Omega = \frac{2}{1 + \cos\theta} > 1\ ,$

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3$.

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)\ , \quad \frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$

$\Omega = \frac{2}{1 + \cosh\theta} < 1\ ,$

where $\alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3\ ,$

$\kappa \equiv -k > 0$.

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or:
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

BLACK-BODY RADIATION:

Whenever $kT \gg mc^2$ for any particle, where k is the Boltzmann constant, T is the temperature, and m is the (rest) mass of the particle, in thermal equilibrium there will be a black-body radiation, in which the particle will make the following contributions to the energy density, mass density, pressure, number density, and energy density:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

The values of g and g^* for photons, neutrinos, and electron-positron pairs are as follows:

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \;,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \;.$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \;.$$

PHYSICAL CONSTANTS:

$$G = 6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} = 6.674 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-23}\,\text{joule/K} \\ &= 1.381 \times 10^{-16}\,\text{erg/K} \\ &= 8.617 \times 10^{-5}\,\text{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-34} \text{ joule} \cdot \text{s} \\ &= 1.055 \times 10^{-27} \text{ erg} \cdot \text{s} \\ &= 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \end{aligned}$$

$$c = 2.998 \times 10^8 \text{ m/s}$$

$$= 2.998 \times 10^{10} \text{ cm/s}$$

$$\hbar c = 197.3 \text{ MeV-fm}, \quad 1 \text{ fm} = 10^{-15} \text{ m}$$

$$1 \text{ yr} = 3.156 \times 10^{7} \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ joule} = 1.602 \times 10^{-12} \text{ erg}$$

$$\begin{aligned} 1 \text{ GeV} = 10^{9} \text{ eV} &= 1.783 \times 10^{-27} \text{ kg (where } c \equiv 1) \\ &= 1.783 \times 10^{-24} \text{ g} \, . \end{aligned}$$

PROBLEM LIST

PROBLEM 1: DID YOU DO THE READING? (2020) *(30 points)*

(a) *(5 points)* What is a necessary condition for two photons in a head-on collision to be able to produce an electron-positron pair?

(i) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 TeV.

(ii) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 GeV.

(iii) The energy of the photons must be below the "rest energy" of the electron-positron pair, which is roughly 1.02 keV.

(iv) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 MeV.

(v) Two colliding massless objects cannot produce massive particles as an outcome of their collision.

(b) *(5 points)* In Chapter 4, *Recipe for a Hot Universe,* Weinberg states that there are three conserved quantities that must be specified in the recipe for the early universe. What are they? [Grading: one correct = 2 pts; two correct = 4 pts; three correct = 5 pts.]

(i) ________________________

(ii) ________________________

(iii) ________________________

(c) *(5 points)* Why are "complicated" Feynman diagrams needed for calculations of the strong interactions, whereas accurate predictions can be made for electromagnetism with only the "simple" diagrams?

(i) Only the simplest Feynman diagrams in electromagnetism give a non-zero contribution; the more complicated diagrams cancel each other out.

(ii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," about 1/137.036. Complicated diagrams therefore give small contributions. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to one, so complicated diagrams are not suppressed.

(iii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," which is about equal to one, meaning that adding internal lines to a diagram doesn't change its value significantly. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to 137, so complicated diagrams are strongly enhanced.

(iv) Simple Feynman diagrams are mathematically ill-defined in the theory of strong interactions, whereas in electromagnetism they give a sensible numerical result.

(v) Calculations for the strong interactions involve six quarks, while there are only three charged leptons. If there were three more electron-like particles (in addition to the muon and the tau lepton), then electromagnetism would be as complicated as the strong interactions.

(d) *(5 points)* Consider the following observable phenomena:

(A) The motion of stars moving in large-radius circular orbits around the centers of spiral galaxies.

(B) The motion of some galaxies relative to the galaxy clusters they inhabit.

(C) Gravitational lensing of light emitted by stars in nearby galaxies.

Which one of the following combinations constitutes observational evidence that dark matter exists?

(i) Only (A).

(ii) Only (B).

(iii) Only (C).

(iv) (A) and (B).

(v) (A), (B), and (C).

(e) *(5 points)* Consider the following claims about hadrons:

(A) Electrons and neutrinos make up 90% of the Universe's hadronic matter content.

(B) They are the class of particles affected by strong interactions (i.e., by the strong nuclear force).

(C) It is a synonym of "baryons."

(D) They are theorized to be made up of more fundamental particles, called "quarks."

(E) They cannot be made up of more fundamental constituents, because if they were, we should be able to break them up into their building blocks and observe the hypothetical "quarks" directly.

Which combination of these claims is true?

(i) Only (A).

(ii) Only (B).

(iii) (B) and (D).

(iv) (B) and (E).

(v) (B), (C), and (D).

(f) *(5 points)* Which of the following is the range of typical binding energies per nucleon in an atomic nucleus? (This is relevant to nuclear fusion and nuclear fission studies, and to the epoch of Big Bang nucleosynthesis.)

(i) 1 eV – 10 eV.

(ii) 100 eV – 1 keV.

(iii) 10 keV – 100 keV.

(iv) 1 MeV – 10 MeV.

(v) 100 MeV – 1 GeV.

PROBLEM 2: DID YOU DO THE READING? (2018) *(25 points)*

For clarity, part (b) has been slightly reworded from the original quiz.

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

(i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.

(ii) The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.

(iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.

(iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

(i) When the temperature of the universe was about 10^{10} °K (t $\sim$ 1 sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.

(ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.

(iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first $3\frac{3}{4}$ minutes, and this is why we call this period the era of nucleosynthesis.

(iv) After the first $3\frac{3}{4}$ minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.

(v) The protons and neutrons became decoupled from the photons after the first $3\frac{3}{4}$ minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.

(vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s (iii) 1940s (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

PROBLEM 3: DID YOU DO THE READING? (2016) *(25 points)*

(a) *(5 points)* In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to Ω from clusters of galaxies as

(i) 0.01 (ii) 0.05 (iii) 0.20 (iv) 0.60 (v) 1.00

(b) *(4 points)* One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring

(i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.

(ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.

(iii) the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.

(iv) the velocity dispersion along the line of sight and also the redshift of the cluster.

(v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

(c) *(4 points)* Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.

(i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.

(ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.

(iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.

(iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.

(v) By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.

(d) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

  (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

  (ii) There was a breakdown in communication between theorists and experimentalists.

  (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

  (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

  (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

  (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

(e) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

  (i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

  (ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

  (iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

  (iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

  (v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

PROBLEM 4: DID YOU DO THE READING? (2013) *(25 points)*

The following problem comes from Quiz 2, 2013.

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity v as a function of radius R. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii?

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

PROBLEM 5: DID YOU DO THE READING? (2011) *(20 points)*

The following problem comes from Quiz 2, 2011.

(a) *(8 points)* During nucleosynthesis, heavier nuclei form from protons and neutrons through a series of two particle reactions.

(i) In *The First Three Minutes,* Weinberg discusses two chains of reactions that, starting from protons and neutrons, end up with helium, He^4. Describe at least one of these two chains.

(ii) Explain briefly what is the *deuterium bottleneck*, and what is its role during nucleosynthesis.

(b) *(12 points)* In Chapter 4 of *The First Three Minutes*, Steven Weinberg makes the following statement regarding the radiation-dominated phase of the early universe:

The time that it takes for the universe to cool from one temperature to another is proportional to the difference of the inverse squares of these temperatures.

In this part of the problem you will explore more quantitatively this statement.

(i) For a radiation-dominated universe the scale-factor $a(t) \propto t^{1/2}$. Find the cosmic time t as a function of the Hubble expansion rate H.

(ii) The mass density stored in radiation ρ_r is proportional to the temperature T to the fourth power: i.e., $\rho_r \simeq \alpha T^4$, for some constant α. For a wide range of temperatures we can take $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg \cdot m^{-3} \cdot K^{-4}}$. If the temperature is measured in degrees Kelvin (K), then ρ_r has the standard SI units, $[\rho_r] =$

$\text{kg} \cdot \text{m}^{-3}$. Use the Friedmann equation for a flat universe ($k = 0$) with $\rho = \rho_r$ to express the Hubble expansion rate H in terms of the temperature T. You will need the SI value of the gravitational constant $G \simeq 6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$. What is the Hubble expansion rate, in inverse seconds, at the start of nucleosynthesis, when $T = T_{\text{nucl}} \simeq 0.9 \times 10^9\,\text{K}$?

(iii) Using the results in (i) and (ii), express the cosmic time t as a function of the temperature. Your result should agree with Weinberg's claim above. What is the cosmic time, in seconds, when $T = T_{\text{nucl}}$?

∗ PROBLEM 6: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time t at which $a/\sqrt{\kappa} = 2\alpha$.

∗ PROBLEM 7: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi \ .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi \ ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi = $ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance $\ell_{\rm phys}$ at time t. You should leave your answer in the form of a definite integral.

The form of $a(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $a(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) \ ,$$
$$a = \alpha(1 - \cos\theta) \ ,$$

where

$$\alpha \equiv \frac{4\pi}{3} \frac{G\rho a^3}{c^2} \ .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 8: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

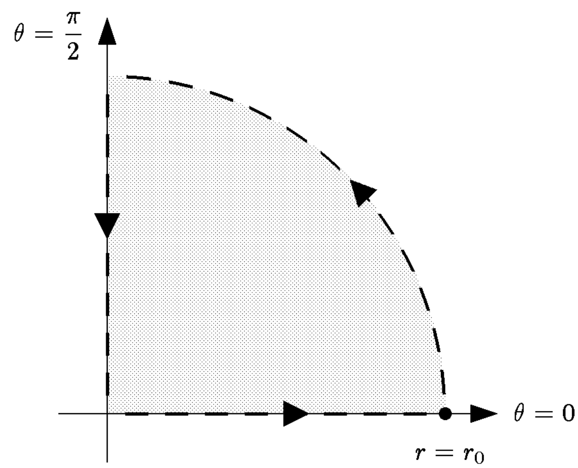
The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2 \, dr^2 + r^2(1 + br)^2 \, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to

$\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

b) *(15 points)* Find the area enclosed by this path.

PROBLEM 9: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \quad .$$

Calculate the volume $V(r_{\text{max}})$ of the sphere described by

$$r \le r_{\text{max}} \ .$$

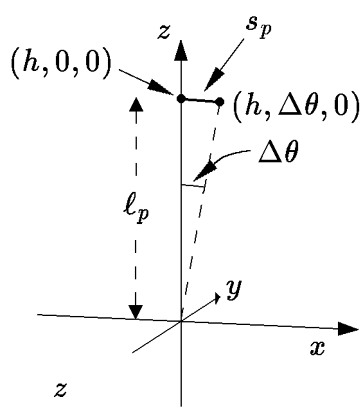
You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

PROBLEM 10: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as a, dropping its explicit t-dependence.

A small rod has one end at the point $(r = h,\ \theta = 0,\ \phi = 0)$ and the other end at the point $(r = h,\ \theta = \Delta\theta,\ \phi = 0)$. Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin $(r = 0)$ to the first end $(h, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1 - r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1 - r^2} = \frac{1}{2} \ln\left(\frac{1 + r}{1 - r}\right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor a, and the coordinates h and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor a.

∗ PROBLEM 11: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2 \ .$$

Express the metric in terms of this new variable.

PROBLEM 12: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$
$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

∗ PROBLEM 13: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} .$$

We will assume that this metric is given, and that $a(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt} dt .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed v_{phys} of the object, since it is the speed that would be measured by a comoving observer. Write an expression for v_{phys} as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A \frac{dr}{d\tau} \right] = B \left(\frac{dt}{d\tau} \right)^2 + C \left(\frac{dr}{d\tau} \right)^2 + D \left(\frac{d\theta}{d\tau} \right)^2 + E \left(\frac{d\phi}{d\tau} \right)^2 ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

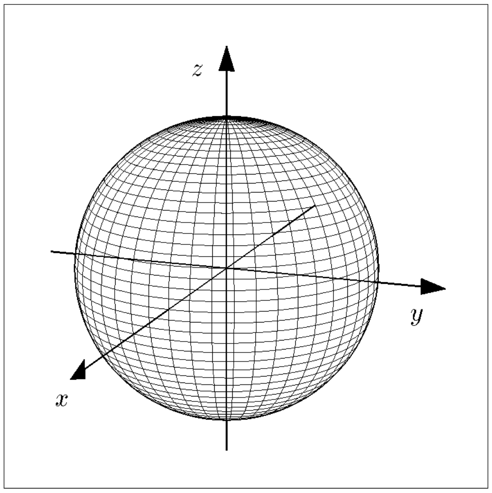
(f) *(5 points EXTRA CREDIT)* On Problem 2 of Problem Set 6 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

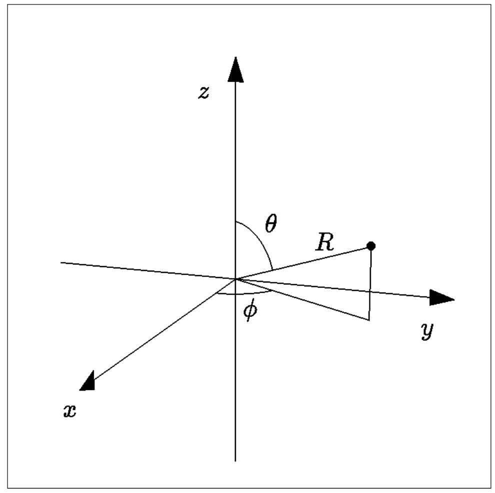
$$p = \frac{m v_{\text{phys}}}{\sqrt{1 - \frac{v_{\text{phys}}^2}{c^2}}} ,$$

falls off as $1/a(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE *(25 points)*

The following problem was Problem 2 on Quiz 2, 2020.





In this problem we will explore the geodesic equation for the metric describing the surface of a sphere. We will describe the sphere as in Lecture Notes 5, with metric given by

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \quad , \tag{14.1}$$

where R is the radius of the sphere. As given on the formula sheet, the geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s} \ . \tag{14.2}$$

(a) *(5 points)* Using the metric of Eq. (14.1), write explicitly the geodesic equation for $i = 1 = \theta$. By "explicitly," we mean that the equation that you give should not contain any instances of g, i, j, or k. All instances of the metric should be replaced using the metric from Eq. (14.1), and all repeated indices should be summed explicitly.

(b) *(5 points)* Again using the metric of Eq. (14.1), write explicitly the geodesic equation for $i = 2 = \phi$.

(c) *(5 points)* Consider a curve that circles the sphere at fixed latitude,

$$\theta(s) = \theta_0 \ , \qquad \phi(s) = \beta s \ , \tag{14.3}$$

where β and θ_0 are constants. Use the geodesic equations you found in parts (a) and (b) to find out if these curves are geodesics (i.e., see if they statisfy the geodesic equations). For what values of θ_0, if any, are these curves geodesics, and for what values of θ_0, if any, are they not?

(d) *(5 points)* Consider a curve that moves along a line of fixed longitude:

$$\theta(s) = \gamma s \ , \qquad \phi(s) = \phi_0 \ , \tag{14.4}$$

where γ and ϕ_0 are constants. Use the equations you found in parts (a) and (b) to find the values of ϕ_0, if any, for which these curves are geodesics, and for what values, if any, they are not.

(e) *(2 points)* The geodesic equations imply that there is a conserved quantity of the form

$$L \equiv F(\theta, \phi) \frac{\mathrm{d}\phi}{\mathrm{d}s} \ . \tag{14.5}$$

(By "conserved," we mean that this quantity is guaranteed to be constant along a geodesic.) Find the function $F(\theta, \phi)$ for which this is true. Note (a) that $F(\theta, \phi)$ is defined by the conservation statement only up to a multiplicative constant; and (b) that a function need not actually vary as its argument is varied (i.e., $q(x) = 5$ is a perfectly valid function of x).

(f) *(3 points)* Use your answers from part (e), (a), and (b) to find an equation for geodesics of the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \text{expression} \ , \tag{14.6}$$

where the "expression" depends only on θ and the conserved quantity L. (This is useful, since differential equations involving just one variable are generally easier to solve than differential equations involving two variables.) You may leave $F(\theta, \phi)$ in your answer, if you have not evaluated it.

∗ PROBLEM 15: ROTATING FRAMES OF REFERENCE *(35 points)*

The following problem was Problem 3, Quiz 2, 2004.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] \ , \tag{P15.1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (P15.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$ *,* $\bar{r}$*,* $\bar{\phi}$*, and* $\bar{z}$*,*

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] \ ,$$

and then introducing new coordinates t*,* r*,* ϕ*, and* z *that are related by*

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z \ ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$*,* $\mathrm{d}\bar{r} = \mathrm{d}r$*,* $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$*, and* $\mathrm{d}\bar{z} = \mathrm{d}z$*.*

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu \ ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \tag{P15.2}$$

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{P15.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

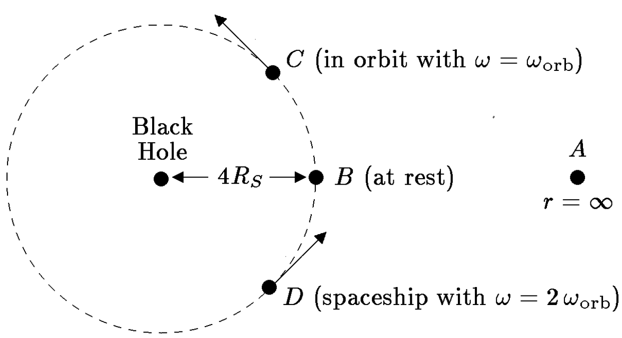
explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for* $\mathrm{d}\tau/\mathrm{d}t$*, in terms of the quantities indicated, and then ask yourself how this result can be used to find* $\mathrm{d}t/\mathrm{d}\tau$*.)*

PROBLEM 16: TRAVEL IN THE SCHWARZSCHILD METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 2020.



Consider a black hole of mass M, described by the Schwarzschild metric as written on the formula sheets:

$$\begin{aligned}\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 \\ + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,\end{aligned} \tag{P16.1}$$

In this problem we will consider a set of different observers in this spacetime, labeled by A, B, C, and D, as shown in the diagram above and described below.

(a) *(5 points)* Let observer A be stationary at an arbitrarily large value of r (you can take $r_A = \infty$), and let observer B be stationary at

$$r_B = 4R_S \ , \tag{P16.2}$$

where R_S is the Schwarzschild radius, $R_S = 2GM/c^2$. (Note that observer B needs to be using rocket propulsion to prevent herself from falling into the black hole.) Suppose that observer B is sending out pulses of electromagnetic radiation, at evenly spaced intervals $\Delta\tau_B$, as measured on B's local clock. What is the time interval $\Delta\tau_A$ that observer A will measure between the pulses that he receives?

(b) *(5 points)* Observer C is in orbit around the black hole, also at radius

$$r_C = 4R_S \; . \tag{P16.3}$$

The orbit is in the equatorial plane, with θ fixed at $\theta = \frac{\pi}{2}$, and ϕ changing with time as

$$\phi(t) = \omega_{\rm orb}\, t \; . \tag{P16.4}$$

In Problem 4 of Problem Set 6 you showed that the orbital angular velocity $\omega_{\rm orb}$, for a circular orbit, must satisfy

$$r\omega_{\rm orb}^2 = \frac{GM}{r^2} \; . \tag{P16.5}$$

As measured on C's local clock, how long does it take for C to complete one orbit (i.e., for ϕ to change by 2π)?

(c) *(5 points)* Now consider an observer D on a spaceship traveling in a circle at the same radius,

$$r_D = 4R_S \; , \tag{P16.6}$$

also in the equatorial plane, $\theta = \frac{\pi}{2}$. Using its rocket power, the spaceship travels twice as fast as observer C, with

$$\phi(t) = \omega_D\, t = 2\,\omega_{\rm orb}\, t \; . \tag{P16.7}$$

As measured on D's local clock, how long does it take for D to make one full circle (i.e., for ϕ to change by 2π).

(d) *(5 points)* If a tape measure, at rest with respect to the Schwarzschild coordinates, were stretched around the circle at radius $r = 4R_S$, what circumference would it measure?

∗ PROBLEM 17: GRAVITATIONAL BENDING OF LIGHT *(30 points)*

When a light ray passes by a massive object, general relativity predicts that it will be bent. Since most celestial objects are nearly spherical, we can use the Schwarzschild metric to calculate the bending. Furthermore, since we are usually interested in objects that are not black holes or anywhere nearly as dense, we can obtain an accurate answer by carrying out the calculation in a weak-field approximation. For a photon that grazes the Sun, for example, the value of $R_{\rm Sch}/R_\odot$, the Schwarzschild radius over the radius of the Sun, is about 4×10^{-6}.

Starting with the Schwarzschild metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{R_{\rm Sch}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{R_{\rm Sch}}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ , \qquad \text{(P17.1)}$$

where $R_{\rm Sch} = 2GM/c^2$, we can expand in powers of $R_{\rm Sch}/r$ and keep only the first order terms:

$$\mathrm{d}s^2 = -\left(1 - \frac{R_{\rm Sch}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 + \frac{R_{\rm Sch}}{r}\right) \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ . \qquad \text{(P17.2)}$$

For this problem it is useful to switch to Cartesian-like coordinates, defined in terms of r, θ, and ϕ by the usual Cartesian formulas,

$$\begin{aligned} x &= r\,\sin\theta\,\cos\phi \ , \\ y &= r\,\sin\theta\,\sin\phi \ , \\ z &= r\cos\theta \ . \end{aligned} \qquad \text{(P17.3)}$$

General relativity allows us to make any coordinate redefinitions that we might want, as long as we calculate the metric in terms of the new coordinates. It is useful to continue to use the quantity r, but now it will be thought of as a function of the coordinates x, y, and z:

$$r = (x^2 + y^2 + z^2)^{1/2} \ . \qquad \text{(P17.4)}$$

The metric can then be rewritten as the Minkowski metric of special relativity, plus small corrections:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \frac{R_{\rm Sch}}{r} c^2 \mathrm{d}t^2 + \frac{R_{\rm Sch}}{r} (\mathrm{d}r)^2 \ , \qquad \text{(P17.5)}$$

where from Eq. (4) one can see that

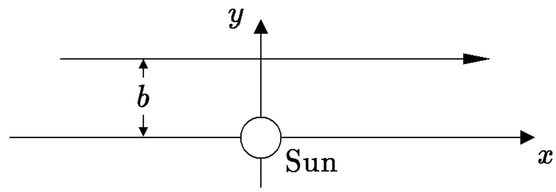
$$\mathrm{d}r = \frac{1}{r}(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z) \ . \qquad \text{(P17.6)}$$

(a) *(6 points)* For the metric as approximated by Eqs. (P17.5) and (P17.6), write the expressions for g_{tt}, g_{xx}, and g_{xy}.

The trajectory of the photon is lightlike, so we cannot use τ to parameterize the trajectory, because proper time intervals along a lightlike trajectory are zero. Nonetheless, it can be shown that one can use an "affine parameter" λ, for which the geodesic equation has the usual form:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right\} = \frac{1}{2}\left[\partial_\mu g_{\sigma\tau}\right]\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\tau}{\mathrm{d}\lambda} . \qquad \text{(P17.7)}$$

To obtain an answer that is accurate to first order in G, we begin by considering the *unperturbed* photon trajectory — the trajectory it would have if G were taken as zero, so $R_{\rm Sch} = 2GM/c^2 = 0$. This would be a straight line in the (x, y, z) coordinates, as shown in the diagram below:



Here b is called the *impact parameter*. We can parameterize this path by

$$x(\lambda) = \lambda , \qquad y(\lambda) = b , \qquad z(\lambda) = 0 , \qquad t(\lambda) = \lambda/c . \qquad \text{(P17.8)}$$

We will calculate the deflection (to first order in G) by assuming that the photon path is accurately described by Eq. (P17.8), and we will calculate the y-velocity that the photon acquires due to the gravitational attraction of the Sun.

(b) *(9 points)* With the goal of calculating $\mathrm{d}^2y/\mathrm{d}\lambda^2$, we evaluate the geodesic equation for $\mu = y$. Start here by evaluating the left-hand side of Eq. (P17.7) for $\mu = y$, to first order in G. Expand the derivative with respect to λ using the product rule, working out explicitly the derivatives of the relevant $g_{\mu\nu}$ with respect to λ. In parts (b) and (c), you may assume that $x(\lambda)$, $y(\lambda)$, $z(\lambda)$, and $t(\lambda)$, as well as $\mathrm{d}x/\mathrm{d}\lambda$, $\mathrm{d}y/\mathrm{d}\lambda$, $\mathrm{d}z/\mathrm{d}\lambda$, and $\mathrm{d}t/\mathrm{d}\lambda$, are all given to sufficient accuracy by Eq. (P17.8) and its derivatives with respect to λ. (Be careful: it is likely that there are more terms than you will at first notice.)

(c) *(9 points)* Evaluate the right-hand side of Eq. (P17.7) for $\mu = y$, to first order in G. Carry out all derivatives explicitly. (It always pays to be careful.)

(d) *(2 points)* Use your answers to parts (c) and (d) to find an equation for $\mathrm{d}^2y/\mathrm{d}\lambda^2$.

(e) *(4 points)* If the photon starts out on the unperturbed trajectory, its initial value of $\mathrm{d}y/\mathrm{d}\lambda$ will be zero. The final value of $\mathrm{d}y/\mathrm{d}\lambda$ will then be

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right|_{\rm final} = \int_{-\infty}^{\infty}\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda . \qquad \text{(P17.9)}$$

Use this fact to express the deflection angle α, to first order in G, as an explicit integral. You need not carry out the integral, but you may wish to use the table of integrals given below to carry it out so that you can check your answer. The correct final answer is

$$\alpha = \frac{4GM}{c^2 b} \,. \tag{P17.10}$$

TABLE OF INTEGRALS:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{b} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^{3/2}}\,\mathrm{d}x = \frac{2}{b^2} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{\pi}{2b^3}$$

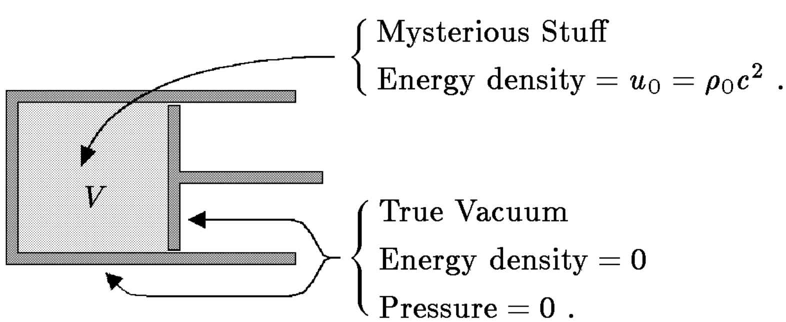
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{\pi}{2b} \qquad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^{5/2}}\,\mathrm{d}x = \frac{2}{3b^2} \qquad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+b^2)^3}\,\mathrm{d}x = \frac{\pi}{8b^3}$$

∗ PROBLEM 18: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

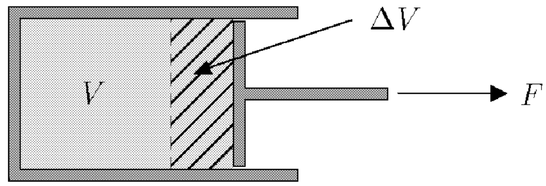
The following problem was Problem 3, Quiz 3, 2002.

In Lecture Notes 6, with further calculations in Problem 1 of Problem Set 7, a thought experiment involving a piston was used to show that $p = \frac{1}{3}\rho c^2$ for radiation. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

PROBLEM 19: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos at temperature T. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

PROBLEM 20: DOUBLING OF ELECTRONS *(10 points)*

The following was on Quiz 3, 2011 (Problem 4):

Suppose that instead of one species of electrons and their antiparticles, suppose there was also another species of electron-like and positron-like particles. Suppose that the new species has the same mass and other properties as the electrons and positrons. If this were the case, what would be the ratio T_ν/T_γ of the temperature today of the neutrinos to the temperature of the CMB photons.

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? (2020) *(30 points)*

(a) *(5 points)* What is a necessary condition for two photons in a head-on collision to be able to produce an electron-positron pair?

(i) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 TeV.

(ii) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 GeV.

(iii) The energy of the photons must be below the "rest energy" of the electron-positron pair, which is roughly 1.02 keV.

(iv) The energy of the photons must exceed the "rest energy" of the electron-positron pair, which is roughly 1.02 MeV.

(v) Two colliding massless objects cannot produce massive particles as an outcome of their collision.

[*Comment:* The condition in item (iv) is necessary, but it is not always sufficient. A total energy exceeding 1.02 MeV will be sufficient to produce an electron-positron pair in a head-on collision if the photons have equal energy. Otherwise one would have to calculate the energy in the center-of-mass frame of reference (also called the zero-momentum frame). If the total center-of-mass energy exceeds 1.02 MeV, then there is enough energy to produce an electron-positron pair.]

(b) *(5 points)* In Chapter 4, *Recipe for a Hot Universe,* Weinberg states that there are three conserved quantities that must be specified in the recipe for the early universe. What are they? [Grading: one correct = 2 pts; two correct = 4 pts; three correct = 5 pts.]

(i) electric charge

(ii) baryon number

(iii) lepton number

[*Comment:* Some students included energy and/or momentum on this list. By looking at Weinberg's Chapter 4, it is easy to verify that these items were not on his list. But it is less obvious why they do not belong on this list.

In the context of general relativity, the energy density and momentum density of everything except the gravitational field are well-defined, but they are obviously not conserved. As in Newtonian mechanics, matter can exchange energy and momentum with the gravitational field. To invoke energy or momentum conservation, we have to be able to include the energy and momentum of the gravitational field. For localized systems the metric approaches the Minkowski metric at large distances, and for such systems it is known how to construct a total energy and momentum, including gravity, that is conserved. But for our cosmological model, the metric does not approach the Minkowski metric anywhere.

We have already constructed the most general possible homogeneous and isotropic universe, described by the Robertson-Walker metric and evolving according to the Friedmann equations. We did not have to specify either the value of the energy or the value of the momentum. Since the Friedmann equations fully describe the evolution, if there was a conserved energy or momentum, the conservation would have to be a consequence of the Friedmann equations. However, the Friedmann equations can in fact be used to show that, if one does not specify the pressure, then any closed universe can evolve into any other, any open universe can evolve into any other, and any flat universe can evolve into any other. So if there is a conserved energy, then all closed universes must have the same energy, all open universes must have the same energy, and all flat universes must have the same energy. Some physicists, including Landau and Lifshitz (*The Classical Theory of Fields, p. 335*), adopt a definition of energy that implies that the total energy of any closed universe is zero. Misner, Thorne, and Wheeler, on the other hand, argue that the total energy of a closed universe is undefined. I (A.G.) side with Landau and Lifshitz.]

(c) *(5 points)* Why are "complicated" Feynman diagrams needed for calculations of the strong interactions, whereas accurate predictions can be made for electromagnetism with only the "simple" diagrams?

(i) Only the simplest Feynman diagrams in electromagnetism give a non-zero contribution; the more complicated diagrams cancel each other out.

(ii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," about 1/137.036. Complicated diagrams therefore give small contributions. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to one, so complicated diagrams are not suppressed.

(iii) When calculating the rate for electromagnetic processes, such as the scattering of two electrons, one needs to add an infinite number of contributions, each one symbolized in a Feynman diagram. The addition of one more internal line to the diagram multiplies its numerical value by a factor roughly equal to the "fine structure constant," which is about equal to one, meaning that adding internal lines to a diagram doesn't change its value significantly. For the strong interactions, however, the constant that plays the role of the fine structure constant is roughly equal to 137, so complicated diagrams are strongly enhanced.

(iv) Simple Feynman diagrams are mathematically ill-defined in the theory of strong interactions, whereas in electromagnetism they give a sensible numerical result.

(v) Calculations for the strong interactions involve six quarks, while there are only three charged leptons. If there were three more electron-like particles (in addition to the muon and the tau lepton), then electromagnetism would be as complicated as the strong interactions.

(d) *(5 points)* Consider the following observable phenomena:

(A) The motion of stars moving in large-radius circular orbits around the centers of spiral galaxies.

(B) The motion of some galaxies relative to the galaxy clusters they inhabit.

(C) Gravitational lensing of light emitted by stars in nearby galaxies.

Which one of the following combinations constitutes observational evidence that dark matter exists?

(i) Only (A).

(ii) Only (B).

(iii) Only (C).

(iv) (A) and (B).

(v) $\boxed{\text{(A), (B), and (C).}}$

[*Comment:* Since (A), (B), and (C) are all correct, answer (v) received full credit. But since the other answers are partially correct, answers (i), (ii), and (iii) received 1 point, and answer (iv) received 3 points.]

(e) *(5 points)* Consider the following claims about hadrons:

(A) Electrons and neutrinos make up 90% of the Universe's hadronic matter content.

(B) They are the class of particles affected by strong interactions (i.e., by the strong nuclear force).

(C) It is a synonym of "baryons."

(D) They are theorized to be made up of more fundamental particles, called "quarks."

(E) They cannot be made up of more fundamental constituents, because if they were, we should be able to break them up into their building blocks and observe the hypothetical "quarks" directly.

Which combination of these claims is true?

(i) Only (A).

(ii) Only (B).

(iii) $\boxed{\text{(B) and (D).}}$

(iv) (B) and (E).

(v) (B), (C), and (D).

[*Comment:* Since (B) and (D) are both correct, answer (iii) received full credit. But since answer (ii) is partially correct, it received 2 points.]

(f) *(5 points)* Which of the following is the range of typical binding energies per nucleon in an atomic nucleus? (This is relevant to nuclear fusion and nuclear fission studies, and to the epoch of Big Bang nucleosynthesis.)

(i) 1 eV – 10 eV.

(ii) 100 eV – 1 keV.

(iii) 10 keV – 100 keV.

(iv) $\boxed{\text{1 MeV – 10 MeV.}}$

(v) 100 MeV – 1 GeV.

PROBLEM 2: DID YOU DO THE READING? (2018) *(25 points)*

(a) *(4 points)* Which of the following statements about deuterium is **NOT** true? Choose one.

(i) The abundance of deuterium in the universe tends to decrease with time, because deuterium is very easily destroyed in stars.

$\boxed{\text{(ii)}}$ The most promising way to find the primordial value of deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium within the quasar itself.

(iii) The Lyman-α transition in deuterium corresponds to a slightly different wavelength than the Lyman-α transition in hydrogen.

(iv) Deuterium plays an important role in forming helium in the early universe mainly by producing tritium or ^{3}He.

[Comment: The most promising way to find the primordial value of the deuterium abundance is to look at the spectra of distant quasars to estimate the abundance of deuterium in ***intergalactic gas clouds*** *that lie between the quasars and us.]*

(b) *(6 points)* In Chapter 5 of *The First Three Minutes*, Steven Weinberg describes the first three minutes (or, more precisely, the first three and three quarter minutes) of the history of the universe. Choose two correct statements about the first three minutes. You can assume the fraction by weight of primordial helium is 26 percent. (3 points for each right answer, no penalty for guessing.)

$\boxed{\text{(i)}}$ When the temperature of the universe was about 10^{10} °K (t $\sim$ 1 sec), neutrinos and antineutrinos started to behave as free particles, no longer having significant interactions with electrons, positrons, or photons.

(ii) After the neutrinos decoupled from the photons, the temperature of the neutrinos was higher than that of the photons because neutrinos interacted less with other particles as the universe expanded.

(iii) Most of the atoms heavier than helium were made through nucleosynthesis during the first $3\frac{3}{4}$ minutes, and this is why we call this period the era of nucleosynthesis.

$\boxed{\text{(iv)}}$ After the first $3\frac{3}{4}$ minutes, the neutron-proton balance was about 13% neutrons, 87% protons, and the ratio of protons to neutrons has been almost preserved until today.

(v) The protons and neutrons became decoupled from the photons after the first $3\frac{3}{4}$ minutes, because the number densities of protons and neutrons were decreased by the formation of helium, and so their interactions with photons became negligible.

(vi) The observed abundance of helium in a galaxy today is much larger than the abundance of primordial helium, because helium is continuously formed inside stars by nuclear fusion.

[Comment: The statement (i) is described in Chapter 5 of Weinberg's book, and it has also been discussed in class. The correctness of statement (iv) can also be seen from Weinberg's Chapter 5, which says that the fraction of neutrons after the first three minutes is around 14%, and it goes down to around 13% "a little later", which he explains is probably about $3\frac{3}{4}$ minutes, depending on the ratio of photons to nuclear particles. Weinberg also explains that most of the neutrons present at this time immdiately combined with protons to form helium, which causes the ratio to be nearly constant until today. If you did not remember Weinberg's numbers, the statement that the fraction by weight of primordial helium is 26% should allow you to determine the fraction of neutrons, provided that you remember that the helium

nucleus consists of 2 protons and two neutrons, that the mass of the proton and neutron are about equal, and that the remaining 74% of the matter is essentially hydrogen, with no neutrons. Thus helium is very nearly half protons and half neutrons by weight, so the neutrons must be about 13% of the matter in the universe. (Note that we are talking about fractions of the total "baryonic" matter, which does not include the dark matter.) (ii) is clearly false, because the temperature of neutrinos becomes lower than that of photons. (iii) is clearly false, because most atoms heavier than helium were made much later in the history of the universe, in the interiors of stars. (v) is false because protons did not decouple from photons until about 350,000 years, and it happened because the plasma of protons and electrons combined to form neutral hydrogen. (vi) is false because most of the helium in the universe today is primordial. Ryden points out, for example, that the abundance of helium in the Sun's atmosphere is only about 28%. Weinberg states, near the end of Chapter 5, that "the 20-30 percent helium abundance could not have been created recently without liberating enormous amounts of radiation that we do not observe."]

(c) *(4 points)* The cosmic microwave background radiation was first discovered by Penzias and Wilson in 1964. However, according to Chapter 6 of *The First Three Minutes*, a team at the MIT Radiation Laboratory led by Robert Dicke was able to set an upper limit on any isotropic extraterrestrial radiation background, showing that the equivalent temperature was less than 20 °K at wavelengths of 1.00, 1.25, and 1.50 centimeters. This measurement was made in the

(i) 1920s (ii) 1930s $\boxed{\text{(iii) 1940s}}$ (iv) 1950s (v) 1960s

(d) *(5 points)* A free neutron can radioactively decay into a proton, plus two other particles. What are these particles? Give the charge, baryon number, and lepton number for each of these particles, verifying that each of these quantities is conserved in this process.

Answer:

The neutron decays through the reaction

$$n \to p + e^- + \bar{\nu}_e \ .$$

The quantum numbers of these particles can be described by the following table:

Particle	Charge	Baryon Number	Lepton Number
Neutron (n)	0	1	0
Proton (p)	$+e$	1	0
Electron (e^-)	-e	0	1
Anti-electron-neutrino ($\bar{\nu}_e$)	0	0	-1

Thus, the total charge of the final state is zero, the total baryon number is 1, and the total lepton number is zero, in all cases matching the initial value of these quantities.

(e) *(6 points)* In Chapter 8 of Ryden's *Introduction to Cosmology*, she discusses three ways to measure the dark matter in clusters. Give a brief, qualitative description of TWO of them. (If you give three descriptions, only the first two will be graded!)

Answer: You should have given two of the following three items.

1) *Virial theorem: The virial theorem relates the total kinetic energy of a steady-state cluster to is gravitational potential energy. Since the kinetic energy is proportional to the mass M of the cluster, while the potential energy is proportional to M^2, the relation will hold for only one value of M. By measuring the velocity dispersion (root mean square of the radial galaxy velocities relative to the mean radial velocity) and the size of the cluster, the mass can be inferred.*

2) *Hot, x-ray emitting gases: By measuring the x-rays emitted by the cluster, it is possible to model the density, temperature, and composition of the hot gas within the cluster (intracluster gas). By assuming that the gas is in hydrostatic equilibrium — i.e., by assuming that the pressure gradients balance the gravitational forces — one can infer the gravitational field, and hence the total mass.*

3) *Gravitational lensing: If one can find a galaxy behind the cluster, so that the image of the galaxy is gravitationally lensed, then the mass of the galaxy can be inferred by the degree to which the galaxy is lensed.*

PROBLEM 3: DID YOU DO THE READING? (2016) *(25 points)*

(a) *(5 points)* In Chapter 8 of Barbara Ryden's *Introduction to Cosmology*, she estimates the contribution to Ω from clusters of galaxies as

(i) 0.01 (ii) 0.05 $\boxed{\text{(iii) 0.20}}$ (iv) 0.60 (v) 1.00

(b) *(4 points)* One method of estimating the total mass of a cluster of galaxies is based on the virial theorem. With this method, one estimates the mass by measuring

(i) the radius containing half the luminosity and also the temperature of the X-ray emitting gas at the center of the galaxy.

(ii) the velocity dispersion perpendicular to the line of sight and also the radius containing half of the luminosity of the cluster.

$\boxed{\text{(iii)}}$ the velocity dispersion along the line of sight and also the radius containing half of the luminosity of the cluster.

(iv) the velocity dispersion along the line of sight and also the redshift of the cluster.

(v) the velocity dispersion perpendicular to the line of sight and also the redshift of the cluster.

Explanation: The virial theorem relates the kinetic energy to the potential energy. The key relationship is

$$\frac{1}{2}M\langle v^2\rangle = \frac{\alpha}{2}\frac{GM^2}{r_h} \ ,$$

where M is the mass of the cluster, $\langle v^2\rangle$ is the average squared velocity of its galaxies, and r_h is the radius containing half the total mass, which is estimated by the radius containing half the luminosity. α is a numerical factor depending on the structure of the cluster, estimated at 0.4 based on observed clusters. Velocities along the line of sight are measured by the spread in Doppler shifts, while velocities perpendicular to the line of sight are essentially impossible to measure, eliminating answers (ii) and (v). Since r_h is needed, neither (i) nor (iv) include enough information. (iii) is exactly right.

(c) *(4 points)* Another method of estimating the total mass of a cluster of galaxies is to make detailed measurements of the x-rays emitted by the hot intracluster gas.

(i) By assuming that this gas is the dominant component of the mass of the cluster, the mass of the cluster can be estimated.

(ii) By assuming that the hot gas comprises about a third of the mass of the cluster, the total mass of the cluster can be estimated.

(iii) By assuming that the gas is heated by stars and supernovae that make up most of the mass of the cluster, the mass of these stars and supernovae can be estimated.

(iv) By assuming that the gas is heated by interactions with dark matter, which dominates the mass of the cluster, the mass of the cluster can be estimated.

$\boxed{\text{(v)}}$ By assuming that this gas is in hydrostatic equilibrium, the temperature, mass density, and even the chemical composition of the cluster can be modeled.

Explanation: The dominant component of the mass is apparently dark matter, so the hot intracluster gas is only a small fraction, and we have no direct way of knowing what fraction. But the gas settles into a state of hydrostatic equilibrium which is determined by pressures and gravitational forces. The gas can be mapped by measuring its x-rays, which allows astronomers to estimate the gravitational forces, and hence the mass.

(d) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

Answer: The correct answers were $\boxed{\text{(ii), (v) and (vi).}}$ The others were incorrect for the following reasons:

(i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe".

(e) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

$\boxed{\text{(ii)}}$ In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

$\boxed{\text{(iv)}}$ In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

PROBLEM 4: DID YOU DO THE READING? (2013) *(25 points)*

(a) *(6 points)* The primary evidence for dark matter in galaxies comes from measuring their rotation curves, i.e., the orbital velocity v as a function of radius R. If stars contributed all, or most, of the mass in a galaxy, what would we expect for the behavior of $v(R)$ at large radii?

Answer: If stars contributed most of the mass, then at large radii the mass would appear to be concentrated as a spherical lump at the center, and the orbits of the stars would be "Keplerian," i.e., orbits in a $1/r^2$ gravitational field. Then $\vec{F} = m\vec{a}$ implies that

$$\frac{1}{R^2} \propto \frac{v^2}{R} \quad \Longrightarrow \quad v \propto \frac{1}{\sqrt{R}} \ .$$

(b) *(5 points)* What is actually found for the behavior of $v(R)$?

Answer: $v(R)$ looks nearly flat at large radii.

(c) *(7 points)* An important tool for estimating the mass in a galaxy is the steady-state virial theorem. What does this theorem state?

Answer: For a gravitationally bound system in equilibrium,

$$\text{Kinetic energy} = -\frac{1}{2}\,(\text{Gravitational potential energy}) \ .$$

(The equality holds whenever $\ddot{I} \approx 0$, where I is the moment of inertia.)

(d) *(7 points)* At the end of Chapter 10, Ryden writes "Thus, the very strong asymmetry between baryons and antibaryons today and the large number of photons per baryon are both products of a tiny asymmetry between quarks and anitquarks in the early universe." Explain in one or a few sentences how a tiny asymmetry between quarks and anitquarks in the early universe results in a strong asymmetry between baryons and antibaryons today.

Answer: When kT was large compared to 150 MeV, the excess of quarks over antiquarks was tiny: only about 3 extra quarks for every 10^9 antiquarks. But there was massive quark-antiquark annihilation as kT fell below 150 MeV, so that today we see the excess quarks, bound into baryons, and almost no sign of antiquarks.

PROBLEM 5: DID YOU DO THE READING? (2011) *(20 points)*†

(a) *(8 points)*

(i) *(4 points)* We will use the notation X^A to indicate a nucleus,* where X is the symbol for the element which indicates the number of protons, while A is the mass number, namely the total number of protons and neutrons. With this notation H^1, H^2, H^3, He^3 and He^4 stand for hydrogen, deuterium, tritium, helium-3 and helium-4 nuclei, respectively. Steven Weinberg, in *The First Three Minutes*, chapter V, page 108, describes two chains of reactions that produce helium, starting from protons and neutrons. They can be written as:

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + p \to He^4 + \gamma,}$$

$$\boxed{p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + n \to He^4 + \gamma.}$$

These are the two examples given by Weinberg. However, different chains of two particle reactions can take place (in general with different probabilities). For example:

$$p + n \to H^2 + \gamma \qquad H^2 + H^2 \to He^4 + \gamma,$$
$$p + n \to H^2 + \gamma \qquad H^2 + n \to H^3 + \gamma \qquad H^3 + H^2 \to He^4 + n,$$
$$p + n \to H^2 + \gamma \qquad H^2 + p \to He^3 + \gamma \qquad He^3 + H^2 \to He^4 + p,$$
$$\ldots$$

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* Notice that some students talked about atoms, while we are talking about nuclei formation. During nucleosynthesis the temperature is way too high to allow electrons and nuclei to bind together to form atoms. This happens much later, in the process called recombination.

Students who described chains different from those of Weinberg, but that can still take place, got full credit for this part. Also, notice that photons in the reactions above carry the additional energy released. However, since the main point was to describe the nuclear reactions, students who didn't include the photons still received full credit.

(ii) *(4 points)* The *deuterium bottleneck* is discussed by Weinberg in *The First Three Minutes*, chapter V, pages 109-110. The key point is that from part (i) it should be clear that deuterium (H^2) plays a crucial role in nucleosynthesis, since it is the starting point for all the chains. However, the deuterium nucleus is extremely loosely bound compared to H^3, He^3, or especially He^4. So, there will be a range of temperatures which are low enough for H^3, He^3, and He^4 nuclei to be bound, but too high to allow the deuterium nucleus to be stable. This is the temperature range where the *deuterium bottleneck* is in action: even if H^3, He^3, and He^4 nuclei could in principle be stable at those temperatures, they do not form because deuterium, which is the starting point for their formation, cannot be formed yet. Nucleosynthesis cannot proceed at a significant rate until the temperature is low enough so that deuterium nuclei are stable; at this point the deuterium bottleneck has been passed.

(b) *(12 points)*

(i) *(3 points)* If we take $a(t) = bt^{1/2}$, for some constant b, we get for the Hubble expansion rate:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \quad \Longrightarrow \quad \boxed{t = \frac{1}{2H}.}$$

(ii) *(6 points)* By using the Friedmann equation with $k = 0$ and $\rho = \rho_r = \alpha T^4$, we find:

$$H^2 = \frac{8\pi}{3} G\rho_r = \frac{8\pi}{3} G\alpha T^4 \quad \Longrightarrow \quad \boxed{H = T^2 \sqrt{\frac{8\pi}{3} G\alpha}\ .}$$

If we substitute the given numerical values $G \simeq 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$ and $\alpha \simeq 4.52 \times 10^{-32}\,\mathrm{kg} \cdot \mathrm{m}^{-3} \cdot \mathrm{K}^{-4}$ we get:

$$\boxed{H \simeq T^2 \times 5.03 \times 10^{-21}\,\mathrm{s}^{-1} \cdot \mathrm{K}^{-2}\ .}$$

Notice that the units correctly combine to give H in units of s^{-1} if the temperature is expressed in degrees Kelvin (K). In detail, we see:

$$[G\alpha]^{1/2} = (\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2} \cdot \mathrm{kg} \cdot \mathrm{m}^{-3} \cdot \mathrm{K}^{-4})^{1/2} = \mathrm{s}^{-1} \cdot \mathrm{K}^{-2}\ ,$$

where we used the fact that $1\ \mathrm{N} = 1\ \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$. At $T = T_{\rm nucl} \simeq 0.9\times 10^9\mathrm{K}$ we get:

$$\boxed{H \simeq 4.07\times 10^{-3}\mathrm{s}^{-1}.}$$

(iii) *(3 points)* Using the results in parts (i) and (ii), we get

$$\boxed{t = \frac{1}{2H} \simeq \left(\frac{9.95\times 10^{19}}{T^2}\right)\mathrm{s}\cdot\mathrm{K}^2\ .}$$

To good accuracy, the numerator in the expression above can be rounded to 10^{20}. The above equation agrees with Weinberg's claim that, for a radiation dominated universe, time is proportional to the inverse square of the temperature. In particular for $T = T_{\rm nucl}$ we get:

$$\boxed{t_{\rm nucl} \simeq 123\ \mathrm{s} \approx 2\ \mathrm{min}.}$$

†Solution written by Daniele Bertolini.

∗ PROBLEM 6: EVOLUTION OF AN OPEN UNIVERSE

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh\theta - \theta)$$
$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)\ .$$

Evaluating the second of these equations at $a/\sqrt{\kappa} = 2\alpha$ yields a solution for θ:

$$2\alpha = \alpha(\cosh\theta - 1) \quad\Longrightarrow\quad \cosh\theta = 3 \quad\Longrightarrow\quad \theta = \cosh^{-1}(3)\ .$$

We can use these results in the first equation to solve for t. Noting that

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{8} = 2\sqrt{2}\ ,$$

we have

$$\boxed{t = \frac{\alpha}{c}\left[2\sqrt{2} - \cosh^{-1}(3)\right]\ .}$$

Numerically, $t \approx 1.06567\,\alpha/c$.

∗ PROBLEM 7: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\,dt^2 + a^2(t)d\psi^2\ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{a^2(t)}\ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{a(t)}\ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{a(t')}dt'\ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} a(t)\,d\psi = \boxed{a(t)\int_0^t \frac{c}{a(t')}dt'\ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)}\ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1-\cos\theta)\ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{a(t)}\ .$$

Then using $a = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1\ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $a = \alpha(1 - \cos\theta)$, one can see that a returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1\ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 5. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2\ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 8: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1+ar)^2\,dr^2$, or $ds = (1+ar)\,dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\,dr = r_0 + \frac{1}{2}ar_0^2\ .$$

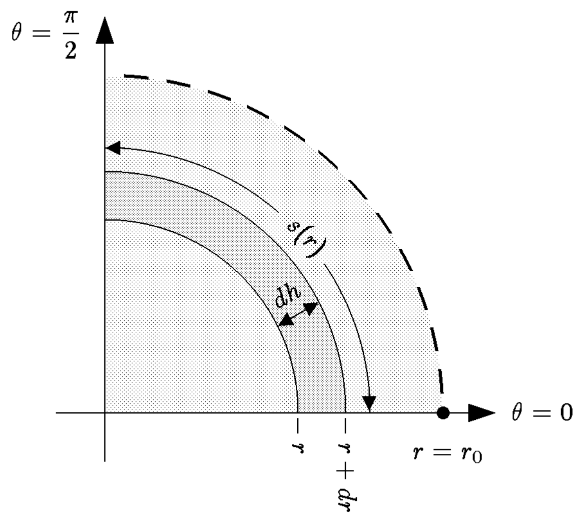
Along the second segment $dr = 0$, so $ds = r(1 + br)\,d\theta$, where $r = r_0$. So the length of the second segment is

$$S_2 = \int_0^{\pi/2} r_0(1 + br_0)\,d\theta = \frac{\pi}{2}r_0(1 + br_0)\ .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$\begin{aligned} S = 2S_1 + S_2 &= 2\left(r_0 + \frac{1}{2}ar_0^2\right) + \frac{\pi}{2}r_0(1 + br_0) \\ &= \boxed{\left(2 + \frac{\pi}{2}\right) r_0 + \frac{1}{2}(2a + \pi b)r_0^2\ .} \end{aligned}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar)\, dr \\ &= \frac{\pi}{2} \int_0^{r_0} [r + (a + b)r^2 + abr^3]\, dr \\ &= \boxed{\frac{\pi}{2} \left[\frac{1}{2} r_0^2 + \frac{1}{3}(a + b) r_0^3 + \frac{1}{4} abr_0^4 \right]} \end{aligned}$$

PROBLEM 9: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = a(t)\frac{dr}{\sqrt{1-kr^2}} \times a(t) r d\theta \times a(t) r \sin\theta d\phi$$

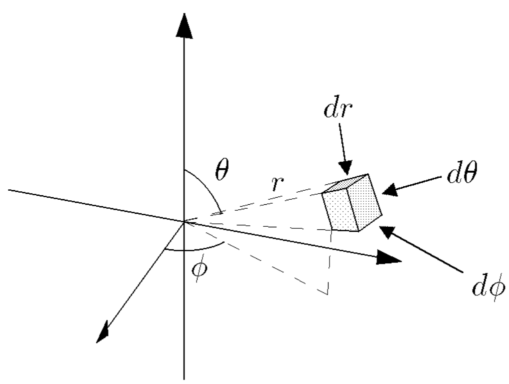
The total volume is then

$$V = \int dV = a^3(t) \int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi a^3(t) \int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}} \ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r + dr$, between θ and $\theta + d\theta$, and between ϕ and $\phi + d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$\begin{aligned} ds_1 &= a(t)\frac{dr}{\sqrt{1-kr^2}} \\ ds_2 &= a(t) r \, d\theta \\ ds_3 &= a(t) r \sin\theta \, d\theta \ . \end{aligned}$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\,d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\sqrt{k}\,r = \sin\psi \quad (\text{if } k > 0)$$
$$\sqrt{-k}\,r = \sinh\psi \quad (\text{if } k > 0).$$

The answer is

$$V = \begin{cases} 2\pi a^3(t)\left[\dfrac{\sin^{-1}\left(\sqrt{k}\,r_{\rm max}\right)}{k^{3/2}} - \dfrac{\sqrt{1-kr_{\rm max}^2}}{k}\right] & (\text{if } k > 0) \\[2ex] 2\pi a^3(t)\left[\dfrac{\sqrt{1-kr_{\rm max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\,r_{\rm max}\right)}{(-k)^{3/2}}\right] & (\text{if } k < 0) \quad .] \end{cases}$$

PROBLEM 10: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(h, 0, 0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{a\,dr}{\sqrt{1-r^2}} \ .$$

So

$$\boxed{\ell_p = \int_0^h \frac{a\,dr}{\sqrt{1-r^2}} = a\sin^{-1} h \ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = ar\,d\theta \ ,$$

so

$$\boxed{s_p = ah\,\Delta\theta \ .}$$

(c) From part (a), one has

$$h = \sin(\ell_p/a) \ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{a\sin(\ell_p/a)} \ .}$$

Note that as $a \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p/\ell_p$.

∗ PROBLEM 11: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from}\\ \text{origin to } r_0}} ds \ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0 \ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\, d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0)\sin\theta\, d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\, d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0)\sin\theta\, d\theta\, d\phi \ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0)\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \\ &= \rho^2(r_0)(2\pi)\left(-\cos\theta\Big|_0^{\pi}\right) \\ &\Longrightarrow \quad \boxed{A = 4\pi\rho^2(r_0) \ .} \end{aligned}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 2 of Problem Set 5, we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\, dr \ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi \int_0^{r_0} \rho^2(r)\, dr \ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma} \ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma} \ .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma}\,)$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}\,)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .}$$

PROBLEM 12: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda} \ .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij} dx^i dx^j = dr^2 + r^2\, d\theta^2 \ ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 \ , \qquad g_{r\theta} = g_{\theta r} = 0 \ .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{ g_{rr} \frac{dr}{d\lambda} \right\} = \frac{1}{2} \left(\partial_r g_{\theta\theta}\right) \frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda} \ ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{ \frac{dr}{d\lambda} \right\} = \frac{1}{2} \left(\partial_r r^2\right) \left(\frac{d\theta}{d\lambda} \right)^2 \ ,$$

or

$$\boxed{\frac{d^2 r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda} \right)^2 \ .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i, j). So

$$\boxed{\frac{d}{d\lambda}\left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 \ .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ ,$$

$$\theta(\lambda) = \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ .$$

Calculating the needed derivatives,*

$$\begin{aligned}
\frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2+1}} \\
\frac{d^2r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3} \\
\frac{d\theta}{d\lambda} &= -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2}\frac{1}{\lambda^2} = -\frac{1}{r^2} \; .
\end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \Longleftrightarrow \frac{1}{r^3} = r\left(-\frac{1}{r^2}\right)^2 \; ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \Longleftrightarrow \frac{d}{d\lambda}\left\{r^2\left(-\frac{1}{r^2}\right)\right\} = 0 \; ,$$

which also checks.

∗ PROBLEM 13: GEODESICS IN A CLOSED UNIVERSE

(a) *(7 points)* For purely radial motion, $d\theta = d\phi = 0$, so the line element reduces do

$$-c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \; .$$

Dividing by dt^2,

$$-c^2\left(\frac{d\tau}{dt}\right)^2 = -c^2 + \frac{a^2(t)}{1-r^2}\left(\frac{dr}{dt}\right)^2 \; .$$

---

* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi/\cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = (1+\tan^2\phi)d\phi \; .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2} \; .$$

Rearranging,

$$\boxed{\frac{d\tau}{dt} = \sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}\ .}$$

(b) *(3 points)*

$$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}} = \boxed{\frac{1}{\sqrt{1 - \frac{a^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}}\ .}$$

(c) *(10 points)* During any interval of clock time dt, the proper time that would be measured by a clock moving with the object is given by $d\tau$, as given by the metric. Using the answer from part (a),

$$d\tau = \frac{d\tau}{dt}\,dt = \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .$$

Integrating to find the total proper time,

$$\boxed{\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{a^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .}$$

(d) *(10 points)* The physical distance $d\ell$ that the object moves during a given time interval is related to the coordinate distance dr by the spatial part of the metric:

$$d\ell^2 = ds^2 = a^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \quad \Longrightarrow \quad d\ell = \frac{a(t)}{\sqrt{1-r^2}}\,dr\ .$$

Thus

$$\boxed{v_{\text{phys}} = \frac{d\ell}{dt} = \frac{a(t)}{\sqrt{1-r^2}}\,\frac{dr}{dt}\ .}$$

Discussion: A common mistake was to include $-c^2\,dt^2$ in the expression for $d\ell^2$. To understand why this is not correct, we should think about how an observer would measure $d\ell$, the distance to be used in calculating the velocity of a passing object.

The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval dt_{meas}. Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the ***same*** time t. Thus, when we compute the distance between the two marks, we set $dt = 0$. To compute the speed she would then divide the distance by dt_{meas}, which is nonzero.

(e) *(10 points)* We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} .$$

This formula is true for each possible value of μ, while the Einstein summation convention implies that the indices ν, λ, and σ are summed. We are trying to derive the equation for r, so we set $\mu = r$. Since the metric is diagonal, the only contribution on the left-hand side will be $\nu = r$. On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when $\lambda = \sigma$. The term will vanish unless $dx^\lambda/d\tau$ is nonzero, so λ must be either r or t (i.e., there is no motion in the θ or ϕ directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r} .$$

Since $g_{tt} = -c^2$, the derivative with respect to r will vanish. Thus, the only nonzero contribution on the right-hand side arises from $\lambda = \sigma = r$. Using

$$g_{rr} = \frac{a^2(t)}{1-r^2} ,$$

the geodesic equation becomes

$$\frac{d}{d\tau}\left\{g_{rr}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left(\partial_r g_{rr}\right)\frac{dr}{d\tau}\frac{dr}{d\tau} ,$$

or

$$\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left[\partial_r\left(\frac{a^2}{1-r^2}\right)\right]\frac{dr}{d\tau}\frac{dr}{d\tau} ,$$

or finally

$$\boxed{\frac{d}{d\tau}\left\{\frac{a^2}{1-r^2}\frac{dr}{d\tau}\right\} = a^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2 .}$$

This matches the form shown in the question, with

$$A = \frac{a^2}{1-r^2} \ , \ \text{and } C = a^2 \frac{r}{(1-r^2)^2} \ ,$$

with $B = D = E = 0$.

(f) *(5 points EXTRA CREDIT)* The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for p. Using the answer from (d),

$$p = \frac{mv_{\text{phys}}}{\sqrt{1 - \frac{v_{\text{phys}}^2}{c^2}}} = \frac{m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt}}{\sqrt{1 - \frac{a^2}{c^2(1-r^2)} \left(\frac{dr}{dt}\right)^2}} \ .$$

Using the answer from (b), this simplifies to

$$p = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{dt} \frac{dt}{d\tau} = m \frac{a(t)}{\sqrt{1-r^2}} \frac{dr}{d\tau} \ .$$

Multiply the geodesic equation by m, and then use the above result to rewrite it as

$$\frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} = ma^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\tau} \right)^2 \ .$$

Expanding the left-hand side,

$$\begin{aligned} LHS = \frac{d}{d\tau} \left\{ \frac{ap}{\sqrt{1-r^2}} \right\} &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ap \frac{r}{(1-r^2)^{3/2}} \frac{dr}{d\tau} \\ &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} + ma^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\tau} \right)^2 \ . \end{aligned}$$

Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}} \frac{d}{d\tau} \{ap\} = 0 \ .$$

Multiplying by $\sqrt{1-r^2}$, one has the desired result:

$$\frac{d}{d\tau} \{ap\} = 0 \quad \Longrightarrow \quad \boxed{p \propto \frac{1}{a(t)} \ .}$$

PROBLEM 14: GEODESICS ON THE SURFACE OF A SPHERE *(25 points)*

(a) Taking $i = \theta$ in Eq. (14.1), and remembering that the metric is diagonal and that only $g_{\phi\phi}$ depends on θ, the equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial\theta}\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} .$$

Using $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2\sin^2\theta$, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{R^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = \frac{1}{2}\frac{\partial(R^2\sin^2\theta)}{\partial\theta}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 .$$

Simplifying,

$$\boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \sin\theta\,\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 .}$$

(b) Taking $i = \phi$ and remembering that the metric is diagonal and that none of the components of $g_{\phi\phi}$ depend on ϕ, the geodesic equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 .$$

Again using $g_{\phi\phi} = R^2\sin^2\theta$, this becomes

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{R^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 .}$$

The equation above is a perfectly acceptable answer. One might also recognize that R^2 can be factored out, giving

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\right\} = 0 .}$$

If one wishes, one could expand the derivative using the product rule, giving

$$\boxed{\sin^2\theta\,\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\sin\theta\,\cos\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\,\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 .}$$

(c) The ϕ equation is trivially satisfied, since

$$\sin^2\theta\frac{d\phi}{ds} = \sin^2(\theta_0)\,\beta\ ,$$

which is constant, so its derivative vanishes. The θ equation becomes

$$0 = \beta^2\sin\theta_0\,\cos\theta_0\ ,$$

which is satisfied for $\theta_0 = \pi/2$. Thus,

$$\boxed{\text{the curves are geodesics if and only if } \theta_0 = \pi/2.}$$

The geodesic equations are formally satisfied if $\theta_0 = 0$ or $\theta_0 = \pi$, since then $\sin\theta_0$ vanishes. These are not actually geodesics, however, due to the singularity of the polar coordinate system at $\theta = 0$ and $\theta = \pi$. For these values of θ, ϕ is undefined, and the curve described by Eq. (14.3) degenerates into a point. (Although $\theta_0 = 0$ or $\theta_0 = \pi$ are not actually geodesics, no points were taken off for including them in the answer.)

(d)

$$\boxed{\text{The curves are geodesics for all values of } \phi_0.}$$

The θ equation is satisfied, since $d^2\theta/ds^2 = 0$ and $d\phi/ds = 0$, while $d\phi/ds = 0$ alone is sufficient to imply that the ϕ equation is satisfied.

(e) The ϕ equation already has this form,

$$\frac{d}{ds}\left\{\sin^2\theta\frac{d\phi}{ds}\right\} = 0 \quad\Longrightarrow\quad \frac{dL}{ds} = 0\ ,$$

for

$$L = \sin^2\theta\,\frac{d\phi}{ds}\ ,$$

so

$$\boxed{F(\theta,\phi) = \sin^2\theta\ .}$$

It would of course also be correct to use $F(\theta,\phi) = R^2\sin^2\theta$, or more generally $F(\theta,\phi)$ can be any constant times $\sin^2\theta$.

(f) Using the definition of L chosen in the solution above for part (e), we can write

$$\frac{d\phi}{ds} = \frac{L}{\sin^2\theta}\ .$$

Substituting this into the geodesic equation for θ, we find

$$\boxed{\frac{d^2\theta}{ds^2} = \frac{\cos\theta}{\sin^3\theta}\,L^2\ .}$$

If L were defined with a different multiplicative constant, then this equation would be modified accordingly.

∗ PROBLEM 15: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\phi + \omega\,\mathrm{d}t\right)^2 + \mathrm{d}z^2\right] \ ,$$

and the metric coefficients are then just read off from this expression:

$$\begin{aligned}
&g_{11} \equiv g_{rr} = 1 \\
&g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = -c^2 + r^2\omega^2 \\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = r^2\omega \\
&g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = r^2 \\
&g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = 1 \ .
\end{aligned}$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\,dx^\mu\,dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} \ .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\, g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned}
RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \ .
\end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 ,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (P15.1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)} ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}}\ .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\ ,$$

adapted to the rotating cylindrical coordinate system.

PROBLEM 16: TRAVEL IN THE SCHWARZSCHILD METRIC *(20 points)*

(a) For B, the coordinate time interval between pulses, Δt_B, is related to the proper time interval between pulses, $\Delta\tau_B$, by the metric relation

$$-c^2\,\Delta\tau_B^2 = -\left(1 - \frac{2GM}{r_B c^2}\right)c^2\Delta t_B^2\ .$$

The other terms vanish, since B is stationary, so $\Delta r = \Delta\theta = \Delta\phi = 0$. Using $r_B = 4R_S$, it follows after a litle algebra that

$$\Delta t_B = \sqrt{\frac{4}{3}}\,\Delta\tau_B\ .$$

$\Delta\tau_B$ is the time interval that would be measured on B's local clock. (If this is not clear, it may help to read again the section *The Generalization from Space to Spacetime*, in Lecture Notes 5. Note also that B's clock is not free-falling, since B is using a rocket to hold her position. In the section on *Accelerating Clocks* in Lecture Notes 1, we discuss why it is reasonable to assume an "ideal clock" for which any corrections due to acceleration can be neglected. If the corrections are not negligible, then they depend on the detailed construction of the clock.)

Since the metric does not depend on time, each pulse traveling from B to A will take the same amount of coordinate time for the trip. So the coordinate time interval at A will be $\Delta t_A = \Delta t_B$, and hence $\Delta t_A = \sqrt{\frac{4}{3}}\,\Delta\tau_B$. Finally, we note that for A, at $r = \infty$, the metric reduces to the Minkowski metric, and since A is stationary, this gives $\Delta\tau_A = \Delta t_A$. So finally,

$$\boxed{\Delta\tau_A = \sqrt{\frac{4}{3}}\,\Delta\tau_B\ .}$$

(b) From Eq. (16.4), it is clear that the coordinate time interval for one orbit is

$$\Delta t_{\rm orb} = \frac{2\pi}{\omega_{\rm orb}} \ ,$$

so the problem is reduced to finding the relation between coordinate time and proper time for C. For C, the metric relation becomes

$$-c^2\Delta\tau_C^2 = -\left(1 - \frac{2GM}{r_C c^2}\right) c^2\Delta t_C^2 + r_C^2 \sin^2\theta_C\,\Delta\phi_C^2 \ .$$

Using $\Delta\phi_C = \omega_{\rm orb}\,\Delta t_C$ and then

$$r_C^2\,\omega_{\rm orb}^2 = \frac{GM}{r_C} = \frac{1}{8}\,c^2 \ ,$$

and also using $\sin^2\theta_C = 1$, one finds

$$-c^2\Delta\tau_C^2 = -\frac{3}{4}c^2\,\Delta t_C^2 + \frac{1}{8}c^2\,\Delta t_C^2 = -\frac{5}{8}c^2\,\Delta t_C^2 \ ,$$

so $\Delta\tau_C = \sqrt{\frac{5}{8}}\,\Delta t_C$. Thus

$$\Delta\tau_C = \sqrt{\frac{5}{2}}\,\frac{\pi}{\omega_{\rm orb}} \ .$$

Using Eq. (16.5) and $r = 4R_S$, one finds that

$$\omega_{\rm orb} = \frac{c^3}{16\sqrt{2}\,GM} \ ,$$

which can be used to replace $\omega_{\rm orb}$ in the previous equation, resulting in

$$\boxed{\Delta\tau_C = 16\sqrt{5}\,\pi\frac{GM}{c^3} \ .}$$

(c) The clock time for one orbit is clearly

$$\Delta t_D = \frac{\pi}{\omega_{\rm orb}} \ ,$$

but we need to convert to proper time. For D, they are related by

$$\begin{aligned} -c^2\Delta\tau_D^2 &= -\left(1 - \frac{2GM}{r_D c^2}\right) c^2\Delta t_D^2 + r_D^2 \sin^2\theta_D\,\Delta\phi_D^2 \\ &= -\frac{3}{4}c^2\Delta t_D^2 + r_D^2(2\omega_{\rm orb}\,\Delta t_D)^2 \ . \end{aligned}$$

Using $r_D^2 \omega_{\rm orb}^2 = \frac{1}{8}\,c^2$, this becomes

$$-c^2 \Delta\tau_D^2 = -\frac{3}{4} c^2 \Delta t_D^2 + \frac{1}{2} c^2 \Delta t_D^2 = -\frac{1}{4} c^2 \Delta t_D^2 \ .$$

So

$$\Delta\tau_D = \frac{1}{2}\,\Delta t_D \ ,$$

and then for one orbit,

$$\Delta\tau_D = \frac{\pi}{2\omega_{\rm orb}} \ .$$

Using the previous equation for $\omega_{\rm orb}$, this becomes

$$\boxed{\Delta\tau_D = \frac{8\sqrt{2}\,\pi G M}{c^3} \ .}$$

(d) For this case the metric reduces to

$$ds^2 = r^2\,\mathrm{d}\phi^2 \ ,$$

so

$$\Delta s = 2\pi r = 8\pi R_S = \boxed{\frac{16\pi G M}{c^2} \ .}$$

∗ PROBLEM 17: GRAVITATIONAL BENDING OF LIGHT *(30 points)*

(a) *(6 points)* Note that

$$\begin{aligned} \mathrm{d}r^2 &= \frac{1}{r^2}\left(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z\right)^2 \\ &= \frac{1}{r^2}\Big(x^2\,\mathrm{d}x^2 + y^2\,\mathrm{d}y^2 + z^2\,\mathrm{d}z^2 + 2xy\,\mathrm{d}x\,\mathrm{d}y + 2xz\,\mathrm{d}x\,\mathrm{d}z + 2yz\,\mathrm{d}y\,\mathrm{d}z\Big) \ . \end{aligned} \tag{S17.1}$$

By using this expression for $(\mathrm{d}r)^2$ in Eq. (P17.5), we have the full expression for $\mathrm{d}s^2$ written out, from which we can read off the components of $g_{\mu\nu}$:

$$\boxed{\begin{aligned} g_{tt} &= \text{coefficient of } \mathrm{d}t^2 = -c^2\left(1 - \frac{R_{\rm Sch}}{r}\right) \\ g_{xx} &= \text{coefficient of } \mathrm{d}x^2 = 1 + \frac{R_{\rm Sch}}{r^3}\,x^2 \\ g_{xy} &= \frac{1}{2} \text{ of coefficient of } \mathrm{d}x\,\mathrm{d}y = \frac{R_{\rm Sch}}{r^3}\,xy \ . \end{aligned}} \tag{S17.2}$$

A number of people missed the factor of 1/2 in the value of g_{xy}. It arises because the general formula is written as $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, which when expanded becomes

$$\mathrm{d}s^2 = g_{xx}\mathrm{d}x^2 + g_{yy}\mathrm{d}y^2 + g_{zz}\mathrm{d}z^2 + g_{tt}\mathrm{d}t^2 + g_{xy}\mathrm{d}x\,\mathrm{d}y + g_{yx}\mathrm{d}y\,\mathrm{d}x + \ldots\ .$$

Since $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}y\,\mathrm{d}x$, the coefficient of $\mathrm{d}x\,\mathrm{d}y$ is $g_{xy} + g_{yx} = 2g_{xy}$.

(b) *(9 points)* It will be useful to know the derivatives of r:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}\\ &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{x}{r}\ .\end{aligned} \tag{S17.3}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \text{and} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}\ , \tag{S17.4}$$

and

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\lambda} &= \frac{\partial r}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{\partial r}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\lambda} + \frac{\partial r}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}\lambda}\\ &= \frac{x}{r}\ .\end{aligned} \tag{S17.5}$$

In the 2nd line I used the value of $\partial r/\partial x$ from Eq. (S17.3), and the derivatives

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda} = 1\ , \qquad \frac{\mathrm{d}y}{\mathrm{d}\lambda} = \frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0 \tag{S17.6}$$

that can be found from Eq. (P17.8).

Now, to expand the left-hand side of the geodesic equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right\} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{g_{yy}\frac{\mathrm{d}y}{\mathrm{d}\lambda} + g_{yx}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right\}\\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\left[1 + \frac{R_{\rm Sch}}{r^3}y^2\right]\frac{\mathrm{d}y}{\mathrm{d}\lambda} + \frac{R_{\rm Sch}}{r^3}xy\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right\}\\ &= \frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2} - 3\frac{R_{\rm Sch}}{r^4}\frac{x}{r}xy\frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{R_{\rm Sch}}{r^3}\frac{\mathrm{d}x}{\mathrm{d}\lambda}y\frac{\mathrm{d}x}{\mathrm{d}\lambda}\\ &= \boxed{\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2} - 3\frac{R_{\rm Sch}b}{r^5}x^2 + \frac{R_{\rm Sch}b}{r^3}\ .}\end{aligned} \tag{S17.7}$$

Note that I dropped a term

$$\frac{R_{\rm Sch}y^2}{r^3}\frac{\mathrm{d}^2y}{\mathrm{d}\lambda^2}$$

and a term

$$\frac{R_{\rm Sch}}{r^3} xy \frac{{\rm d}^2 x}{{\rm d}\lambda^2} \ ,$$

which is justified because the acceleration $\frac{{\rm d}^2 y}{{\rm d}\lambda^2}$ will be proportional to G, and $R_{\rm Sch}$ is proportional to G, so this term is 2nd order in G. The problem stated that we are to work to first order in G. No points were taken off, however, from students who retained these or other negligible terms.

Note, however, that ${\rm d}^2 y\ {\rm d}\lambda^2$ is not negligible, and appears in the answer. This is because ${\rm d}y/{\rm d}\lambda$ is not actually zero, but is of order G. ${\rm d}y/{\rm d}\lambda$ is zero for the unperturbed path, but in reality the photon picks up a small velocity in the y-direction, caused by the gravitational attraction of the Sun and proportional to G. ${\rm d}^2 y/{\rm d}\lambda^2$ will also be proportional to G. When ${\rm d}y/{\rm d}\lambda$ multiplies a factor proportional to $R_{\rm Sch}$, the product is of order G^2 and hence negligible. But ${\rm d}^2 y/{\rm d}\lambda^2$ by itself is of order G and is not negligible.

Note on propagation of errors: I normally do not take off points for propagating errors, so for example a student who forgot the factor of 1/2 in determining g_{xy} would get full credit on part (b), even though the answer would contain terms that are wrong by a factor of 1/2. However, it seems right to me to make an exception to this rule in cases where an error on part (a) causes the consequent answer on a later part to become trivial. For example, if a student described a metric in part (a) which had no dependence on r, then many of the terms in parts (b) and (c) would not be present. In such cases I still took off points in parts (b) and (c), because it didn't seem fair to me to give such a student credit for calculating these terms, when the student exhibited no such capability.

(c) *(9 points)*

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial y}\left(g_{\sigma\tau}\right)\frac{{\rm d}x^\sigma}{{\rm d}\lambda}\frac{{\rm d}x^\tau}{{\rm d}\lambda} &= \frac{1}{2}\frac{\partial}{\partial y}(g_{xx})\left(\frac{{\rm d}x}{{\rm d}\lambda}\right)^2 + \frac{1}{2}\frac{\partial}{\partial y}(g_{tt})\left(\frac{{\rm d}t}{{\rm d}\lambda}\right)^2 \\ &= \frac{1}{2}\frac{\partial}{\partial y}\left(1+\frac{R_{\rm Sch}}{r^3}x^2\right) - \frac{1}{2}c^2\frac{\partial}{\partial y}\left(1-\frac{R_{\rm Sch}}{r}\right)\left(\frac{1}{c^2}\right) \\ &= -\frac{1}{2}\left(3\frac{R_{\rm Sch}}{r^4}\frac{y}{r}x^2\right) - \frac{1}{2}\left(\frac{R_{\rm Sch}}{r^2}\frac{y}{r}\right) \\ &= \boxed{-\frac{3}{2}\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{1}{2}\frac{R_{\rm Sch}b}{r^3}\ .} \end{aligned} \tag{S17.8}$$

(d) *(2 points)* Combining Eqs. (S17.7) and (S17.8), we find

$$\begin{aligned} \frac{{\rm d}^2 y}{{\rm d}\lambda^2} &= -\frac{3}{2}\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{1}{2}\frac{R_{\rm Sch}b}{r^3} + 3\frac{R_{\rm Sch}b}{r^5}x^2 - \frac{R_{\rm Sch}b}{r^3} \\ &= \boxed{\frac{3}{2}R_{\rm Sch}b\left[\frac{x^2}{r^5}-\frac{1}{r^3}\right]\ .} \end{aligned} \tag{S17.9}$$

(e) *(4 points)* The final value of $\mathrm{d}y/\mathrm{d}\lambda$ is given by Eq. (P17.9), while the final value of $\mathrm{d}x/\mathrm{d}\lambda$ will be equal to 1, at least up to possible corrections proportional to G. Thus, the final velocity will make an angle α relative to the horizontal, where

$$\tan\alpha = \frac{\mathrm{d}y/d\lambda|_{\text{final}}}{\mathrm{d}x/d\lambda|_{\text{final}}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda\ .$$

Since $\tan\alpha$ will be proportional to G, the small angle approximation $\tan\alpha = \alpha$ will apply, and

$$\boxed{\alpha \approx \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\,\mathrm{d}\lambda\ .} \tag{S17.10}$$

Then, using Eqs. (S17.9) and combining with Eqs. (P17.4) and (P17.8),

$$\begin{aligned}\alpha &= \frac{3}{2}R_{\text{Sch}}b\int_{-\infty}^{\infty}\left[\frac{x^2}{r^5} - \frac{1}{r^3}\right]\mathrm{d}\lambda \\ &= \boxed{\frac{3}{2}R_{\text{Sch}}b\int_{-\infty}^{\infty}\left[\frac{\lambda^2}{(\lambda^2+b^2)^{5/2}} - \frac{1}{(\lambda^2+b^2)^{3/2}}\right]\mathrm{d}\lambda\ .}\end{aligned} \tag{S17.11}$$

You were not asked to carry out these integrals, but using the table of integrals given with the problem, one finds

$$\alpha = \frac{3}{2}R_{\text{Sch}}b\left[\frac{2}{3b^2} - \frac{2}{b^2}\right] = -\frac{2R_{\text{Sch}}}{b} = -\frac{4GM}{c^2 b}\ . \tag{S17.12}$$

The minus sign indicates that the deflection is downward, as one would expect.

* PROBLEM 18: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V+\Delta V) = u_0\sqrt{\frac{V}{V+\Delta V}}\ .$$

(The above expression is proportional to $1/\sqrt{V+\Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1+\frac{\Delta V}{V}}} = \frac{u_0}{1+\frac{1}{2}\frac{\Delta V}{V}} = u_0\left(1 - \frac{1}{2}\frac{\Delta V}{V}\right)\ .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 \ .}$$

PROBLEM 19: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)} kT \ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT \ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)} k \ .}$$

Numerically, this gives $3.602\, k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$E = \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)} kT \ .}$$

Numerically, $E = 3.1514\, kT$.

> Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$\begin{aligned} S &= \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{7\pi^4}{135\zeta(3)}\,k\ .} \end{aligned}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 20: DOUBLING OF ELECTRONS *(10 points)*

The entropy density of black-body radiation is given by

$$\begin{aligned} s &= g\left[\frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3}\right]T^3 \\ &= g\,C\,T^3\ , \end{aligned}$$

where C is a constant. At the time when the electron-positron pairs disappear, the neutrinos are decoupled, so their entropy is conserved. All of the entropy from electron-positron pairs is given to the photons, and none to the neutrinos. The same will be true here, for both species of electron-positron pairs.

The conserved neutrino entropy can be described by $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation n^- and n^+ for the new electron-like and positron-like particles, and also the convention that

Primed quantities:	values after $e^+e^-n^+n^-$ annihilation
Unprimed quantities:	values before $e^+e^-n^+n^-$ annihilation.

For the neutrinos,

$$S'_\nu = S_\nu \quad \Longrightarrow \quad g_\nu C\left(a'T'_\nu\right)^3 = g_\nu C\left(aT_\nu\right)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\nu = aT_\nu \ .}$$

For the photons, before $e^+e^-n^+n^-$ annihilation we have

$$T_\gamma = T_{e^+e^-n^+n^-} = T_\nu \ ; \qquad g_\gamma = 2, \ g_{e^+e^-} = g_{n^+n^-} = 7/2 \ .$$

When the e^+e^- and n^+n^- pairs annihilate, their entropy is added to the photons:

$$S'_\gamma = S_{e^+e^-} + S_{n^+n^-} + S_\gamma \quad \Longrightarrow \quad 2C\left(a'T'_\gamma\right)^3 = \left(2 + 2\cdot\frac{7}{2}\right)C\left(aT_\gamma\right)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\gamma = \left(\frac{9}{2}\right)^{1/3} aT_\gamma \ ,}$$

so aT_γ increases by a factor of $(9/2)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{2}{9}\right)^{1/3} T'_\gamma \ .}$$

This ratio would remain unchanged until the present day.

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 9, 2022
Prof. Alan Guth

QUIZ 2

BLANK PAGES ARE AT THE END OF THE EXAM.

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins, and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	15	
3	30	
4	35	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

State whether each of the following statements is true (T) or false (F):

(a) We say that the universe went through a period where it was opaque because there were no photons at that time; they were created later. **T** or **F** .

(b) In thermal equilibrium the number of any type of particle, whose threshold temperature is below the actual temperature (i.e., for which $mc^2 < kT$), is about equal to the number of photons. **T** or **F** .

(c) An electron and a positron can collide and emit 2 photons. **T** or **F** .

(d) Baryon number is not a conserved quantity in the early Universe, since baryons are not elementary particles. **T** or **F** .

(e) The most likely explanation for the apparent predominance of matter over anti-matter in the Universe today is that the Universe segregated into domains of almost pure matter and almost pure anti-matter, and we happen to live in a matter domain. **T** or **F** .

(f) The neutron is an unstable particle and can decay into a proton, an electron, and an anti-neutrino. **T** or **F** .

(g) The first particles to decouple in the early Universe are the electrons because they are the lightest elementary particles. **T** or **F** .

(h) The current mass density of stars, brown dwarfs, and stellar remnants, such as white dwarfs, neutron stars, and black holes is less than 0.5% of the total mass density of the universe. **T** or **F** .

(i) Even if dark matter doesn't absorb, emit, or scatter light, it can still affect the trajectory of photons. **T** or **F** .

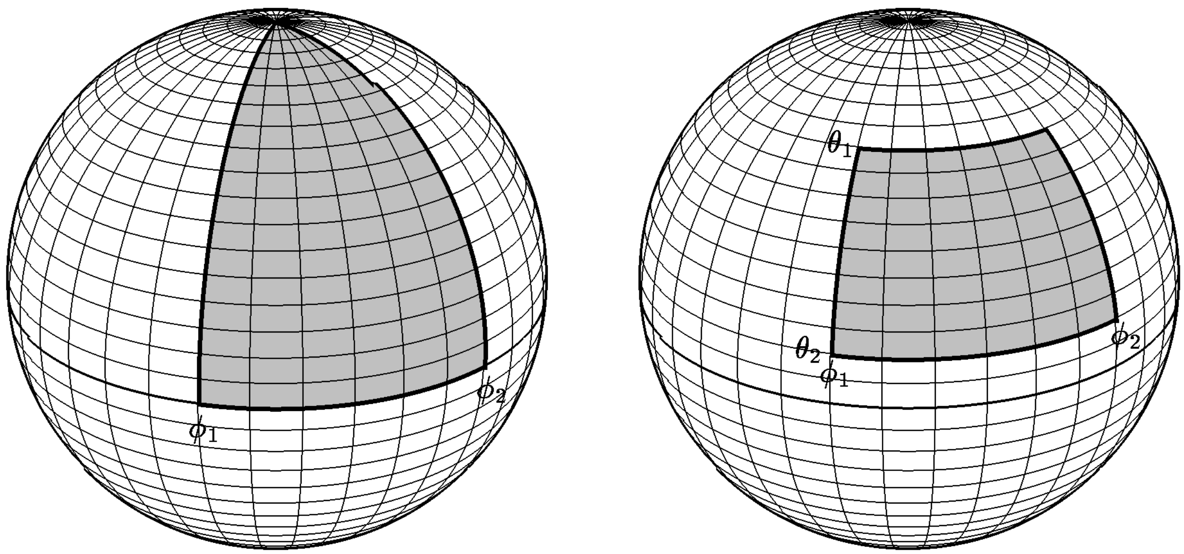
(j) The amounts of the light elements produced in the early Universe depend on the number of neutrino species. (By neutrino species, we mean electron neutrinos, muon neutrinos, etc.) **T** or **F** .

PROBLEM 2: Areas on the Surface of a Sphere *(15 points)*

Consider the surface of a sphere of radius R, with a metric on the surface given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

Here θ and ϕ are the usual polar angles, where θ is measured from the north pole, and ϕ increases in the counterclockwise direction as seen from above the north pole.



(a) (b)

(a) *(5 points)* Consider the three-sided region indicated in Fig. (a) above. One vertex is at the north pole, while the other two are on the equator, at the ϕ values ϕ_1 and ϕ_2, with $\phi_1 < \phi_2 < 2\pi$. Find the area of this region. (You can find it by integration, or by relating it to an area that you know.)

(b) *(10 points)* Now consider the four-sided region shown in Fig. (b). It is bounded at the top by the curve $\theta = \theta_1$, at the bottom by $\theta = \theta_2$, on the left by the curve $\phi = \phi_1$, and on the right by $\phi = \phi_2$. Again we assume that $\phi_1 < \phi_2 < 2\pi$, and also that $\theta_1 < \theta_2 < \pi$. Find the area of this region. *[Curiosity: the states of Colorado and Wyoming have boundaries described exactly in this way.]*

PROBLEM 3: WHAT IF THE PARTICLES WERE DIFFERENT? *(30 points)*

Imagine a fictional universe — I will call it Narnia — where the principles of physics are the same as in our world, but the types of particles that exist, and they way in which they interact, are different. Assume that the early history of Narnia is essentially the same as the big bang past that we believe our universe underwent, except for the details of the particles and their interactions.

In particular, suppose that in Narnia there exists two species of spin-$\frac{1}{2}$ fermions, $\psi^{\pm}$ and $\chi^{\pm}$, both with mass m_1. Spin-$\frac{1}{2}$ means that there are two spin states. The ψ^+ and ψ^- are antiparticles of each other, as are the χ^+ and χ^-. In addition there are photons, with exactly the same properties as in our universe. Finally, there are neutrinos, which are massless and left-handed. Unlike our universe, in Narnia there are four flavors of neutrino: ν_e, ν_μ, ν_τ, and ν_σ. There are also antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$, and $\bar{\nu}_\sigma$. No other particles exist.

(a) *(10 points)* When $kT \gg m_1c^2$, what is the energy density u as a function of the temperature T? (Here k is the Boltzmann constant, and c is the speed of light.)

(b) *(5 points)* Again when $kT \gg m_1c^2$, what are the particle number densities $n_{\psi^+\,\psi^-}$, n_γ, and n_ν? Here $n_{\psi^+\,\psi^-}$ refers to the total number density of ψ^+ and ψ^- particles, n_γ is the number density of photons, and n_ν represents the total number density of neutrinos and antineutrinos.

(c) *(15 points)* Assume, as we do for our universe, that at high temperatures all of the particles were in thermal equilibrium. Unlike our universe, however, in Narnia the photons interact very weakly. As the temperature falls below m_1c^2/k, and the $\psi^{\pm}$ and $\chi^{\pm}$ particles disappear from the thermal equilibrium gas, the photons are interacting so weakly that they obtain none of the energy or entropy that is released by the annihilating $\psi^{\pm}$ and $\chi^{\pm}$ pairs. Instead, all of the energy and entropy goes into heating the neutrinos, heating all 4 species equally. In this universe, after the $\psi^{\pm}$ and $\chi^{\pm}$ particles have disappeared, what is the ratio T_ν/T_γ?

PROBLEM 4: Rotating Frames of Reference *(35 points)*

This problem was on Quiz 2 of 2004, and this year it was Problem 15 in the Review Problems for Quiz 2.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] , \tag{4.1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \leq r < \infty$, $-\infty < z < \infty$, and $0 \leq \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (4.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, and $\bar{z}$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] ,$$

and then introducing new coordinates t, r, ϕ, and z that are related by

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, and $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \tag{4.2}$$

— *Problem 3 continues on the next page.* —

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_2(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{4.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r. Your answer should have the form

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \ldots \ .$$

where $\ldots$ is an expression written in terms of $r, \omega, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$. Any derivatives with respect to the coordinates t, r, θ, or ϕ can be carried out, so the only derivatives that appear in your equation should be derivatives with respect to τ.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ. Again, your answer should be written in terms of $r, \omega, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$, and the only derivatives should be with respect to τ. It is okay to leave expressions such as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\cdots\Big)$$

in your answer.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for* $\mathrm{d}\tau/\mathrm{d}t$*, in terms of the quantities indicated, and then ask yourself how this result can be used to find* $\mathrm{d}t/\mathrm{d}\tau$*.)*

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 28, 2022
Prof. Alan Guth

QUIZ 2 SOLUTIONS

Quiz Date: November 9, 2022

BLANK PAGES ARE AT THE END OF THE EXAM.

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins,
and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	15	
3	30	
4	35	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

State whether each of the following statements is true (T) or false (F):

(a) We say that the universe went through a period where it was opaque because there were no photons at that time; they were created later. **T** or $\boxed{\textbf{F}}$.

Comment: See Weinberg, Chapter 4, or Ryden, Sections 8.1 and 8.2. During the opaque period the universe was already filled with the photons that now make up the cosmic microwave background. The matter was ionized during the opaque phase; the opacity was caused by the large cross section for the scattering of photons by free electrons.

(b) In thermal equilibrium the number of any type of particle, whose threshold temperature is below the actual temperature (i.e., for which $mc^2 < kT$), is about equal to the number of photons. $\boxed{\textbf{T}}$ or **F** .

Comment: Weinberg uses almost exactly these words in Chapter 4, p. 84. When he says "about equal to," he means it in the sense of orders of magnitude, because the actual formula involves the factor that we called g^*, which is 2 for photons and 9/2 for neutrinos.

(c) An electron and a positron can collide and emit 2 photons. $\boxed{\textbf{T}}$ or **F** .

Comment: This is mentioned by Ryden, as Eq. (9.8), and also by Weinberg on p. 84. Any particle-antiparticle pair can annihilate into photons. Annihilation into a single photon is not possible, because it cannot conserve energy and momentum — the total energy-momentum four-vector for the particle-antiparticle pair is always timelike, while the energy-momentum vector for a single photon is lightlike.

(d) Baryon number is not a conserved quantity in the early Universe, since baryons are not elementary particles. **T** or $\boxed{\textbf{F}}$.

Comment: Baryon number is one of the three conserved quantities, along with electric charge and lepton number, that need to be specified in the recipe for the universe, as described by Weinberg in Chapter 4. In the Afterword, on pp. 183-184, Weinberg discusses the modern point of view that baryon number is not exactly conserved, but the nonconservation has nothing to do with whether or not baryons are elementary particles.

(e) The most likely explanation for the apparent predominance of matter over anti-matter in the Universe today is that the Universe segregated into domains of almost pure matter and almost pure anti-matter, and we happen to live in a matter domain. **T** or $\boxed{\textbf{F}}$.

Comment: In Chapter 4, on p. 96, Weinberg explains that the possibility that matter and antimatter are segregated in the universe seems contrary to the fact that we do not see antimatter cosmic rays coming from the proposed regions of antimatter, and we also don't see annihilation radiation coming from the interfaces of matter and antimatter regions.

(f) The neutron is an unstable particle and can decay into a proton, an electron, and an antineutrino. $\boxed{\textbf{T}}$ or **F** .

Comment: The decay of free neutrons is an important part of the story of big bang nucleosynthesis, which Weinberg mentions a number of times in Chapter 5. He describes the process in Chapter 4, on p. 93. Note that the antineutrino is needed to conserve lepton number. (Ryden discusses neutron decay in Sec. 9.2, specifically at Eq. (9.7), but questions based on this section were off-limits for this quiz.) Neutrons inside of nuclei that are not radioactive are nonetheless stable, because the nuclear binding energies cause it to be energetically impossible for the neutrons to decay.

(g) The first particles to decouple in the early Universe are the electrons because they are the lightest elementary particles. **T** or $\boxed{\textbf{F}}$.

Comment: An electron is certainly not the lightest particle: photons (zero mass) and neutrinos ($m_\nu c^2 \leq 0.1$ eV for average of three species) are certainly lighter. They are also certainly not the first particles to decouple. Neither Weinberg nor Ryden discuss the decoupling of electrons, but the electrons are certainly interacting significantly until the time of decoupling, at about 380,000 years. Neutrinos, by contrast, decouple at about 1 second (Weinberg, 4th frame of Chapter 5; Ryden, p. 172).

(h) The current mass density of stars, brown dwarfs, and stellar remnants, such as white dwarfs, neutron stars, and black holes is less than 0.5% of the total mass density of the universe. $\boxed{\textbf{T}}$ or **F** .

Comment: See Ryden, Sec. 7.1, p. 126.

(i) Even if dark matter doesn't absorb, emit, or scatter light, it can still affect the trajectory of photons. $\boxed{\textbf{T}}$ or **F** .

Comment: Yes, dark matter contributes significantly to the gravitational lensing of light, as discussed by Ryden in Sec. 7.4.

(j) The amounts of the light elements produced in the early Universe depend on the number of neutrino species. (By neutrino species, we mean electron neutrinos, muon neutrinos, etc.) $\boxed{\textbf{T}}$ or **F** .

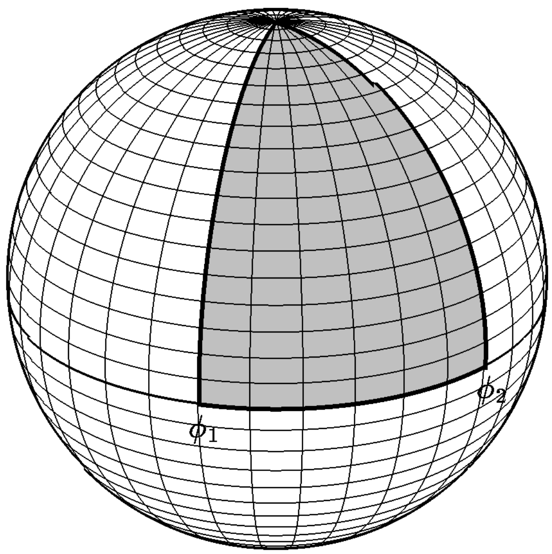
Comment: See Weinberg, Afterword, p. 183. The more species of neutrinos, the faster the early universe expands, so there is less time for the neutron to proton ratio to decrease before nucleosynthesis occurs. Thus, more species of neutrinos implies more neutrons available for nucleosynthesis, and hence more helium.

PROBLEM 2: Areas on the Surface of a Sphere *(15 points)*

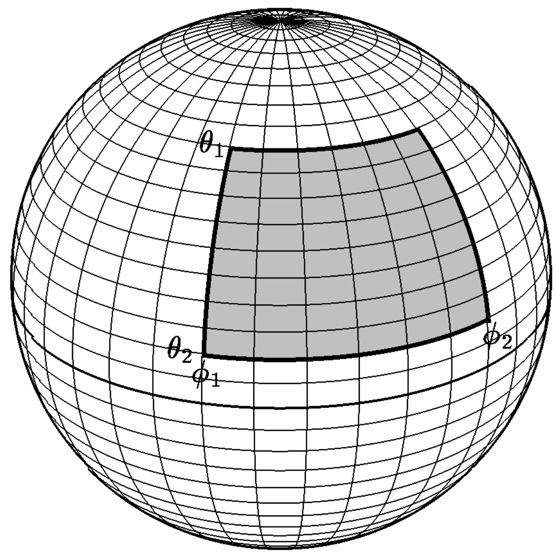
Consider the surface of a sphere of radius R, with a metric on the surface given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

Here θ and ϕ are the usual polar angles, where θ is measured from the north pole, and ϕ increases in the counterclockwise direction as seen from above the north pole.



(a)



(b)

(a) *(5 points)* Consider the three-sided region indicated in Fig. (a) above. One vertex is at the north pole, while the other two are on the equator, at the ϕ values ϕ_1 and ϕ_2, with $\phi_1 < \phi_2 < 2\pi$. Find the area of this region. (You can find it by integration, or by relating it to an area that you know.)

Answer: Clearly, if $\phi_2 - \phi_1 = 2\pi$, so the grey region extended over the entire northern hemisphere, the area would be $2\pi R^2$. If $\phi_2 - \phi_1 \leq 2\pi$, then the area is proportional to $\phi_2 - \phi_1$. Thus

$$\boxed{A = (\phi_2 - \phi_1)R^2 \ .}$$

(b) *(10 points)* Now consider the four-sided region shown in Fig. (b). It is bounded at the top by the curve $\theta = \theta_1$, at the bottom by $\theta = \theta_2$, on the left by the curve $\phi = \phi_1$, and on the right by $\phi = \phi_2$. Again we assume that $\phi_1 < \phi_2 < 2\pi$, and also that $\theta_1 < \theta_2 < \pi$. Find the area of this region. *[Curiosity: the states of Colorado and Wyoming have boundaries described exactly in this way.]*

Answer: According to the metric, if θ is varied by $d\theta$, the corresponding length is $ds_\theta = R\,d\theta$. If ϕ is varied by $d\phi$, the corresponding length is $ds_\phi = R\sin\theta\,d\phi$. Furthermore, these displacements are perpendicular to each other — this fact can be visualized from the diagram, or can be deduced from the fact that the metric implies a Pythagorean relation: $ds^2 = ds_\theta^2 + ds_\phi^2$, with no cross term $ds_\theta\,ds_\phi$. Thus, the area of an infinitesimal rectangle with sides $d\theta$ and $d\phi$ is given by

$$dA = ds_\theta\,ds_\phi = R^2 \sin\theta\,d\theta\,d\phi\ .$$

Integrating over the indicated region,

$$A = R^2 \int_{\theta_1}^{\theta_2} \sin\theta\,d\theta \int_{\phi_1}^{\phi_2} d\phi\ .$$

But $\sin\theta\,d\theta = -d(\cos\theta)$, so the integral is easily carried out, giving

$$\boxed{A = (\cos\theta_1 - \cos\theta_2)(\phi_2 - \phi_1)R^2\ .}$$

Note that this agrees with the answer in part (a), for which $\theta_1 = 0$ and $\theta_2 = \pi/2$.

PROBLEM 3: WHAT IF THE PARTICLES WERE DIFFERENT? *(30 points)*

Imagine a fictional universe — I will call it Narnia — where the principles of physics are the same as in our world, but the types of particles that exist, and they way in which they interact, are different. Assume that the early history of Narnia is essentially the same as the big bang past that we believe our universe underwent, except for the details of the particles and their interactions.

In particular, suppose that in Narnia there exists two species of spin-$\frac{1}{2}$ fermions, $\psi^{\pm}$ and $\chi^{\pm}$, both with mass m_1. Spin-$\frac{1}{2}$ means that there are two spin states. The ψ^{+} and ψ^{-} are antiparticles of each other, as are the χ^{+} and χ^{-}. In addition there are photons, with exactly the same properties as in our universe. Finally, there are neutrinos, which are massless and left-handed. Unlike our universe, in Narnia there are four flavors of neutrino: ν_e, ν_μ, ν_τ, and ν_σ. There are also antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$, and $\bar{\nu}_\sigma$. No other particles exist.

(a) *(10 points)* When $kT \gg m_1 c^2$, what is the energy density u as a function of the temperature T? (Here k is the Boltzmann constant, and c is the speed of light.)

Answer: The general formula for the energy density of black-body radiation is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ .$$

For the particles of Narnia,

$$g_{\psi^+\psi^-} = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}$$

$$g_{\chi^+\chi^-} = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}$$

$$g_\gamma = \underbrace{1}_{\text{No fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{1}_{\text{No antiparticles}} \times \underbrace{2}_{\text{Spin states}} = 2$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{4}_{\substack{\text{4 species}\\ \nu_e,\nu_\mu,\nu_\tau,\nu_\sigma}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = 7\ .$$

Thus, $g_{\text{tot}} = 16$, so finally

$$\boxed{u = \frac{8\pi^2}{15}\,\frac{(kT)^4}{(\hbar c)^3}\ .}$$

(b) *(5 points)* Again when $kT \gg m_1 c^2$, what are the particle number densities $n_{\psi^+\psi^-}$, n_γ, and n_ν? Here $n_{\psi^+\psi^-}$ refers to the total number density of ψ^+ and ψ^- particles, n_γ is the number density of photons, and n_ν represents the total number density of neutrinos and antineutrinos.

Answer: The general formula for the number density of particles in black-body radiation is

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} ,$$

where $\zeta(3) \approx 1.202$ is the Riemann zeta function evaluated at 3. For the particles of Narnia,

$$g^*_{\psi^+\psi^-} = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = 3$$

$$g^*_{\chi^+\chi^-} = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = 3$$

$$g^*_\gamma = \underbrace{1}_{\text{No fermion factor}} \times \underbrace{1}_{\text{1 species}} \times \underbrace{1}_{\text{No antiparticles}} \times \underbrace{2}_{\text{Spin states}} = 2$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{4}_{\substack{\text{4 species} \\ \nu_e, \nu_\mu, \nu_\tau, \nu_\sigma}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = 6 .$$

Thus,

$$\boxed{\begin{aligned} n_{\psi^+\psi^-} &= 3\frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} , \\ n_\gamma &= 2\frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} , \\ n_\nu &= 6\frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} . \end{aligned}}$$

(c) *(15 points)* Assume, as we do for our universe, that at high temperatures all of the particles were in thermal equilibrium. Unlike our universe, however, in Narnia the photons interact very weakly. As the temperature falls below $m_1 c^2/k$, and the $\psi^\pm$ and $\chi^\pm$ particles disappear from the thermal equilibrium gas, the photons are interacting so weakly that they obtain none of the energy or entropy that is released by the annihilating $\psi^\pm$ and $\chi^\pm$ pairs. Instead, all of the energy and entropy goes into heating the neutrinos, heating all 4 species equally. In this universe, after the $\psi^\pm$ and $\chi^\pm$ particles have disappeared, what is the ratio T_ν/T_γ?

Answer: The conservation of energy is generally difficult to apply in an expanding universe, since

$$\dot{u} = -3H\,(u+p)\ ,$$

where u is the energy density, p is the pressure, and H is the Hubble expansion rate. The complication is that the pressure p is difficult to calculate, especially when kT and mc^2 are comparable. Entropy, however, to an excellent approximation is simply conserved: the total entropy in a comoving volume is constant:

$$S \equiv a^3 s = \text{constant},$$

where s is the entropy density and a is the scale factor.

When kT falls below $m_1 c^2$, all of the entropy of the $\psi^\pm$ and $\chi^\pm$ pairs is given to the neutrinos. Using the subscript 1 to indicate the epoch when $kT \gg m_1 c^2$, and the subscript 2 to indicate $kT \ll m_1 c^2$, we have

$$\begin{aligned} S_{\nu,2} &= S_{\nu,1} + S_{\psi^\pm,1} + S_{\chi^\pm,1}\ , \\ S_{\gamma,2} &= S_{\gamma,1}\ . \end{aligned}$$

Thus,

$$\frac{S_{\nu,2}}{S_{\gamma,2}} = \frac{S_{\nu,1} + S_{\psi^\pm,1} + S_{\chi^\pm,1}}{S_{\gamma,1}} = \frac{g_\nu + g_{\psi^+\psi^-} + g_{\chi^+\chi^-}}{g_\gamma} = \frac{7 + \frac{7}{2} + \frac{7}{2}}{2} = 7\ .$$

But

$$\frac{S_{\nu,2}}{S_{\gamma,2}} = \frac{g_\nu T_\nu^3}{g_\gamma T_\gamma^3} = \frac{7}{2}\frac{T_\nu^3}{T_\gamma^3}\ .$$

By comparing the two equations above, one sees that

$$\boxed{\frac{T_\nu}{T_\gamma} = 2^{1/3}\ .}$$

PROBLEM 4: Rotating Frames of Reference *(35 points)*

This problem was on Quiz 2 of 2004, and this year it was Problem 15 in the Review Problems for Quiz 2.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] , \tag{4.1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (4.1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, and $\bar{z}$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] ,$$

and then introducing new coordinates t, r, ϕ, and z that are related by

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, and $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned}
g_{11} &\equiv g_{rr} = 1\\
g_{00} &\equiv g_{tt} = ?\\
g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ?\\
g_{22} &\equiv g_{\phi\phi} = ?\\
g_{33} &\equiv g_{zz} = ?
\end{aligned} \tag{4.2}$$

Answer: The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\phi + \omega\,\mathrm{d}t\right)^2 + \mathrm{d}z^2\right]\ ,$$

and the metric coefficients are then just read off from this expression:

$$g_{11} \equiv g_{rr} = 1$$

$$g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = \boxed{-c^2 + r^2\omega^2}$$

$$g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = \boxed{r^2\omega}$$

$$g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = \boxed{r^2}$$

$$g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = \boxed{1}\ .$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\,dx^\mu\,dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned}
g_{11} &\equiv g_{rr} = 1 \\
g_{00} &\equiv g_{tt} = f_1(r) \\
g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_2(r) \\
g_{22} &\equiv g_{\phi\phi} = f_3(r) \\
g_{33} &\equiv g_{zz} = f_4(r)\ ,
\end{aligned} \tag{4.3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r. Your answer should have the form

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \dots \ .$$

where $\dots$ is an expression written in terms of $r, \omega, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$. Any derivatives with respect to the coordinates t, r, θ, or ϕ can be carried out, so the only derivatives that appear in your equation should be derivatives with respect to τ.

Answer: Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} \ .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\, g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned} RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ &= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ &= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \ . \end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \ .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\ ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ. Again, your answer should be written in terms of r, ω, $\frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}t}{\mathrm{d}\tau}$, and the only derivatives should be with respect to τ. It is okay to leave expressions such as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\dots\Big)$$

in your answer.

Answer: Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0\ ,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0\ .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for $\mathrm{d}\tau/\mathrm{d}t$, in terms of the quantities indicated, and then ask yourself how this result can be used to find $\mathrm{d}t/\mathrm{d}\tau$.)*

Answer: If Eq. (4.1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}+\omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)} ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}+\omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}} .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} ,$$

adapted to the rotating cylindrical coordinate system.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 2, 2022
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

Revised December 3, 2022*

QUIZ DATE: Wedneday, December 7, 2022, during the normal class time, but in Walker Memorial Room 50-340.

COVERAGE: Lecture Notes 6 (pp. 23–end), Lecture Notes 7 Lecture Notes 8, and *Notes on Thermal Equilibrium.* Material from Lecture Notes 9 will be fair game for reading-type questions — multiple choice, true/false, or maybe short answer questions — but you will not be expected to use this material for calculations. Problem Sets 8 and 9; Barbara Ryden, *Introduction to Cosmology*, Chapter 8 (*The Cosmic Microwave Background*) and Chapter 9 (*Nucleosynthesis and the Early Universe*), especially Sections 8.3 (*The Physics of Recombination*), 9.2 (*Neutrons and Protons*), and 9.3 (*Deuterium Synthesis*) — with the help of *Notes on Thermal Equilibrium*, you are now expected to understand these sections in detail. Ryden, Chapter 11 *(Structure Formation: Gravitational Instability)*; Alan Guth, *Inflation and the New Era of High-Precision Cosmology,*

https://physics.mit.edu/news/journal/physicsatmit_02_cosmology/ .

The data quoted in this article about the nonuniformities of the cosmic microwave background radiation has since been superceded by much better data, but the conclusions have not changed. They have only gotten stronger. Like Lecture Notes 9, the material in this article will be fair game for Quiz 3 only in the context of reading-type questions.

One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 7, 9, 10, 12, 16, and 19.

QUIZ LOGISTICS: The logistics will be identical to Quiz 2. The quiz will take place during the regular class time, 11:05 am to 12:25 pm. It will not be in our regular classroom, but instead in Room 50-340 (Walker Memorial). The quiz will be closed book, no calculators, no internet, and 80 minutes long.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some

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* The original version lacked the solution to Problem 20, and failed to state that it was Problem 3 on Quiz 3, 2020.

cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

QUIZZES FROM PREVIOUS YEARS: In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2007, 2009, 2011, 2013, 2016, 2018, and 2020. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

REVIEW SESSION AND OFFICE HOURS: To help you study for the quiz, Marianne Moore will hold a review session for the quiz on Monday, December 5, at 6:15 pm in Room 8-320, right after Alan's office hour, which is in the same room, starting at 5:15 pm. You can still fill out the Piazza poll (https://piazza.com/class/l7roppm5yg856n/post/84) to suggest topics to cover at the review session. Marianne will also hold a special office hour on Tuesday December 6, 5:00-6:00 pm, in room 8-316.

INFORMATION TO BE GIVEN ON QUIZ:

For the third quiz, the following information will be made available to you:

DOPPLER SHIFT (Definition:)

$$1 + z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} = \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} \ ,$$

where $\Delta t_{\text{observer}}$ and Δt_{source} are the period of the wave as measured by the observer and by the source, respectively, and $\lambda_{\text{observer}}$ and λ_{source} are the wavelength of the wave, as measured by the observer and by the source, respectively.

DOPPLER SHIFT (For motion along a line):

Nonrelativistic, u = wave speed, source moving at speed v away from observer:

$$z = v/u$$

Nonrelativistic, observer moving at speed v away from source:

$$z = \frac{v/u}{1 - v/u}$$

Doppler shift for light (special relativity), $\beta \equiv v/c$, where c is the speed of light and v is the velocity of recession, as measured by either the source or the observer:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation. A clock that is moving at speed v relative to an inertial reference frame appears to be running slowly, as measured in that frame, by a factor γ:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction. A rod that is moving along its length, relative to an inertial frame, appears to be contracted, as measured in that frame, by the same factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Relativity of Simultaneity. If two clocks that are synchronized in their own reference frame, and separated by a distance ℓ_0 in their own frame, are moving together, in the direction of the line separating them, at speed v relative to an inertial frame, then measurements in the inertial frame will show the trailing clock reading later by an amount

$$\Delta t = \frac{\beta \ell_0}{c}$$

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

KINEMATICS OF A HOMOGENEOUSLY EXPANDING UNIVERSE:

Hubble's Law: $v = Hr$,

where v = recession velocity of a distant object, H = Hubble expansion rate, and r = distance to the distant object.

Present Value of Hubble Expansion Rate (Planck 2018):

$$H_0 = 67.66 \pm 0.42 \text{ km-s}^{-1}\text{-Mpc}^{-1}$$

Scale Factor: $\ell_p(t) = a(t)\ell_c$,

where $\ell_p(t)$ is the physical distance between any two objects, $a(t)$ is the scale factor, and ℓ_c is the coordinate distance between the objects, also called the comoving distance.

Hubble Expansion Rate: $H(t) = \dfrac{1}{a(t)} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t}$.

Light Rays in Comoving Coordinates: Light rays travel in straight lines with physical speed c relative to any observer. In Cartesian coordinates, coordinate speed $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{c}{a(t)}$. In general, $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$.

Horizon Distance:

$$\begin{aligned}\ell_{p,\mathrm{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).}\end{cases}\end{aligned}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \quad \ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right) a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\,\rho_m(t_i) \ \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\,\rho_r(t_i) \ \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \ \ \rho_c = \frac{3H^2}{8\pi G} \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Flat ($k = 0$): $\quad a(t) \propto t^{2/3}$

$$\Omega = 1 \ .$$

Closed ($k > 0$): $\quad ct = \alpha(\theta - \sin\theta)\ , \quad \dfrac{a}{\sqrt{k}} = \alpha(1-\cos\theta)\ ,$

$$\Omega = \frac{2}{1+\cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$): $\quad ct = \alpha\left(\sinh\theta - \theta\right)\ , \quad \dfrac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$

$$\Omega = \frac{2}{1+\cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

MINKOWSKI METRIC (Special Relativity):

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

ROBERTSON-WALKER METRIC:

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \equiv -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \tilde{a}^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

SCHWARZSCHILD METRIC:

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2$$
$$+ r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,$$

GEODESIC EQUATION:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^\ell}{\mathrm{d}s}$$

or: $$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

BLACK-BODY RADIATION:

Whenever $kT \gg mc^2$ for any particle, where k is the Boltzmann constant, T is the temperature, and m is the (rest) mass of the particle, in thermal equilibrium there will be a black-body radiation, in which the particle will make the following contributions to the energy density, mass density, pressure, number density, and energy density:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

The values of g and g^* for photons, neutrinos, and electron-positron pairs are as follows:

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}\ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3\ .$$

Spectrum of Black-Body Radiation:

The energy density for radiation in the frequency interval between ν and $\nu + \mathrm{d}\nu$ is given by

$$\rho_\nu(\nu)d\nu = \frac{8\pi^2\, g\, \hbar\nu^3}{c^3}\frac{1}{e^{2\pi\hbar\nu/kT}-1}d\nu\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860\ \text{MeV}}{\sqrt{t\ (\text{in sec})}}\left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}\ .$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}}c^2 = \frac{\Lambda c^4}{8\pi G}\ ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 = -\frac{\Lambda c^4}{8\pi G}\ .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \Omega_{\mathrm{vac},0}x^4 + \Omega_{k,0}x^2} \ ,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z} \ ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\mathrm{rad},0} - \Omega_{\mathrm{vac},0} \ .$$

Age of universe:

$$\begin{aligned} t_0 &= \frac{1}{H_0}\int_0^1 \frac{x\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\mathrm{rad},0} + \Omega_{\mathrm{vac},0}x^4 + \Omega_{k,0}x^2}} \\ &= \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{\mathrm{vac},0} + \Omega_{k,0}(1+z)^2}} \ . \end{aligned}$$

Look-back time:

$t_{\text{look-back}}(z) =$

$$\frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\mathrm{rad},0}(1+z')^4 + \Omega_{\mathrm{vac},0} + \Omega_{k,0}(1+z')^2}} \ .$$

PHYSICAL CONSTANTS:

$$G = 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2} = 6.674 \times 10^{-8}\ \mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-23}\,\mathrm{joule/K} \\ &= 1.381 \times 10^{-16}\,\mathrm{erg/K} \\ &= 8.617 \times 10^{-5}\,\mathrm{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-34}\ \mathrm{joule} \cdot \mathrm{s} \\ &= 1.055 \times 10^{-27}\ \mathrm{erg} \cdot \mathrm{s} \\ &= 6.582 \times 10^{-16}\ \mathrm{eV} \cdot \mathrm{s} \end{aligned}$$

$$\begin{aligned} c &= 2.998 \times 10^8\ \mathrm{m/s} \\ &= 2.998 \times 10^{10}\ \mathrm{cm/s} \end{aligned}$$

$$\hbar c = 197.3\ \text{MeV-fm}, \quad 1\ \mathrm{fm} = 10^{-15}\ \mathrm{m}$$

$$1\ \mathrm{yr} = 3.156 \times 10^7\ \mathrm{s}$$

$$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{joule} = 1.602 \times 10^{-12}\ \mathrm{erg}$$

$$\begin{aligned} 1\ \mathrm{GeV} = 10^9\ \mathrm{eV} &= 1.783 \times 10^{-27}\ \ \mathrm{kg}\ (\text{where } c \equiv 1) \\ &= 1.783 \times 10^{-24}\ \mathrm{g}\ . \end{aligned}$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\begin{aligned}
\ell_P &= \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-35} \text{ m} , \\
&= 1.616 \times 10^{-33} \text{ cm} , \\
t_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44} \text{ s} , \\
m_P &= \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-8} \text{ kg} , \\
&= 2.177 \times 10^{-5} \text{ g} , \\
E_P &= \sqrt{\frac{\hbar c^5}{G}} = 1.221 \times 10^{19} \text{ GeV} .
\end{aligned}$$

We do not have a complete quantum theory of gravity, but we expect the Planck scale to be the scale at which the effects of quantum gravity become significant. That is, we expect the effects of quantum gravity to be important for processes that involve distances of order ℓ_P or less, times of order t_P or less, or particles with masses of order m_P or greater, or energies of order E_P or greater.

CHEMICAL EQUILIBRIUM:

(This topic will NOT be included on Quiz 3, but the formulas are nonetheless included here for logical completeness. They will be relevant to Problem Set 9.)

General Ideal Gas, Relativistic or Not, Bosons or Fermions:

The number density of particles of type i with momenta within a box of size d^3p centered at $\vec{p}$ is given by

$$n_{i,\vec{p}}(\vec{p})\,\mathrm{d}^3p = \frac{\bar{g}_i}{(2\pi\hbar)^3}\,\frac{\mathrm{d}^3p}{\left[\exp\left(\frac{E_i(p)-\mu_i}{kT}\right) \pm 1\right]} ,$$

where

$$\begin{aligned}
\bar{g}_i &= \text{number of spin states of particle} \\
E_i(p) &= \sqrt{m_i^2 c^4 + p^2 c^2} = \text{energy of particle with momentum } p \\
m_i &= \text{mass of particle} \\
\mu_i &= \text{chemical potential} \\
\pm &= + \text{ for fermions, and } - \text{ for bosons.}
\end{aligned}$$

Note that unlike the quantities g and g^* defined in the section on black-body radiation, $\bar{g}_i$ simply counts spin states, with no correction factor associated with fermions.

Chemical potentials are assigned initially to conserved quantities (e.g., electric charge, baryon number, or lepton number), and are a way of specifying how much of these quantities are present. The chemical potential of any type of particle is the sum of the chemical potentials of its conserved quantities. For example, a proton has one unit of baryon number and one unit of electric charge, so $\mu_p = \mu_{\text{baryon}} + \mu_{\text{charge}}$.

The number density of particle i is given by

$$n_i = \frac{\bar{g}_i}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 \, \mathrm{d}p}{\left[\exp\left(\frac{E(p)-\mu_i}{kT}\right) \pm 1\right]} \ ,$$

and the energy density is given by

$$u_i = \frac{\bar{g}_i}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 \, E(p) \, \mathrm{d}p}{\left[\exp\left(\frac{E(p)-\mu_i}{kT}\right) \pm 1\right]} \ .$$

Ideal Dilute Gas of Nonrelativistic Particles:

The nonrelativistic, dilute gas limit of the formula above for the number density n_i is given by

$$n_i = \bar{g}_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \ .$$

$$\begin{aligned} \text{where } n_i &= \text{number density of particle} \\ \bar{g}_i &= \text{number of spin states of particle} \\ m_i &= \text{mass of particle} \\ \mu_i &= \text{chemical potential} \end{aligned}$$

The formula above assumes that the gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($e^{(\mu_i - m_i c^2)/kT} \ll 1$).

For any reaction that is consistent with all conservation laws, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Consequently, the product of the number densities on the left-hand side, divided by the product of the number densities on the right-hand side, is always independent of all chemical potentials. For example, since $H + \gamma \longleftrightarrow p + e^-$ (hydrogen atom + photon $\longleftrightarrow$ proton + electron) is a possible reaction, $\mu_H + \mu_\gamma = \mu_p + \mu_{e^-}$, and therefore

$$\frac{n_H\, n_\gamma}{n_p\, n_{e^-}}$$

can be evaluated using the formula above for number densities, and all chemical potentials will cancel out. (Photons have no conserved quantities, so $\mu_\gamma \equiv 0$, so $n_H/(n_p\, n_{e^-})$ is also independent of any chemical potentials.)

PROBLEM LIST

PROBLEM 1: DID YOU DO THE READING (2020)? *(25 points)*

(a) *(5 points)* In what sense is the Big Bang theory incomplete?

(i) It doesn't explain the origin of our universe. It only explains what happened after the "big bang."

(ii) It doesn't explain how matter clumped into galaxies.

(iii) It is incompatible with our observation of a homogeneous cosmic microwave background (CMB).

(iv) It always leads to a Big Crunch at the end of time, which is incompatible with the accelerated expansion we observe today.

(v) It isn't incomplete in any way; it is a fully self-contained theory that explains why the universe is as flat and homogeneous as we observe it.

(b) *(5 points)* The cosmic microwave background is nearly isotropic, up to some small fluctuations. Is our observation of these fluctuations from Earth affected by the motion of our galaxy?

(i) No, because our galaxy isn't located at any special point in space, so the universe we observe must be statistically homogeneous in every direction. That is to say, after averaging over small patches of the sky, every direction looks exactly the same.

(ii) No, because we can always consider the reference frame where our galaxy is at rest, and the fact that there is local thermal equilibrium in our universe means that we will observe a homogeneous pattern of radiation once we go to the locally equilibrated frame.

(iii) Yes. The rapid motion of our galaxy, because of its massive kinetic energy, leads to a strong gravitational field that distorts the CMB in such a way that most CMB photons we observe had their trajectory bent by more than 90 degrees.

(iv) Yes. Galaxies generally have "peculiar" velocities (small departures from a completely uniform Hubble expansion). This means that the CMB we observe is not perfectly isotropic because we are moving relative to the frame of reference in which the CMB is isotropic, leading to what is known as the "dipole distortion" of the CMB.

(v) No, because the peculiar velocity of the Milky way is too small to produce any significant distortion in our observations of the CMB.

(c) *(5 points)* Which of the following sequences of events is correctly ordered from earliest to latest?

(i) Radiation-matter equality, Inflation, Nucleosynthesis

(ii) Last scattering surface, Recombination, Formation of galaxies

(iii) Radiation-matter equality, Recombination, Photon decoupling

(iv) Nucleosynthesis, Recombination, Inflation

(v) Inflation, Formation of galaxies, Last scattering surface

(d) *(5 points)* What problematic aspects of the conventional Big Bang theory does the inflationary theory explain? Consider the following possibilities:

(A) The flatness problem

(B) The horizon problem

(C) The monopole problem

Which one of the following combinations is the best answer to the question?

(i) Only (A)

(ii) Only (B)

(iii) Only (C)

(iv) (A) and (B)

(v) (A), (B), and (C)

(e) *(5 points)* Ryden gives the equation of motion for the inflaton field as

$$\ddot{\phi} + 3H(t)\,\dot{\phi} = -\hbar c^3 \frac{dV}{d\phi} \ .$$

Ryden explains that the inflaton field normally reaches terminal velocity. Explain in a sentence or two what this means.

PROBLEM 2: DID YOU DO THE READING (2018)? *(20 points)*

(a) *(5 points)* Which one of the following statements about CMB is NOT correct?

(i) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(ii) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725$K.

(iii) After the dipole distortion of the CMB is subtracted away, the temperature of the CMB varies by 0.3 microKelvin across the sky.

(iv) The photons of the CMB have mostly been traveling on straight lines since they were last scattered at $t \approx 370,000$ yr, at a location called the surface of last scattering.

(b) *(5 points)* The nonuniformities in the cosmic microwave background allow us to measure the ripples in the mass density of the universe at the time when the plasma combined to form neutral atoms, about 300,000 - 400,000 years after the big bang. These ripples are crucial for understanding what happened later, since they are the seeds which led to the complicated tapestry of galaxies, clusters of galaxies, and voids. Which of the following sentences describes how these ripples are created in the context of inflationary models:

(i) Magnetic monopoles can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(ii) Cosmic strings, which are linelike topological defects, can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(iii) They are generated by quantum fluctuations during inflation.

(iv) Since the early universe was very hot, there were large thermal fluctuations which ultimately evolved into the ripples in the mass density.

(c) *(5 points)* In Chapter 8 of *The First Three Minutes*, Steven Weinberg describes the future of the universe (assuming, as was thought then to be the case, that the cosmological constant is zero). One possibility that he discusses is that the cosmic matter density could be greater than the critical density. Assuming that we live in such a universe, which of the following statements is NOT true?

(i) The universe is finite and its expansion will eventually cease, giving way to an accelerating contraction.

(ii) Three minutes after the temperature reaches a thousand million degrees (10^9 K), the laws of physics guarantee that the universe will crunch, and time will stop.

(iii) During at least the early part of the contracting phase, we will be able to observe both redshifts and blueshifts.

(iv) When the universe has recontracted to one-hundredth its present size, the radiation background will begin to dominate the sky, with a temperature of about 300 K.

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(iv) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(v) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(vi) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

PROBLEM 3: DID YOU DO THE READING? (2016) *(25 points)*

Except for part (d), you should answer these questions by circling the one statement that is correct.

(a) *(5 points)* In the Epilogue of *The First Three Minutes*, Steve Weinberg wrote: "The more the universe seems comprehensible, the more it also seems pointless." The sentence was qualified, however, by a closing paragraph that points out that

(i) the quest of the human race to create a better life for all can still give meaning to our lives.

(ii) if the universe cannot give meaning to our lives, then perhaps there is an afterlife that will.

(iii) the complexity and beauty of the laws of physics strongly suggest that the universe must have a purpose, even if we are not aware of what it is.

(iv) the effort to understand the universe gives human life some of the grace of tragedy.

(b) *(5 points)* In the Afterword of *The First Three Minutes*, Weinberg discusses the baryon number of the universe. (The baryon number of any system is the total number of protons and neutrons (and certain related particles known as hyperons) minus the number of their antiparticles (antiprotons, antineutrons, antihyperons) that are contained in the system.) Weinberg concluded that

(i) baryon number is exactly conserved, so the total baryon number of the universe must be zero. While nuclei in our part of the universe are composed of protons and neutrons, the universe must also contain antimatter regions in which nuclei are composed of antiprotons and antineutrons.

(ii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. Since baryon number is conserved, this can only be explained by assuming that the excess baryons were put in at the beginning.

(iii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. This can be taken as a positive hint that baryon number is not conserved, which can happen if there exist as yet undetected heavy "exotic" particles.

(iv) it is possible that baryon number is not exactly conserved, but even if that is the case, it is not possible that the observed excess of matter over antimatter can be explained by the very rare processes that violate baryon number conservation.

(c) *(5 points)* In discussing the COBE measurements of the cosmic microwave background, Ryden describes a dipole component of the temperature pattern, for which the temperature of the radiation from one direction is found to be hotter than the temperature of the radiation detected from the opposite direction.

(i) This discovery is important, because it allows us to pinpoint the direction of the point in space where the big bang occurred.

(ii) This is the largest component of the CMB anisotropies, amounting to a 10% variation in the temperature of the radiation.

(iii) In addition to the dipole component, the anisotropies also includes contributions from a quadrupole, octupole, etc., all of which are comparable in magnitude.

(iv) This pattern is interpreted as a simple Doppler shift, caused by the net motion of the COBE satellite relative to a frame of reference in which the CMB is almost isotropic.

(d) *(5 points)* (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

(e) *(5 points)* Inflation is driven by a field that is by definition called the *inflaton* field. In standard inflationary models, the field has the following properties:

(i) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its potential energy.

(ii) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its potential energy.

(iii) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(iv) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(v) The inflaton is a tensor field, which is responsible for only a small fraction of the energy density of the universe during inflation.

PROBLEM 4: DID YOU DO THE READING (2013)? *(35 points)*

This was Problem 1 of Quiz 3, 2013.

(a) *(5 points)* Ryden summarizes the results of the COBE satellite experiment for the measurements of the cosmic microwave background (CMB) in the form of three important results. The first was that, in any particular direction of the sky, the spectrum of the CMB is very close to that of an ideal blackbody. The FIRAS instrument on the COBE satellite could have detected deviations from the blackbody spectrum as small as $\Delta\epsilon/\epsilon \approx 10^{-n}$, where n is an integer. To within ± 1, what is n?

(b) *(5 points)* The second result was the measurement of a dipole distortion of the CMB spectrum; that is, the radiation is slightly blueshifted to higher temperatures in one direction, and slightly redshifted to lower temperatures in the opposite direction. To what physical effect was this dipole distortion attributed?

(c) *(5 points)* The third result concerned the measurement of temperature fluctuations after the dipole feature mentioned above was subtracted out. Defining

$$\frac{\delta T}{T}(\theta, \phi) \equiv \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} \ ,$$

where $\langle T \rangle = 2.725$ K, the average value of T, they found a root mean square fluctuation,

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} \ ,$$

equal to some number. To within an order of magnitude, what was that number?

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(e) *(5 points)* The flatness problem refers to the extreme fine-tuning that is needed in Ω at early times, in order for it to be as close to 1 today as we observe. Starting with the assumption that Ω today is equal to 1 within about 1%, one concludes that at one second after the big bang,

$$|\Omega - 1|_{t=1 \text{ sec}} < 10^{-m} \ ,$$

where m is an integer. To within $\pm$ 3, what is m?

(f) *(5 points)* The total energy density of the present universe consists mainly of baryonic matter, dark matter, and dark energy. Give the percentages of each, according to the best fit obtained from the Planck 2013 data. You will get full credit if the first (baryonic matter) is accurate to $\pm 2\%$, and the other two are accurate to within $\pm 5\%$.

(g) *(5 points)* Within the conventional hot big bang cosmology (without inflation), it is difficult to understand how the temperature of the CMB can be correlated at angular separations that are so large that the points on the surface of last scattering was separated from each other by more than a horizon distance. Approximately what angle, in degrees, corresponds to a separation on the surface last scattering of one horizon length? You will get full credit if your answer is right to within a factor of 2.

PROBLEM 5: DID YOU DO THE READING (2009)? *(25 points)*

This problem was Problem 1, Quiz 3, 2009.

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%).

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB.

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\text{bary}}/n_\gamma$.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct.

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5 (ii) 1/50 (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or F ?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. T or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or F ?

(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. T or F ?

PROBLEM 6: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

The following problem was Problem 2 of Quiz 3, 2016. It was also Problem 2 of Problem Set 7 (2016), except that some numerical constants have been changed, so the answers will not be identical.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^8(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{2}{3}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} \ .$$

Find the power n.

*** PROBLEM 7: TIME EVOLUTION OF A UNIVERSE INCLUDING A HYPOTHETICAL KIND OF MATTER** *(30 points)*

The following problem was Problem 2, Quiz 3, 2018.

Suppose that a flat universe includes nonrelativistic matter, radiation, and also mysticium, where the mass density of mysticium behaves as

$$\rho_{\text{myst}} \propto \frac{1}{a^5(t)}$$

as the universe expands. In this problem we will define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

where t_0 is the present time. For the following questions, you need not evaluate any of the integrals that might arise, but they must be integrals of explicit functions with explicit limits of integration; remember that $a(t)$ is not given. You may express your answers in terms of the present value of the Hubble expansion rate, H_0, and the various contributions to the present value of Ω: $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{myst},0}$.

(a) *(7 points)* Write an expression for the Hubble expansion rate $H(t)$.

(b) *(7 points)* Write an expression for the current age of the universe.

(c) *(3 points)* Write an expression for the time $t(x)$ in terms of the value of x.

(d) *(3 points)* Write an expression for the total mass density $\rho(x)$ as a function of x.

(e) *(10 points)* Write an expression for the physical horizon distance, $\ell_{p,\text{hor}}$.

PROBLEM 8: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos at temperature T. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗ PROBLEM 9: THE CONSEQUENCES OF AN ALT-PHOTON *(25 points)*

Suppose that, in addition to the particles that are known to exist, there also existed an alt-photon, which has exactly the properties of a photon: it is massless, has two spin states (or polarization states), and has the same interactions with other particles that photons do. Like photons, it is its own antiparticle.

(a) *(5 points)* In thermal equilibrium at temperature T, what is the total energy density of alt-photons?

(b) *(5 points)* In thermal equilibrium at temperature T, what is the number density of alt-photons?

(c) *(10 points)* In this situation, what would be the temperature ratios T_ν/T_γ and $T_\nu/T_{\text{alt}\gamma}$ today?

(d) *(5 points)* Would the existence of this particle increase or decrease the abundance of helium, or would it have no effect?

∗ PROBLEM 10: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X *(25 points)*

The following problem was Problem 3 of Quiz 3, 2016.

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, X^+, X^-, and X^0, all with masses 0.511 MeV/c^2, exactly the same as the electron. The X^- is the antiparticle of the X^+, and the X^0 is its own antiparticle. Since the X's are spin-1 particles with nonzero mass, each particle has three spin states.

The X's do not interact with neutrinos any more strongly than the electrons and positrons do, so when the X's freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total energy density of the X^+, X^-, and X^0 particles?

(b) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total number density of the X^+, X^-, and X^0 particles?

(c) *(10 points)* The X particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as kT decreases from values large compared to 0.511 MeV/c^2 to values that are small compared to it. If the X's, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio T_ν/T_γ, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

(d) *(5 points)* If the mass of the X's was, for example, 0.100 MeV/c^2, so that the electron-positron pairs froze out first, and then the X's froze out, would the final ratio T_ν/T_γ be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

PROBLEM 11: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that $0.75 \text{ MeV} \ll kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of g/cm^3.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

∗ PROBLEM 12: A NEW THEORY OF THE WEAK INTERACTIONS *(40 points)*

This problem was Problem 3, Quiz 3, 2009.

Suppose a New Theory of the Weak Interactions (NTWI) was proposed, which differs from the standard theory in two ways. First, the NTWI predicts that the weak interactions are somewhat weaker than in the standard model. In addition, the theory implies the existence of new spin-$\frac{1}{2}$ particles (fermions) called the R^+ and R^-, with a rest energy of 50 MeV (where 1 MeV $= 10^6$ eV). This problem will deal with the cosmological consequences of such a theory.

The NTWI will predict that the neutrinos in the early universe will decouple at a higher temperature than in the standard model. Suppose that this decoupling takes place at $kT \approx 200$ MeV. This means that when the neutrinos cease to be thermally coupled to the rest of matter, the hot soup of particles would contain not only photons, neutrinos, and e^+-e^- pairs, but also μ^+, μ^-, π^+, π^-, and π^0 particles, along with the R^+-R^- pairs. (The muon is a particle which behaves almost identically to an electron, except that its rest energy is 106 MeV. The pions are the lightest of the mesons, with zero angular momentum and rest energies of 135 MeV and 140 MeV for the neutral and charged pions, respectively. The π^+ and π^- are antiparticles of each other, and the π^0 is its own antiparticle. Zero angular momentum implies a single spin state.) You may assume that the universe is flat.

(a) *(10 points)* According to the standard particle physics model, what is the mass density ρ of the universe when $kT \approx 200$ MeV? What is the value of ρ at this temperature, according to NTWI? Use either g/cm^3 or kg/m^3. (If you wish, you can save time by not carrying out the arithmetic. If you do this, however, you should give the answer in "calculator-ready" form, by which I mean an expression involving pure numbers (no units), with any necessary conversion factors included, and with the units of the answer specified at the end. For example, if asked how far light travels in 5 minutes, you could answer $2.998 \times 10^8 \times 5 \times 60$ m.)

(b) *(10 points)* According to the standard model, the temperature today of the thermal neutrino background should be $(4/11)^{1/3} T_\gamma$, where T_γ is the temperature of the thermal photon background. What does the NTWI predict for the temperature of the thermal neutrino background?

(c) *(10 points)* According to the standard model, what is the ratio today of the number density of thermal neutrinos to the number density of thermal photons? What is this ratio according to NTWI?

(d) *(10 points)* Since the reactions which interchange protons and neutrons involve neutrinos, these reactions "freeze out" at roughly the same time as the neutrinos decouple. At later times the only reaction which effectively converts neutrons to protons is the free decay of the neutron. Despite the fact that neutron decay is a weak interaction, we will assume that it occurs with the usual 15 minute mean lifetime. Would the helium abundance predicted by the NTWI be higher or lower than the prediction of the standard model? To within 5 or 10%, what would the NTWI predict for the percent abundance (by weight) of helium in the universe? (As in part (a), you can either carry out the arithmetic, or leave the answer in calculator-ready form.)

Useful information: The proton and neutron rest energies are given by $m_p c^2 =$ 938.27 MeV and $m_n c^2 = 939.57$ MeV, with $(m_n - m_p)c^2 = 1.29$ MeV. The mean lifetime for the neutron decay, $n \to p + e^- + \bar{\nu}_e$, is given by $\tau = 886$ s.

PROBLEM 13: DOUBLING OF ELECTRONS *(10 points)*

The following was on Quiz 3, 2011 (Problem 4):

Suppose that instead of one species of electrons and their antiparticles, suppose there was also another species of electron-like and positron-like particles. Suppose that the new species has the same mass and other properties as the electrons and positrons. If this were the case, what would be the ratio T_ν/T_γ of the temperature today of the neutrinos to the temperature of the CMB photons.

PROBLEM 14: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 15: THE TIME t_d OF DECOUPLING *(25 points)*

The following problem was Problem 4 of Quiz 3, 2016.

The process by which the photons of the cosmic microwave background stop scattering and begin to travel on straight lines is called *decoupling*, and it happens at a photon temperature of about $T_d \approx 3,000$ K. In Lecture Notes 6 we estimated the time t_d of decoupling, working in the approximation that the universe has been matter-dominated from that time to the present. We found a value of 370,000 years. In this problem we will remove this approximation, although we will not carry out the numerical evaluation needed to compare with the previous answer.

(a) *(5 points)* Let us define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

as on the formula sheets, where t_0 is the present time. What is the value of $x_d \equiv x(t_d)$? Assume that the entropy of photons is conserved from time t_d to the present, and let T_0 denote the present photon temperature.

(b) *(5 points)* Assume that the universe is flat, and that $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ denote the present contributions to Ω from nonrelativistic matter, radiation, and vacuum energy, respectively. Let H_0 denote the present value of the Hubble expansion rate. Write an expression in terms of these quantities for $\mathrm{d}x/\mathrm{d}t$, the derivative of x with respect to t. *Hint: you may use formulas from the formula sheet without derivation, so this problem should require essentially no work. To receive full credit, your answer should include only terms that make a nonzero contribution to the answer.*

(c) *(5 points)* Write an expression for t_d. If your answer involves an integral, you need not try to evaluate it, but you should be sure that the limits of integration are clearly shown.

(d) *(10 points)* Now suppose that in addition to the constituents described in part (b), the universe also contains some of the fictitious material from part (a) of Problem 6 (Quiz 3 Review Problems, 2022), with

$$\rho(t) \propto \frac{1}{a^8(t)} .$$

Denote the present contribution to Ω from this fictitious material as $\Omega_{f,0}$. The universe is still assumed to be flat, so the numerical values of $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ must sum to a smaller value than in parts (b) and (c). With this extra contribution to the mass density of the universe, what is the new expression for t_d?

∗ PROBLEM 16: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

The following problem was Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\text{phys},0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\text{phys,e}}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

PROBLEM 17: SECOND HUBBLE CROSSING *(40 points)*

This problem was Problem 3, Quiz 3, 2007. In 2018 we have not yet talked about Hubble crossings and the evolution of density perturbations, so this problem would not be fair as worded. Actually, however, you have learned how to do these calculations, so the problem would be fair if it described in more detail what needs to be calculated.

In Problem Set 9 (2007) we calculated the time $t_{H1}(\lambda)$ of the first Hubble crossing for a mode specified by its (physical) wavelength λ at the present time. In this problem we will calculate the time $t_{H2}(\lambda)$ of the second Hubble crossing, the time at which the growing Hubble length $cH^{-1}(t)$ catches up to the physical wavelength, which is also growing. At the time of the second Hubble crossing for the wavelengths of interest, the universe can be described very simply: it is a radiation-dominated flat universe. However, since λ is defined as the present value of the wavelength, the evolution of the universe between $t_{H2}(\lambda)$ and the present will also be relevant to the problem. We will need to use methods, therefore, that allow for both the matter-dominated era and the onset of the dark-energy-dominated era. As in Problem Set 9 (2007), the model universe that we consider will be described by the WMAP 3-year best fit parameters:

Hubble expansion rate	H_0	$= 73.5\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$
Nonrelativistic mass density	Ω_m	$= 0.237$
Vacuum mass density	$\Omega_{\rm vac}$	$= 0.763$
CMB temperature	$T_{\gamma,0}$	$= 2.725$ K

The mass densities are defined as contributions to Ω, and hence describe the mass density of each constituent relative to the critical density. Note that the model is exactly flat, so you need not worry about spatial curvature. Here you are not expected to give a numerical answer, so the above list will serve only to define the symbols that can appear in your answers, along with λ and the physical constants G, $\hbar$, c, and k.

(a) *(5 points)* For a radiation-dominated flat universe, what is the Hubble length $\ell_H(t) \equiv cH^{-1}(t)$ as a function of time t?

(b) *(10 points)* The second Hubble crossing will occur during the interval

$$30 \text{ sec} \ll t \ll 50{,}000 \text{ years},$$

when the mass density of the universe is dominated by photons and neutrinos. During this era the neutrinos are a little colder than the photons, with $T_\nu = (4/11)^{1/3} T_\gamma$. The total energy density of the photons and neutrinos together can be written as

$$u_{\rm tot} = g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{(\hbar c)^3} \ .$$

What is the value of g_1? (For the following parts you can treat g_1 as a given variable that can be left in your answers, whether or not you found it.)

(c) *(10 points)* For times in the range described in part (b), what is the photon temperature $T_\gamma(t)$ as a function of t?

(d) *(15 points)* Finally, we are ready to find the time $t_{H2}(\lambda)$ of the second Hubble crossing, for a given value of the physical wavelength λ today. Making use of the previous results, you should be able to determine $t_{H2}(\lambda)$. If you were not able to answer some of the previous parts, you may leave the symbols $\ell_H(t)$, g_1, and/or $T_\gamma(t)$ in your answer.

PROBLEM 18: EVOLUTION OF FLATNESS *(15 points)*

The following problem was Problem 3, Quiz 3, 2004.

The "flatness problem" is related to the fact that during the evolution of the standard cosmological model, Ω is always driven away from 1.

(a) *(9 points)* During a period in which the universe is matter-dominated (meaning that the only relevant component is nonrelativistic matter), the quantity

$$\frac{\Omega - 1}{\Omega}$$

grows as a power of t, provided that Ω is near 1. Show that this is true, and derive the power. (Stating the right power without a derivation will be worth 3 points.)

(b) *(6 points)* During a period in which the universe is radiation-dominated, the same quantity will grow like a different power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will again be worth 3 points.)

In each part, you may assume that the universe was *always* dominated by the specified form of matter.

PROBLEM 19: THE EVENT HORIZON FOR OUR UNIVERSE *(25 points)*

The following problem was Problem 3 from Quiz 3, 2013.

We have learned that the expansion history of our universe can be described in terms of a small set of numbers: $\Omega_{m,0}$, the present contribution to Ω from nonrelativistic matter; $\Omega_{\text{rad},0}$, the present contribution to Ω from radiation; Ω_{vac}, the present contribution to Ω from vacuum energy; and H_0, the present value of the Hubble expansion rate. The best estimates of these numbers are consistent with a flat universe, so we can take $k = 0$, $\Omega_{m,0} + \Omega_{\text{rad},0} + \Omega_{\text{vac}} = 1$, and we can use the flat Robertson-Walker metric,

$$\mathrm{d}s^2 = -c^2\,dt^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \; .$$

(a) *(5 points)* Suppose that we are at the origin of the coordinate system, and that at the present time t_0 we emit a spherical pulse of light. It turns out that there is a maximum coordinate radius $r = r_{\rm max}$ that this pulse will ever reach, no matter how long we wait. (The pulse will never actually reach $r_{\rm max}$, but will reach all r such that $0 < r < r_{\rm max}$.) $r_{\rm max}$ is the coordinate of what is called the *event horizon*: events that happen now at $r \geq r_{\rm max}$ will never be visible to us, assuming that we remain at the origin. Assuming for this part that the function $a(t)$ is a known function, write an expression for $r_{\rm max}$. Your answer should be expressed as an integral, which can involve $a(t)$, t_0, and any of the parameters defined in the preamble. [*Advice:* If you cannot answer this, you should still try part (c).]

(b) *(10 points)* Since $a(t)$ is not known explicitly, the answer to the previous part is difficult to use. Show, however, that by changing the variable of integration, you can rewrite the expression for $r_{\rm max}$ as a definite integral involving only the parameters specified in the preamble, without any reference to the function $a(t)$, except perhaps to its present value $a(t_0)$. You are not expected to evaluate this integral. [*Hint:* One method is to use

$$x = \frac{a(t)}{a(t_0)}$$

as the variable of integration, just as we did when we derived the first of the expressions for t_0 shown in the formula sheets.]

(c) *(10 points)* Astronomers often describe distances in terms of redshifts, so it is useful to find the redshift of the event horizon. That is, if a light ray that originated at $r = r_{\rm max}$ arrived at Earth today, what would be its redshift $z_{\rm eh}$ (eh = event horizon)? You are not asked to find an explicit expression for $z_{\rm eh}$, but instead an equation that could be solved numerically to determine $z_{\rm eh}$. For this part you can treat $r_{\rm max}$ as given, so it does not matter if you have done parts (a) and (b). You will get half credit for a correct answer that involves the function $a(t)$, and full credit for a correct answer that involves only explicit integrals depending only on the parameters specified in the preamble, and possibly $a(t_0)$.

PROBLEM 20: A MESSAGE FROM A DISTANT GALAXY *(35 points)*

This problem was Problem 3 on Quiz 3 of 2020.

Our universe, at the present time, is well-described as a flat universe with a Hubble expansion rate H_0. Nonrelativistic matter comprises a fraction $\Omega_{m,0}$ of the critical density, radiation comprises a fraction $\Omega_{r,0}$, and vacuum energy comprises a fraction $\Omega_{v,0}$. For the following questions, you may leave your answers in terms of integrals that you do not evaluate, but you should be sure to specify the limits of integration. The answer to each part can be expressed in terms of the given variables, and/or the variables that represent the answers to any previous part.

(a) *(5 points)* Suppose that a galaxy G is observed at a redshift z_G. What was the cosmic time of emission t_e of the light that we are now receiving from G? ("Cosmic time" is the time variable of the Robertson-Walker metric.)

(b) *(10 points)* What is the present value of the physical distance $\ell_{p,0}$ to the galaxy G?

(c) *(10 points)* Suppose that a civilization in galaxy G sends us a light signal, originating at the present cosmic time. The light signal will reach us at a time called t_r. Write an equation that would determine the value of $a_r \equiv a(t_r)$, the scale factor at the time that we receive the signal. You will be happy to know that you are not expected to solve this equation.

(d) *(10 points)* At what cosmic time t_r would we receive the signal from the galaxy G?

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING (2020)? *(25 points)*

(a) *(5 points)* In what sense is the Big Bang theory incomplete?

(i) It doesn't explain the origin of our universe. It only explains what happened after the "big bang."

(ii) It doesn't explain how matter clumped into galaxies.

(iii) It is incompatible with our observation of a homogeneous cosmic microwave background (CMB).

(iv) It always leads to a Big Crunch at the end of time, which is incompatible with the accelerated expansion we observe today.

(v) It isn't incomplete in any way; it is a fully self-contained theory that explains why the universe is as flat and homogeneous as we observe it.

(b) *(5 points)* The cosmic microwave background is nearly isotropic, up to some small fluctuations. Is our observation of these fluctuations from Earth affected by the motion of our galaxy?

(i) No, because our galaxy isn't located at any special point in space, so the universe we observe must be statistically homogeneous in every direction. That is to say, after averaging over small patches of the sky, every direction looks exactly the same.

(ii) No, because we can always consider the reference frame where our galaxy is at rest, and the fact that there is local thermal equilibrium in our universe means that we will observe a homogeneous pattern of radiation once we go to the locally equilibrated frame.

(iii) Yes. The rapid motion of our galaxy, because of its massive kinetic energy, leads to a strong gravitational field that distorts the CMB in such a way that most CMB photons we observe had their trajectory bent by more than 90 degrees.

(iv) Yes. Galaxies generally have "peculiar" velocities (small departures from a completely uniform Hubble expansion). This means that the CMB we observe is not perfectly isotropic because we are moving relative to the frame of reference in which the CMB is isotropic, leading to what is known as the "dipole distortion" of the CMB.

(v) No, because the peculiar velocity of the Milky way is too small to produce any significant distortion in our observations of the CMB.

(c) *(5 points)* Which of the following sequences of events is correctly ordered from earliest to latest?

(i) Radiation-matter equality, Inflation, Nucleosynthesis

(ii) Last scattering surface, Recombination, Formation of galaxies

(iii) $\boxed{\text{Radiation-matter equality, Recombination, Photon decoupling}}$

(iv) Nucleosynthesis, Recombination, Inflation

(v) Inflation, Formation of galaxies, Last scattering surface

(d) *(5 points)* What problematic aspects of the conventional Big Bang theory does the inflationary theory explain? Consider the following possibilities:

(A) The flatness problem

(B) The horizon problem

(C) The monopole problem

Which one of the following combinations is the best answer to the question?

(i) Only (A)

(ii) Only (B)

(iii) Only (C)

(iv) (A) and (B)

(v) $\boxed{\text{(A), (B), and (C)}}$

(e) *(5 points)* Ryden gives the equation of motion for the inflaton field as

$$\ddot{\phi} + 3H(t)\,\dot{\phi} = -\hbar c^3 \frac{dV}{d\phi} \ .$$

Ryden explains that the inflaton field normally reaches terminal velocity. Explain in a sentence or two what this means.

If we pretend that ϕ represents the position coordinate of a particle moving in one dimension, the equation would describe a particle of mass 1 moving in the potential energy function $\hbar c^3 V(\phi)$, with a drag term $3H(t)\,\dot{\phi}$. If the force is changing slowly or not at all, the particle will accelerate until the drag force is equal in magnitude to the force due to the potential energy, and then the $\ddot{\phi}$ term will become negligible:

$$3H(t)\,\dot{\phi} \simeq -\hbar c^3 \frac{dV}{d\phi} \ .$$

PROBLEM 2: DID YOU DO THE READING (2018)? *(20 points)*

(a) *(5 points)* Which one of the following statements about CMB is NOT correct?

(i) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(ii) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is <T> = 2.725K.

$\boxed{\text{(iii)}}$ After the dipole distortion of the CMB is subtracted away, the temperature of the CMB varies by 0.3 microKelvin across the sky.

(iv) The photons of the CMB have mostly been traveling on straight lines since they were last scattered at $t \approx 370,000$ yr, at a location called the surface of last scattering.

[Comment: The actual variation is about 30 microKelvin, or maybe a few times that much. Ryden quotes the COBE root mean square fractional variation of the CMB temperature as

$$< \left(\frac{\delta T}{T} \right)^2 >^{1/2} = 1.1 \times 10^{-5}$$

as Eq. (8.8) (2nd Edition), which gives a value of about 30 microKelvin, given that $T \approx 3\,K$. In Lecture Notes 2 we quoted a value of 4.14×10^{-5} computed from Planck data. The root mean square fluctuations increase with better angular resolution, because fluctuations with small angular wavelengths are not seen unless the resolution is high.

(b) *(5 points)* The nonuniformities in the cosmic microwave background allow us to measure the ripples in the mass density of the universe at the time when the plasma combined to form neutral atoms, about 300,000 - 400,000 years after the big bang. These ripples are crucial for understanding what happened later, since they are the seeds which led to the complicated tapestry of galaxies, clusters of galaxies, and voids. Which of the following sentences describes how these ripples are created in the context of inflationary models:

(i) Magnetic monopoles can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

(ii) Cosmic strings, which are linelike topological defects, can form randomly during the grand unified theory phase transition, resulting in nonuniformities in the mass density.

$\boxed{\text{(iii)}}$ They are generated by quantum fluctuations during inflation.

(iv) Since the early universe was very hot, there were large thermal fluctuations which ultimately evolved into the ripples in the mass density.

(c) *(5 points)* In Chapter 8 of *The First Three Minutes*, Steven Weinberg describes the future of the universe (assuming, as was thought then to be the case, that the cosmological constant is zero). One possibility that he discusses is that the cosmic matter density could be greater than the critical density. Assuming that we live in such a universe, which of the following statements is NOT true?

(i) The universe is finite and its expansion will eventually cease, giving way to an accelerating contraction.

$\boxed{\text{(ii)}}$ Three minutes after the temperature reaches a thousand million degrees (10^9 K), the laws of physics guarantee that the universe will crunch, and time will stop.

(iii) During at least the early part of the contracting phase, we will be able to observe both redshifts and blueshifts.

(iv) When the universe has recontracted to one-hundredth its present size, the radiation background will begin to dominate the sky, with a temperature of about 300 K.

[Comment: Weinberg is very clear no speculations about the end of the universe are guaranteed to be true: "Does time really have to stop some three minutes after the temperature reaches a thousand million degrees? Obviously, we cannot be sure. All the uncertainties that we met in the preceding chapter, in trying to explore the first hundredth of a second, will return to perplex us as we look into the last hundredth of a second."]

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(iv) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

$\boxed{\text{(v)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(vi) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

[Comment: Ryden discusses the Sachs-Wolfe effect on pp. 161–162 (2nd Edition).]

PROBLEM 3: DID YOU DO THE READING (2016)? *(25 points)*

Except for part (d), you should answer these questions by circling the one statement that is correct.

(a) *(5 points)* In the Epilogue of *The First Three Minutes*, Steve Weinberg wrote: "The more the universe seems comprehensible, the more it also seems pointless." The sentence was qualified, however, by a closing paragraph that points out that

(i) the quest of the human race to create a better life for all can still give meaning to our lives.

(ii) if the universe cannot give meaning to our lives, then perhaps there is an afterlife that will.

(iii) the complexity and beauty of the laws of physics strongly suggest that the universe must have a purpose, even if we are not aware of what it is.

$\boxed{\text{(iv)}}$ the effort to understand the universe gives human life some of the grace of tragedy.

(b) *(5 points)* In the Afterword of *The First Three Minutes*, Weinberg discusses the baryon number of the universe. (The baryon number of any system is the total number of protons and neutrons (and certain related particles known as hyperons) minus the number of their antiparticles (antiprotons, antineutrons, antihyperons) that are contained in the system.) Weinberg concluded that

(i) baryon number is exactly conserved, so the total baryon number of the universe must be zero. While nuclei in our part of the universe are composed of protons and neutrons, the universe must also contain antimatter regions in which nuclei are composed of antiprotons and antineutrons.

(ii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. Since baryon number is conserved, this can only be explained by assuming that the excess baryons were put in at the beginning.

$\boxed{\text{(iii)}}$ there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. This can be taken as a positive hint that baryon number is not conserved, which can happen if there exist as yet undetected heavy "exotic" particles.

(iv) it is possible that baryon number is not exactly conserved, but even if that is the case, it is not possible that the observed excess of matter over antimatter can be explained by the very rare processes that violate baryon number conservation.

Explanation: All students were given credit for this part, whether they answered it correctly or not. I was in San Francisco when I made up this quiz, and due to poor planning I did not have my copy of *The First Three Minutes.* So I found a version online, but I could only find the British version, published by Flamingo/Fontana Paperbacks, rather than the US version published by Basic Books. I assumed that the "Afterword" in the two versions would be the same, but I was wrong! So this question was based on a different "Afterword" than the one that you read. 55% of you still got it right, but obviously the question was not fair. Apologies.

(c) *(5 points)* In discussing the COBE measurements of the cosmic microwave background, Ryden describes a dipole component of the temperature pattern, for which the temperature of the radiation from one direction is found to be hotter than the temperature of the radiation detected from the opposite direction.

(i) This discovery is important, because it allows us to pinpoint the direction of the point in space where the big bang occurred.

(ii) This is the largest component of the CMB anisotropies, amounting to a 10% variation in the temperature of the radiation.

(iii) In addition to the dipole component, the anisotropies also include contributions from a quadrupole, octupole, etc., all of which are comparable in magnitude.

$\boxed{\text{(iv)}}$ This pattern is interpreted as a simple Doppler shift, caused by the net motion of the COBE satellite relative to a frame of reference in which the CMB is almost isotropic.

Explanation: (i) is nonsense, since the conventional big bang theory descibes a completely homogeneous universe, which has no single point at which the big bang occurred. (ii) is wrong, because the variations in the temperature of the CMB are much smaller than 10%. The dipole term has a magnitude of about 1/1000 of the mean temperature. (iii) is wrong because the dipole is not comparable to the other terms, because they have magnitudes of only about 1/100,000 of the mean.

(d) *(5 points)* (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

$\boxed{\text{(ii)}}$ After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Explanation: The right value is

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5} \ .$$

(e) *(5 points)* Inflation is driven by a field that is by definition called the *inflaton* field. In standard inflationary models, the field has the following properties:

$\boxed{\text{(i)}}$ The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its potential energy.

(ii) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its potential energy.

(iii) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(iv) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(v) The inflaton is a tensor field, which is responsible for only a small fraction of the energy density of the universe during inflation.

Explanation: These facts were mentioned in both Section 11.5 (*The Physics of Inflation)* of Ryden's book, and also in the article that you were asked to read called *Inflation and the New Era of High-Precision Cosmology,* written by me for the Physics Department 2002 newsletter.

PROBLEM 4: DID YOU DO THE READING (2013)? *(35 points)*

(a) *(5 points)* Ryden summarizes the results of the COBE satellite experiment for the measurements of the cosmic microwave background (CMB) in the form of three important results. The first was that, in any particular direction of the sky, the spectrum of the CMB is very close to that of an ideal blackbody. The FIRAS instrument on the COBE satellite could have detected deviations from the blackbody spectrum as small as $\Delta\epsilon/\epsilon \approx 10^{-n}$, where n is an integer. To within ± 1, what is n?

Answer: $n = 4$

(b) *(5 points)* The second result was the measurement of a dipole distortion of the CMB spectrum; that is, the radiation is slightly blueshifted to higher temperatures in one direction, and slightly redshifted to lower temperatures in the opposite direction. To what physical effect was this dipole distortion attributed?

Answer: The large dipole in the CMB is attributed to the motion of the satellite relative to the frame in which the CMB is very nearly isotropic. (The entire Local Group is moving relative to this frame at a speed of about $0.002c$.)

(c) *(5 points)* The third result concerned the measurement of temperature fluctuations after the dipole feature mentioned above was subtracted out. Defining

$$\frac{\delta T}{T}(\theta, \phi) \equiv \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} \ ,$$

where $\langle T \rangle = 2.725$ K, the average value of T, they found a root mean square fluctuation,

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} \ ,$$

equal to some number. To within an order of magnitude, what was that number?

Answer:

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5} \ .$$

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

$\boxed{\text{(iii)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(e) *(5 points)* The flatness problem refers to the extreme fine-tuning that is needed in Ω at early times, in order for it to be as close to 1 today as we observe. Starting with the assumption that Ω today is equal to 1 within about 1%, one concludes that at one second after the big bang,

$$|\Omega - 1|_{t=1\ \text{sec}} < 10^{-m} \ ,$$

where m is an integer. To within $\pm$ 3, what is m?

Answer: $m = 18$. (See the derivation in Lecture Notes 8.)

(f) *(5 points)* The total energy density of the present universe consists mainly of baryonic matter, dark matter, and dark energy. Give the percentages of each, according to the best fit obtained from the Planck 2013 data. You will get full credit if the first (baryonic matter) is accurate to $\pm 2\%$, and the other two are accurate to within $\pm 5\%$.

Answer: Baryonic matter: 5%. Dark matter: 26.5%. Dark energy: 68.5%. The Planck 2013 numbers were given in Lecture Notes 7. To the requested accuracy, however, numbers such as Ryden's Benchmark Model would also be satisfactory.

(g) *(5 points)* Within the conventional hot big bang cosmology (without inflation), it is difficult to understand how the temperature of the CMB can be correlated at angular separations that are so large that the points on the surface of last scattering was separated from each other by more than a horizon distance. Approximately what angle, in degrees, corresponds to a separation on the surface last scattering of one horizon length? You will get full credit if your answer is right to within a factor of 2.

Answer: Ryden gives 1° as the angle subtended by the Hubble length on the surface of last scattering. For a matter-dominated universe, which would be a good model for our universe, the horizon length is twice the Hubble length. Any number from 1° to 5° was considered acceptable.

PROBLEM 5: DID YOU DO THE READING (2009)? *(25 points)*

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%). $\boxed{2.725\,\text{K}}$

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

The dipole anisotropy corresponds to a "peculiar velocity" (that is, velocity which is not due to the expansion of the universe) of $630 \pm 20\,\mathrm{km\,s^{-1}}$*, or in terms of the speed of light,* $\boxed{v/c \approx 2 \times 10^{-3}}$.

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB. $\boxed{1.1 \times 10^{-5}}$

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\rm bary}/n_\gamma$.

The WMAP 5-year value for $\eta = n_b/n_\gamma = \boxed{(6.225 \pm 0.170) \times 10^{-10}}$*, which to closest order of magnitude is* 10^{-9}.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct. $\boxed{\sim 1^\circ}$

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5 $\boxed{\text{(ii) } 1/50}$ (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

This is not a number one has to commit to memory if one can remember the temperature of (re)combination in eV*, or if only in* K *along with the conversion factor (*$k \approx 10^{-4}\,\mathrm{eV\,K^{-1}}$*). One can then calculate that near recombination,* $kT_\gamma/Q_H \approx (10^{-4}\,\mathrm{eV\,K^{-1}})(3000\,\mathrm{K})/(13.6\,\mathrm{eV}) \approx 1/45$.

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

$\boxed{\text{(iii)}}$ Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

Explanation: Denser regions have a deeper (more negative) gravitational potential. Photons which travel through a spatially varying potential acquire a redshift or blueshift depending on whether they are going up or down the potential, respectively. Photons originating in the denser regions start at a lower potential and must climb out, so they end up being redshifted relative to their original energies.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or $\boxed{\text{F}}$?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. $\boxed{\text{T}}$ or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or $\boxed{\text{F}}$?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or $\boxed{\text{F}}$?

(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. $\boxed{\text{T}}$ or F ?

PROBLEM 6: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

(a) *(8 points)* This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho c^2 a^3\right) = -p\frac{d}{dt}\left(a^3\right) \ .$$

Setting

$$\rho = \frac{\alpha}{a^8}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{a^5}\right) = -p\frac{d}{dt}\left(a^3\right) \ ,$$

which implies

$$-5\frac{\alpha c^2}{a^6}\frac{da}{dt} = -3pa^2\frac{da}{dt} \ .$$

Thus

$$p = \frac{5}{3}\frac{\alpha c^2}{a^8} = \boxed{\frac{5}{3}\rho c^2 \ .}$$

Alternatively, one may start from the equation for the time derivative of ρ,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Since $\rho = \frac{\alpha}{a^8}$, we take the time derivative to find $\dot{\rho} = -8(\dot{a}/a)\rho$, and therefore

$$-8\frac{\dot{a}}{a}\rho = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

and therefore

$$p = \frac{5}{3}\rho c^2 .$$

(b) *(9 points)* For a flat universe, the Friedmann equation reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

Using $\rho \propto 1/a^8$, this implies that

$$\dot{a} = \frac{\beta}{a^3} \ ,$$

for some constant β. Rewriting this as

$$a^3\, da = \beta\, dt \ ,$$

we can integrate the equation to give

$$\frac{1}{4}a^4 = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $a = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{a \propto t^{1/4} \ .}$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that "You should be able to determine the function $a(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α that we used in part (a), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then a is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) *(8 points)* We start from the conservation of energy equation in the form

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Substituting $\dot{\rho} = -n(\dot{a}/a)\rho$ and $p = (2/3)\rho c^2$, we have

$$-nH\rho = -3H\left(\frac{5}{3}\rho\right)$$

and therefore

$$\boxed{n = 5.}$$

PROBLEM 7: TIME EVOLUTION OF A UNIVERSE INCLUDING A HYPOTHETICAL KIND OF MATTER *(30 points)*

Suppose that a flat universe includes nonrelativistic matter, radiation, and also mysticium, where the mass density of mysticium behaves as

$$\rho_{\rm myst} \propto \frac{1}{a^5(t)}$$

as the universe expands. In this problem we will define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

where t_0 is the present time. For the following questions, you need not evaluate any of the integrals that might arise, but they must be integrals of explicit functions with explicit limits of integration; remember that $a(t)$ is not given. You may express your answers in terms of the present value of the Hubble expansion rate, H_0, and the various contributions to the present value of Ω: $\Omega_{m,0}$, $\Omega_{\rm rad,0}$, and $\Omega_{\rm myst,0}$.

(a) *(7 points)* Write an expression for the Hubble expansion rate $H(x)$.

(b) *(7 points)* Write an expression for the current age of the universe.

(c) *(3 points)* Write an expression for the time $t(x)$ in terms of the value of x.

(d) *(3 points)* Write an expression for the total mass density $\rho(x)$ as a function of x.

(e) *(10 points)* Write an expression for present value of the physical horizon distance, $\ell_{p,\text{hor}}(t_0)$.

Solution:

(a) Since the universe is flat, the first Friedmann equation becomes

$$H^2 = \frac{8\pi}{3} G\rho \ ,$$

but then we can write ρ as

$$H^2 = \frac{8\pi}{3} G \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{\text{rad},0} \left[\frac{a(t_0)}{a(t)}\right]^4 + \rho_{\text{myst},0} \left[\frac{a(t_0)}{a(t)}\right]^5 \right\} \ .$$

Now use

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \quad \text{and} \quad \Omega \equiv \frac{\rho}{\rho_c} \ ,$$

so

$$\begin{aligned} H^2 &= \frac{H_0^2}{\rho_{c,0}} \left\{ \rho_{m,0} \left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{\text{rad},0} \left[\frac{a(t_0)}{a(t)}\right]^4 + \rho_{\text{myst},0} \left[\frac{a(t_0)}{a(t)}\right]^5 \right\} \\ &= H_0^2 \left\{ \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \frac{\Omega_{\text{myst},0}}{x^5} \right\} \ . \end{aligned}$$

Finally,

$$\boxed{H(x) = \frac{H_0}{x^2} \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \frac{\Omega_{\text{myst},0}}{x}} \ .}$$

(b) To find the current age t_0, we start with

$$H = \frac{\dot{a}}{a} = \frac{\dot{x}}{x} \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = xH \quad \Longrightarrow \quad \mathrm{d}t = \frac{\mathrm{d}x}{xH} \ .$$

So t_0 can be found by integrating over the range of x, from 0 to 1:

$$\begin{aligned} t_0 &= \int_0^1 \frac{\mathrm{d}x}{xH(x)} \\ &= \boxed{\frac{1}{H_0} \int_0^1 \frac{x \, \mathrm{d}x}{\sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \frac{\Omega_{\text{myst},0}}{x}}} \ .} \end{aligned}$$

(c) To find the time t corresponding to some value of x other than 1, one simply integrates $\mathrm{d}t$ from $x' = 0$ to $x' = x$:

$$t(x) = \int_0^x \frac{\mathrm{d}x'}{x' H(x')}$$

$$= \boxed{\frac{1}{H_0} \int_0^x \frac{x'\,\mathrm{d}x'}{\sqrt{\Omega_{m,0} x' + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x'}}}\ .}$$

(d) From the first Friedmann equation,

$$H^2 = \frac{8\pi}{3} G\rho \quad \Longrightarrow \quad \rho = \frac{3}{8\pi G} H^2(x)\ .$$

Given the answer in part (a), this becomes

$$\boxed{\rho(x) = \frac{3}{8\pi G} \frac{H_0^2}{x^4} \left[\Omega_{m,0} x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}\right]\ .}$$

(e) The general formula for the physical horizon distance is given on the formula sheet:

$$\ell_{p,\mathrm{hor}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt'\ .$$

Here we are not given the function $a(t)$, but we can change the variable of integration to integrate over x:

$$\mathrm{d}t' = \frac{\mathrm{d}t'}{\mathrm{d}a} \mathrm{d}a = \frac{1}{\dot{a}} \mathrm{d}a = \frac{1}{a} \frac{a}{\dot{a}} \mathrm{d}a = \frac{\mathrm{d}a}{aH(x)}\ .$$

So

$$\ell_{p,\mathrm{hor}}(t_0) = a(t_0) \int_0^{a(t_0)} \frac{c\,\mathrm{d}a}{a^2 H(a)}$$

$$= \int_0^1 \frac{c\,\mathrm{d}x}{x^2 H(x)}$$

$$= \boxed{\frac{c}{H_0} \int_0^1 \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0} x + \Omega_{\mathrm{rad},0} + \frac{\Omega_{\mathrm{myst},0}}{x}}}\ .}$$

PROBLEM 8: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$\begin{aligned} E &= \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{7\pi^4}{180\zeta(3)} kT \ .} \end{aligned}$$

Numerically, $E = 3.1514\, kT$.

> Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{135\zeta(3)} k \ .}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 9: THE CONSEQUENCES OF AN ALT-PHOTON *(25 points)*

Suppose that, in addition to the particles that are known to exist, there also existed an alt-photon, which has exactly the properties of a photon: it is massless, has two spin states (or polarization states), and has the same interactions with other particles that photons do. Like photons, it is its own antiparticle.

(a) *(5 points)* In thermal equilibrium at temperature T, what is the total energy density of alt-photons?

(b) *(5 points)* In thermal equilibrium at temperature T, what is the number density of alt-photons?

(c) *(10 points)* In this situation, what would be the temperature ratios T_ν / T_γ and $T_\nu / T_{\text{alt}\gamma}$ today?

(d) *(5 points)* Would the existence of this particle increase or decrease the abundance of helium, or would it have no effect?

Solution:

(a) The energy density will be the same as for photons, since there is no difference. The general formula is

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \ ,$$

as given on the formula sheets, and $g = 2$ for alt-photons (or photons), since there are two polarization states, and the particles are bosons. So

$$\boxed{u_{\text{alt}\gamma} = \frac{\pi^2}{15}\,\frac{(kT)^4}{(\hbar c)^3}\ .} \tag{S9.1}$$

(b) For the number density, the general formula is

$$n = g^* \frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\ ,$$

where $g^* = 2$ since again the alt-photons are bosons with two polarization states. So

$$\boxed{n_{\text{alt}\gamma} = 2\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\ .} \tag{S9.2}$$

(c) As in the actual scenario, the event that causes a temperature difference is the disappearance of the electron-positron pairs from the thermal equilibrium mix, which occurs as kT changes from values large compared to $m_e c^2 = 0.511$ MeV to values that are small compared to it. The key point is that this disappearance occurs after the neutrinos have decoupled from the other particles, so all of the entropy from the electron-positron pairs is given to the photons, and none is given to the neutrinos. In this case the entropy is given to both the photons and the alt-photons.

The general formula for entropy density is on the formula sheet, and it can be rewritten as

$$s = AgT^3\ , \tag{S9.3}$$

where

$$A = \frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3}\ . \tag{S9.4}$$

The value of A will in fact not be needed for this problem.

Since the neutrinos have decoupled by the time the e^+e^- pairs disappear, the entropy of neutrinos and the entropy of everything else will be separately conserved. Entropy conservation means that the entropy per comoving volume does not change. During the period before e^+e^- freeze-out, g is constant, so the constancy of entropy per comoving volume implies that

$$S = sV_{\text{phys}} = gT^3AV_{\text{phys}} = ga^3T^3AV_{\text{coord}}\ , \tag{S9.5}$$

so $S/V_{\rm coord}$ = const implies that a^3T^3 is constant, and so aT is constant. Here T is the common temperature of photons, alt-photons, electrons and positrons, and neutrinos, all of which were in thermal equilibrium during this period. Since aT is constant during this period, we can give the constant a name,

$$aT = [aT]_{\rm before} \ . \tag{S9.6}$$

For the neutrinos, the formula sheet tells us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \ , \tag{S9.7}$$

while

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ . \tag{S9.8}$$

Thus

$$g_{\rm else} = g_\gamma + g_{{\rm alt}\gamma} + g_{e^+e^-} = 2 + 2 + \frac{7}{2} = \frac{15}{2} \ . \tag{S9.9}$$

Thus before the e^+e^- freezeout, the two conserved quantities were

$$\frac{S_\nu}{V_{\rm coord}} = A g_\nu [aT]^3_{\rm before} \ , \quad \frac{S_{\rm else}}{V_{\rm coord}} = A g_{\rm else} [aT]^3_{\rm before} \ . \tag{S9.10}$$

After e^+e^- freezeout, the temperature of the neutrinos T_ν will no longer be the same as the temperature T_γ of the photons and alt-photons, and of course e^+e^- pairs will no longer be present. But T_γ and $T_{{\rm alt}\gamma}$ will be equal to each other, since they have the same interactions; we know that the interactions of the photons keep them in thermal equilibrium until $t_{\rm decoupling} \sim 380,000$ years, so both the photons and the alt-photons will remain in thermal equilibrium until long after the era of e^+e^- freezeout, which is of order 1–10 seconds. Thus the two conserved quantities will be

$$\frac{S_\nu}{V_{\rm coord}} = A g_\nu [aT_\nu]^3_{\rm after} \ , \quad \frac{S_{\rm else}}{V_{\rm coord}} = A (g_\gamma + g_{{\rm alt}\gamma}) [aT_\gamma]^3_{\rm after} \ . \tag{S9.11}$$

By equating the values of $S_\nu/V_{\rm coord}$ before and after, we see that

$$[aT_\nu]_{\rm after} = [aT]_{\rm before} \ , \tag{S9.12}$$

and then by equating the values of $S_{\rm else}/V_{\rm coord}$ before and after, we see that

$$[aT_\gamma]_{\rm after} = \left(\frac{g_{\rm else}}{g_\gamma + g_{{\rm alt}\gamma}}\right)^{1/3} [aT]_{\rm before} = \left(\frac{g_{\rm else}}{g_\gamma + g_{{\rm alt}\gamma}}\right)^{1/3} [aT_\nu]_{\rm after} \ , \qquad \text{(S9.13)}$$

where we used Eq. (S9.12) in the last step. It follows that

$$\left[\frac{T_\nu}{T_\gamma}\right]_{\rm after} = \left(\frac{g_\gamma + g_{{\rm alt}\gamma}}{g_{\rm else}}\right)^{1/3} = \left(\frac{2+2}{\frac{15}{2}}\right)^{1/3} = \boxed{\left(\frac{8}{15}\right)^{1/3} \ .} \qquad \text{(S9.14)}$$

(d) It would increase the abundance of helium. The main effect of the alt-photon would be to increase the expansion rate of the universe, which in turn would cause the neutrinos to decouple earlier from the thermal equilibrium mix, which in turn would mean that the ratio n_n/n_p, the ratio of neutrons to protons, would become frozen at a larger value. The increased expansion rate would also mean less time available for free neutron decay, which further increases the number of neutrons that remain when the temperature falls low enough for helium formation to complete. Essentially all the neutrons become bound into helium, so more neutrons implies more helium.

PROBLEM 10: THE FREEZE-0UT OF A FICTITIOUS PARTICLE X *(25 points)*

(a) *(5 points)* The formula sheet tells us that the energy density of black-body radiation is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases} .$$

Since the X is spin-1, and 1 is an integer, the X particles are bosons and $g = 1$ per spin state. There are 3 species, X^+, X^-, and X^0, and each species we are told has three spin states, so there are a total of 9 spin states, so $g = 9$. Thus,

$$\boxed{u = 9\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \ .}$$

Alternatively, one could count the X^+ and X^- as one species with a distinct particle and antiparticle, so $g_{X^+X^-}$ is given by

$$g_{X^+X^-} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 6 \ .$$

The X^0 is its own antiparticle, which means that the particle/antiparticle factor is one, so

$$g_{X^0} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{1}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 3 \ ,$$

so the total g for X^+, X^-, and X^0 is again equal to 9.

(b) *(5 points)* The formula sheet tells us that the number density of particles in black-body radiation is

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ ,$$

where

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} \ . \end{cases}$$

For bosons $g^* = g$, so g^* for the X particles is 9. Then

$$\boxed{n_X = 9 \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ .}$$

(c) *(10 points)* We are told that, when the X particles freeze out, all of their energy and entropy is given to the photons. We use entropy rather than energy to determine the final temperature of the photons, because the entropy in a comoving volume is simply conserved, while the energy density varies as

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Thus, to track the energy, we need to know exactly how p behaves, and the behavior of p during freeze-out is complicated, and we have not calculated it in this course.

The formula sheet tells us that the entropy density of a constituent of black-body radiation is given by

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \ .$$

If we consider some fixed coordinate volume $V_{\rm coord}$, the corresponding physical volume is $V_{\rm phys} = V_{\rm coord}\, a^3(t)$, where $a(t)$ is the scale factor. The total entropy of neutrinos in $V_{\rm coord}$ is then

$$S_\nu = g_\nu \frac{2\pi^2}{45}\ \frac{k^4 T_\nu^3(t)}{(\hbar c)^3}\, V_{\rm coord}\, a^3(t)\ .$$

The quantities $T_\nu(t)$ and $a(t)$ depend on time, but the expression on the right-hand-side does not, since entropy is conserved. For brevity I will write

$$S_\nu = g_\nu A(t) T_\nu^3(t)\ , \tag{S10.1}$$

where

$$A(t) \equiv \frac{2\pi^2}{45}\ \frac{k^4}{(\hbar c)^3} V_{\rm coord}\, a^3(t)\ .$$

The e^+e^- pairs and the X's contribute to the black-body radiation only before the freeze-out, when $kT \gg 0.511$ MeV/c^2. Let t_b denote any time before the freeze-out. Before the freeze-out, the total entropy of photons, e^+e^- pairs, and X particles is given by

$$S_{{\rm before},\gamma e X} = (g_\gamma + g_{e^+e^-} + g_X) A(t_b) T_\gamma^3(t_b)\ . \tag{S10.2}$$

I can call the temperature T_γ, because the e^+e^- pairs and the X's (as well as the neutrinos) are all in thermal equilibrium at this point, so they all have the same temperature.

Using t_a to denote an arbitrary time after the freeze-out, the entropy of the photons during this time period can be written

$$S_{{\rm after},\gamma} = g_\gamma A(t_a) T_\gamma^3(t_a)\ . \tag{S10.3}$$

But since the e^+e^- pairs and X particles give all their entropy to the photons, we have

$$S_{{\rm after},\gamma} = S_{{\rm before},\gamma e X}\ . \tag{S10.4}$$

Then using Eqs. (S10.2) and (S10.3) we find

$$g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_b) T_\gamma^3(t_b)\ . \tag{S10.5}$$

We can rewrite the last factor in Eq. (S10.5) by remembering that Eq. (S10.1) holds at all times, and that $T_\nu(t_b) = T_\gamma(t_b)$. So,

$$A(t_b) T_\gamma^3(t_b) = A(t_b) T_\nu^3(t_b) = \frac{S_\nu}{g_\nu} = A(t_a) T_\nu^3(t_a)\ . \tag{S10.6}$$

Substituting Eq. (S10.6) into Eq. (S10.5), we have

$$g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_a) T_\nu^3(t_a) \ ,$$

from which we see that

$$T_\gamma^3(t_a) = \frac{g_\gamma + g_{e^+e^-} + g_X}{g_\gamma} T_\nu^3(t_a) \ ,$$

and therefore

$$\begin{aligned} \frac{T_\nu(t_a)}{T_\gamma(t_a)} &= \left(\frac{g_\gamma}{g_\gamma + g_{e^+e^-} + g_X} \right)^{1/3} \\ &= \left(\frac{2}{2 + \frac{7}{2} + 9} \right)^{1/3} = \boxed{\left(\frac{4}{29} \right)^{1/3}} \ . \end{aligned}$$

(d) *(5 points)* The answer would be the same, since it was completely determined by the conservation equation, Eq. (S10.4) in the above answer. Regardless of the order in which the freeze-outs occurred, the total entropy from the e^+e^- pairs and the X's would ultimately be given to the photons, so the amount of heating of the photons would be the same.

PROBLEM 11: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$\begin{aligned} n &= 3 \frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \ \cancel{\text{MeV}}}{6.582 \times 10^{-16} \ \cancel{\text{eV}}\text{-}\cancel{\text{sec}} \times 2.998 \times 10^{10} \ \cancel{\text{cm}}\text{-}\cancel{\text{sec}}^{-1}} \right)^3 \\ &\quad \times \left(\frac{10^6 \ \cancel{\text{eV}}}{1 \ \cancel{\text{MeV}}} \right)^3 \times \left(\frac{10^2 \ \cancel{\text{cm}}}{1 \text{ m}} \right)^3 \\ &= 3 \frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}} \right)^3 \text{m}^{-3} \ . \end{aligned}$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}}\right)^3 \quad .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $a(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{a}}{a} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion – anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/a$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\rm rest} T^3(t) a^3(t)$$

where $g_{\rm rest} = g_{\rm tot} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 a^3(t) = g_{\rm rest} T^3(t) a^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\rm rest}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 12: A NEW THEORY OF THE WEAK INTERACTIONS *(40 points)*

(a) In the standard model, the black-body radiation at $kT \approx 200$ MeV contains the following contributions:

$$\left.\begin{array}{ll} \text{Photons:} & g = 2 \\ e^+e^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \nu_e, \nu_\mu, \nu_\tau: & g = 6 \times \frac{7}{8} = 5\frac{1}{4} \\ \mu^+\mu^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \pi^+\pi^-\pi^0 & g = 3 \end{array}\right\} g_{\rm TOT} = 17\frac{1}{4}$$

The mass density is then given by

$$\rho = \frac{u}{c^2} = g_{\rm TOT} \frac{\pi^2}{30} \frac{(kT)^4}{\hbar^3 c^5} \quad .$$

In kg/m^3, one can evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right) \frac{\pi^2}{30} \frac{\left[200 \times 10^6 \,\text{eV} \times \dfrac{1.602 \times 10^{-19}\,\text{J}}{\text{eV}}\right]^4}{(1.055 \times 10^{-34}\,\text{J-s})^3\,(2.998 \times 10^8\,\text{m/s})^5} \quad .$$

Checking the units,

$$[\rho] = \frac{\mathrm{J}^4}{\mathrm{J}^3\text{-}\mathrm{s}^3\text{-}\mathrm{m}^5\text{-}\mathrm{s}^{-5}} = \frac{\mathrm{J}\text{-}\mathrm{s}^2}{\mathrm{m}^5}$$

$$= \frac{\left(\mathrm{kg}\text{-}\mathrm{m}^2\text{-}\mathrm{s}^{-2}\right)\mathrm{s}^2}{\mathrm{m}^5} = \mathrm{kg/m}^3 \ .$$

So, the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6 \times 1.602\times 10^{-19}\right]^4}{(1.055\times 10^{-34})^3\,(2.998\times 10^8)^5}\frac{\mathrm{kg}}{\mathrm{m}^3} \ .}$$

You were not expected to evaluate this, but with a calculator one would find

$$\rho = 2.10\times 10^{18}\ \mathrm{kg/m}^3 \ .$$

In $\mathrm{g/cm}^3$, one would evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6\,\mathrm{eV}\times\dfrac{1.602\times 10^{-12}\,\mathrm{erg}}{\mathrm{eV}}\right]^4}{(1.055\times 10^{-27}\,\mathrm{erg\text{-}s})^3\,(2.998\times 10^{10}\,\mathrm{cm/s})^5} \ .$$

Checking the units,

$$[\rho] = \frac{\mathrm{erg}^4}{\mathrm{erg}^3\text{-}\mathrm{s}^3\text{-}\mathrm{cm}^5\text{-}\mathrm{s}^{-5}} = \frac{\mathrm{erg}\text{-}\mathrm{s}^2}{\mathrm{cm}^5}$$

$$= \frac{\left(\mathrm{g}\text{-}\mathrm{cm}^2\text{-}\mathrm{s}^{-2}\right)\mathrm{s}^2}{\mathrm{cm}^5} = \mathrm{g/cm}^3 \ .$$

So, in this case the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6 \times 1.602\times 10^{-12}\right]^4}{(1.055\times 10^{-27})^3\,(2.998\times 10^{10})^5}\frac{\mathrm{g}}{\mathrm{cm}^3} \ .}$$

No evaluation was requested, but with a calculator you would find

$$\rho = 2.10\times 10^{15}\ \mathrm{g/cm}^3 \ ,$$

which agrees with the answer above.

Note: A common mistake was to leave out the conversion factor 1.602×10^{-19} J/eV (or 1.602×10^{-12} erg/eV), and instead to use $\hbar = 6.582 \times 10^{-16}$ eV-s. But if one works out the units of this answer, they turn out to be eV-sec^2/m^5 (or eV-sec^2/cm^5), which is a most peculiar set of units to measure a mass density.

In the NTWI, we have in addition the contribution to the mass density from R^+-R^- pairs, which would act just like e^+-e^- pairs or μ^+-μ^- pairs, with $g = 3\frac{1}{2}$. Thus $g_{\rm TOT} = 20\frac{3}{4}$, so

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-19}\right]^4}{(1.055 \times 10^{-34})^3\,(2.998 \times 10^8)^5}\frac{\rm kg}{\rm m^3}}$$

or

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-12}\right]^4}{(1.055 \times 10^{-27})^3\,(2.998 \times 10^{10})^5}\frac{\rm g}{\rm cm^3}\ .}$$

Numerically, the answer in this case would be

$$\rho_{\rm NTWI} = 2.53 \times 10^{18}\ {\rm kg/m}^3 = 2.53 \times 10^{15}\ {\rm g/cm}^3\ .$$

(b) As long as the universe is in thermal equilibrium, entropy is conserved. The entropy in a given volume of the comoving coordinate system is

$$a^3(t)s\,V_{\rm coord}\quad ,$$

where s is the entropy density and $a^3 V_{\rm coord}$ is the physical volume. So

$$a^3(t)s$$

is conserved. After the neutrinos decouple,

$$a^3 s_\nu \quad \text{and} \quad a^3 s_{\rm other}$$

are separately conserved, where $s_{\rm other}$ is the entropy of everything except neutrinos.

Note that s can be written as

$$s = gAT^3\quad ,$$

where A is a constant. Before the disappearance of the e, μ, R, and π particles from the thermal equilibrium radiation,

$$s_\nu = \left(5\frac{1}{4}\right) AT^3$$

$$s_{\text{other}} = \left(15\frac{1}{2}\right) AT^3 \quad .$$

So

$$\frac{s_\nu}{s_{\text{other}}} = \frac{5\frac{1}{4}}{15\frac{1}{2}} \quad .$$

If $a^3 s_\nu$ and $a^3 s_{\text{other}}$ are conserved, then so is s_ν/s_{other}. By today, the entropy previously shared among the various particles still in equilibrium after neutrino decoupling has been transfered to the photons so that

$$s_{\text{other}} = s_{\text{photons}} = 2AT_\gamma^3 \quad .$$

The entropy in neutrinos is still

$$s_\nu = \left(5\frac{1}{4}\right) AT_\nu^3 \quad .$$

Since s_ν/s_{other} is constant we know that

$$\frac{\left(5\frac{1}{4}\right) T_\nu^3}{2T_\gamma^3} = \frac{s_\nu}{s_{\text{other}}} = \frac{5\frac{1}{4}}{15\frac{1}{2}}$$

$$\Longrightarrow \quad \boxed{T_\nu = \left(\frac{4}{31}\right)^{1/3} T_\gamma} \quad .$$

(c) One can write

$$n = g^* BT^3 \quad ,$$

where B is a constant. Here $g_\gamma^* = 2$, and $g_\nu^* = 6 \times \frac{3}{4} = 4\frac{1}{2}$. In the standard model, one has today

$$\frac{n_\nu}{n_\gamma} = \frac{g_\nu^* T_\nu^3}{g_\gamma^* T_\gamma^3} = \frac{\left(4\frac{1}{2}\right)}{2}\frac{4}{11} = \boxed{\frac{9}{11} \ .}$$

In the NTWI,

$$\frac{n_\nu}{n_\gamma} = \frac{\left(4\frac{1}{2}\right)}{2}\,\frac{4}{31} = \boxed{\frac{9}{31}\ .}$$

(d) At $kT = 200$ MeV, the thermal equilibrium ratio of neutrons to protons is given by

$$\frac{n_\mathrm{n}}{n_\mathrm{p}} = e^{-1.29\,\mathrm{MeV}/200\,\mathrm{MeV}} \approx 1 \quad .$$

In the standard theory this ratio would decrease rapidly as the universe cooled and kT fell below the p-n mass difference of 1.29 MeV, but in the NTWI the ratio freezes out at the high temperature corresponding to $kT = 200$ MeV, when the ratio is about 1. When kT falls below 200 MeV in the NTWI, the neutrino interactions

$$n + \nu_e \leftrightarrow p + e^- \qquad \text{and} \qquad n + e^+ \leftrightarrow p + \bar{\nu}_e$$

that maintain the thermal equilibrium balance between protons and neutrons no longer occur at a significant rate, so the ratio $n_/n_p$ is no longer controlled by thermal equilibrium. After kT falls below 200 MeV, the only process that can convert neutrons to protons is the rather slow process of free neutron decay, with a decay time τ_d of about 890 s. Thus, when the deuterium bottleneck breaks at about 200 s, the number density of neutrons will be considerably higher than in the standard model. Since essentially all of these neutrons will become bound into He nuclei, the higher neutron abundance of the NTWI implies a

$$\boxed{\text{higher predicted He abundance.}}$$

To estimate the He abundance, note that if we temporarily ignore free neutron decay, then the neutron-proton ratio would be frozen at about 1 and would remain 1 until the time of nucleosynthesis. At the time of nucleosynthesis essentially all of these neutrons would be bound into He nuclei (each with 2 protons and 2 neutrons). For an initial 1:1 ratio of neutrons to protons, all the neutrons and protons can be bound into He nuclei, with no protons left over in the form of hydrogen, so Y would equal 1. However, the free neutron decay process will cause the ratio n_n/n_p to fall below 1 before the start of nucleosynthesis, so the predicted value of Y would be less than 1.

To calculate how much less, note that Ryden estimates the start of nucleosynthesis at the time when the temperature reaches T_nuc, which is the temperature for which a thermal equilibrium calculation gives $n_D/n_n = 1$. This corresponds to what Weinberg refers to as the breaking of the deuterium bottleneck. The temperature

$T_{\rm nuc}$ is calculated in terms of $\eta = n_B/n_\gamma$ and physical constants, so it would not be changed by the NTWI. The time when this temperature is reached, however, would be changed slightly by the change in the ratio T_ν/T_γ. Since this effect is rather subtle, no points will be taken off if you omitted it. However, to be as accurate as possible, one should recognize that nucleosynthesis occurs during the radiation-dominated era, but long after the e^+-e^- pairs have disappeared, so the black-body radiation consists of photons at temperature T_γ and neutrinos at a lower temperature T_ν. The energy density is given by

$$u = \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3}\left[2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4\right] \equiv g_{\rm eff}\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} \ ,$$

where

$$g_{\rm eff} = 2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4 \ .$$

For the standard model

$$g_{\rm eff}^{\rm sm} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{11}\right)^{4/3} \ ,$$

and for the NTWI

$$g_{\rm eff}^{\rm NTWI} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{31}\right)^{4/3} \ .$$

The relation between time and temperature in a flat radiation-dominated universe is given in the formula sheets as

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}} \ .$$

Thus,

$$t \propto \frac{1}{g_{\rm eff}^{1/2} T^2} \ .$$

In the standard model Ryden estimates the time of nucleosynthesis as $t_{\rm nuc}^{\rm sm} \approx 200$ s, so in the NTWI it would be longer by the factor

$$t_{\rm nuc}^{\rm NTWI} = \sqrt{\frac{g_{\rm eff}^{\rm sm}}{g_{\rm eff}^{\rm NTWI}}}\, t_{\rm nuc}^{\rm sm} \ .$$

While of coure you were not expected to work out the numerics, this gives

$$t_{\rm nuc}^{\rm NTWI} = 1.20\, t_{\rm nuc}^{\rm sm} \ .$$

Note that Ryden gives $t_{\rm nuc} \approx 200s$, while Weinberg places it at $3\frac{3}{4}$ minutes ≈ 225 s, which is close enough.

To follow the effect of this free decay, it is easiest to do it by considering the ratio neutrons to baryon number, n_n/n_B, since n_B does not change during this period. At freeze-out, when $kT \approx 200$ MeV,

$$\frac{n_n}{n_B} \approx \frac{1}{2} \ .$$

Just before nucleosynthesis, at time $t_{\rm nuc}$, the ratio will be

$$\frac{n_n}{n_B} \approx \frac{1}{2} e^{-t_{\rm nuc}/\tau_d} \ .$$

If free decay is ignored, we found $Y = 1$. Since all the surviving neutrons are bound into He, the corrected value of Y is simply deceased by multiplying by the fraction of neutrons that do not undergo decay. Thus, the prediction of NTWI is

$$\boxed{Y = e^{-t_{\rm nuc}/\tau_d} = \exp\left\{ -\frac{\sqrt{\frac{g_{\rm eff}^{\rm sm}}{g_{\rm eff}^{\rm NTWI}}}\,200}{890} \right\}} \ ,$$

where $g_{\rm eff}^{\rm sm}$ and $g_{\rm eff}^{\rm NTWI}$ are given above. When evaluated numerically, this would give

$$Y = \text{Predicted He abundance by weight} \approx 0.76 \ .$$

PROBLEM 13: DOUBLING OF ELECTRONS *(10 points)*

The entropy density of black-body radiation is given by

$$\begin{aligned} s &= g \left[\frac{2\pi^2}{45} \frac{k^4}{(\hbar c)^3} \right] T^3 \\ &= g\, C\, T^3 \ , \end{aligned}$$

where C is a constant. At the time when the electron-positron pairs disappear, the neutrinos are decoupled, so their entropy is conserved. All of the entropy from electron-positron pairs is given to the photons, and none to the neutrinos. The same will be true here, for both species of electron-positron pairs.

The conserved neutrino entropy can be described by $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation n^- and n^+ for the new electron-like and positron-like particles, and also the convention that

$$\begin{aligned} \text{Primed quantities:} \quad & \text{values after } e^+e^-n^+n^- \text{ annihilation} \\ \text{Unprimed quantities:} \quad & \text{values before } e^+e^-n^+n^- \text{ annihilation.} \end{aligned}$$

For the neutrinos,

$$S'_\nu = S_\nu \quad \Longrightarrow \quad g_\nu C\left(a'T'_\nu\right)^3 = g_\nu C\left(aT_\nu\right)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\nu = aT_\nu \ .}$$

For the photons, before $e^+e^-n^+n^-$ annihilation we have

$$T_\gamma = T_{e^+e^-n^+n^-} = T_\nu \ ; \qquad g_\gamma = 2,\ g_{e^+e^-} = g_{n^+n^-} = 7/2 \ .$$

When the e^+e^- and n^+n^- pairs annihilate, their entropy is added to the photons:

$$S'_\gamma = S_{e^+e^-} + S_{n^+n^-} + S_\gamma \quad \Longrightarrow \quad 2C\left(a'T'_\gamma\right)^3 = \left(2 + 2\cdot\frac{7}{2}\right)C\left(aT_\gamma\right)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\gamma = \left(\frac{9}{2}\right)^{1/3} aT_\gamma \ ,}$$

so aT_γ increases by a factor of $(9/2)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{2}{9}\right)^{1/3} T'_\gamma \ .}$$

This ratio would remain unchanged until the present day.

PROBLEM 14: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto a^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 15: THE TIME t_d OF DECOUPLING *(25 points)*

(a) *(5 points)* If the entropy of photons is conserved, then the entropy density falls as

$$s \propto \frac{1}{a^3(t)} \ .$$

Since $s \propto T^3$, it follows that

$$T \propto \frac{1}{a(t)} \ .$$

Thus, the ratio of the scale factors is equal to the inverse of the ratio temperatures:

$$\boxed{x_d = \frac{T_0}{T_d} \ .}$$

(b) *(5 points)* The formula sheet reminds us that

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2} \ ,$$

where

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\rm rad,0} - \Omega_{\rm vac,0} \ .$$

So for a flat universe $\Omega_{k,0} = 0$, and we have

$$\boxed{\frac{dx}{dt} = \frac{H_0}{x}\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}\ .}$$

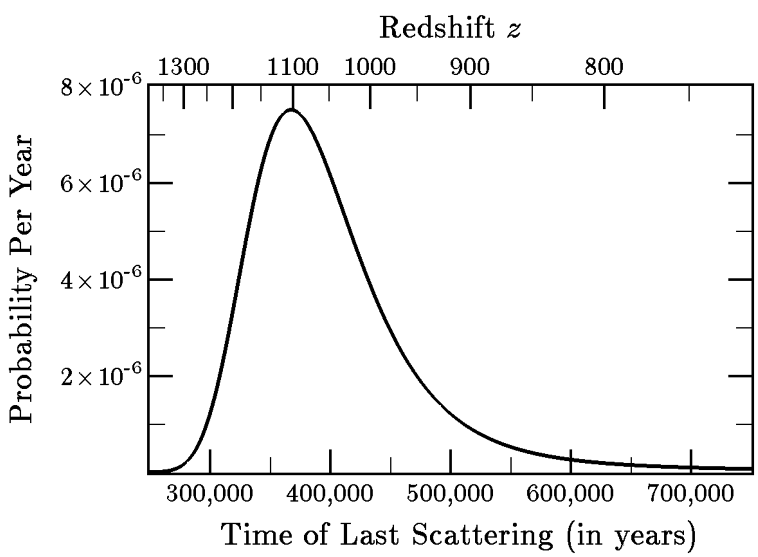
(c) *(5 points)* The answer to part (b) can be rewritten as

$$dt = \frac{x\,dx}{H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}}\ .$$

t_d is the time that elapses from when the universe has $x = 0$ to when it has $x = x_d$, so

$$\boxed{t_d = \frac{1}{H_0}\int_0^{x_d} \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}}\ .}$$

You were of course not asked to evaluate this integral numerically, but we will do that now. We take $T_0 = 2.7255$ K from Fixsen et al. (cited in Lecture Notes 6) and the Planck 2015 best fit values of $H_0 = 67.7$ km-s^{-1}-Mpc^{-1}, $\Omega_{m,0} = 0.309$, $\Omega_{\text{vac},0} = 0.691$. The energy density of radiation (photons plus neutrinos) can then be calculated to give $\Omega_{\text{rad},0} = 9.2 \times 10^{-5}$ (see Eq. (6.23) of Lecture Notes 6 and the text of the 2nd paragraph of p. 12 of Lecture Notes 7). To keep our model universe exactly flat, I am modifying $\Omega_{\text{vac},0}$ to set it equal to $0.691 - \Omega_{\text{rad},0}$, which is well within the uncertainties. Numerical integration then gives 366,000 years, very close to our original estimate. Of course this number is still approximate, since we started with $T_d \approx 3000$ K. In any case, the decoupling of the photons in the CMB is actually a gradual process. In 2003 I modified a standard program called CMBFast to calculate the probability distribution of the time of last scattering (published in https://arxiv.org/abs/astro-ph/0306275), with the following results:



The parameters used were $\Omega_{\rm vac,0} = 0.70$, $\Omega_{m,0} = 0.30$, $H_0 = 68$ km-s^{-1}-Mpc^{-1}. The peak of the curve is at 367,000 years, and the median is at 388,000 years.

(d) *(10 points)* The derivation starts with the first-order Friedmann equation. Since we are describing a flat universe, we can start with the Friedmann equation for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho \ .$$

Now we use the facts that $\rho_m \propto 1/a^3$, $\rho_{\rm rad} \propto 1/a^4$, $\rho_{\rm vac} \propto 1$, and $\rho_f \propto 1/a^8$ to write

$$H^2 = \frac{8\pi}{3} G \left[\frac{\rho_{m,0}}{x^3} + \frac{\rho_{\rm rad,0}}{x^4} + \rho_{\rm vac,0} + \frac{\rho_{f,0}}{x^8} \right] \ .$$

Then we use

$$\rho_{m,0} = \rho_c \Omega_{m,0} = \frac{3H_0^2}{8\pi G} \Omega_{m,0} ,$$

with similar relations for the other components of the mass density, to rewrite the Friedmann equation as

$$H^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\rm rad,0}}{x^4} + \Omega_{\rm vac,0} + \frac{\Omega_{f,0}}{x^8} \right] \ .$$

Next we rewrite H^2 as

$$H^2 = \left(\frac{\dot a}{a} \right)^2 = \left(\frac{\dot x}{x} \right)^2 \ ,$$

so

$$\left(\frac{\dot x}{x} \right)^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\rm rad,0}}{x^4} + \Omega_{\rm vac,0} + \frac{\Omega_{f,0}}{x^8} \right] \ ,$$

which can be rewritten as

$$x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0} x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0} x^4 + \frac{\Omega_{f,0}}{x^4}} \ .$$

From here the derivation is identical to that in part (c), leading to

$$\boxed{t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x\, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0} x^4 + \frac{\Omega_{f,0}}{x^4}}} \ ,}$$

which can also be written more neatly as

$$\boxed{t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x^3\, dx}{\sqrt{\Omega_{m,0} x^5 + \Omega_{\rm rad,0} x^4 + \Omega_{\rm vac,0} x^8 + \Omega_{f,0}}} \ .}$$

PROBLEM 16: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) \ ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/a^3(t)$ and $\rho_\Lambda = \text{const}$, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{a(t)}{a(t_0)} \ ,$$

one has

$$\begin{aligned}
\left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3} G \left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\
&= \frac{8\pi}{3} G \rho_{c,0} \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\
&= H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \ .
\end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} \ ,$$

and

$$H_0^2 = \frac{8\pi}{3} G \rho_{c,0} \ .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0 \, x \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} \ ,$$

which in turn implies that

$$\mathrm{d}t = \frac{1}{H_0} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}}\ .}$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}}\ ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z}\ ; \qquad \mathrm{d}x = -\frac{\mathrm{d}z}{(1+z)^2}\ .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}}\ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right)\ .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0}\ \mathrm{yr}\ ,$$

where $H_0 = 100\,h_0\,\text{km-sec}^{-1}\text{-Mpc}^{-1}$, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9\ \mathrm{yr}\ .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \text{ yr} .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} ,$$

since the cosmological redshift is given by

$$1 + z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} .$$

Using the expression for $\mathrm{d}t$ from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0}\int_z^{\infty} \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}x^3_{\max}} + \sqrt{\Omega_{\Lambda,0}}\, x^{3/2}_{\max}}{\sqrt{\Omega_m}}\right) .$$

Using $x_{\rm max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr} \ .}$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $a(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\} \ ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{a(t)} \ .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t \ .$$

This is not the final answer, however, because we don't explicitly know $a(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}} \ .$$

So

$$\ell_c = \frac{c}{a(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} \ ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 a(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Finally, then

$$\boxed{\ell_{\rm phys,0} = a(t_0)\,\ell_c = \frac{c}{H_0}\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0}\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}}\ ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0}\int_{0}^{z} \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{\mathrm{d}z'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099\ ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\text{phys},0} = 27.23 \text{ light-yr}\ .}$$

Right again.

(d) $\ell_{\text{phys},e} = a(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{a(t_e)}{a(t_0)}\,\ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z}\ .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr}\ .}$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\,\ell_{\text{phys},0} = c\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099\ .}$$

The SDSS people get an A.

PROBLEM 17: SECOND HUBBLE CROSSING *(40 points)*

(a) From the formula sheets, we know that for a flat radiation-dominated universe,

$$a(t) \propto t^{1/2}\ .$$

Since

$$H = \frac{\dot{a}}{a}\ ,$$

(which is also on the formula sheets),

$$H = \frac{1}{2t}\ .$$

Then

$$\ell_H(t) \equiv cH^{-1}(t) = \boxed{2ct\ .}$$

(b) We are told that the energy density is dominated by photons and neutrinos, so we need to add together these two contributions to the energy density. For photons, the formula sheet reminds us that $g_\gamma = 2$, so

$$u_\gamma = 2\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{(\hbar c)^3}\ .$$

For neutrinos the formula sheet reminds us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

so

$$u_\nu = \frac{21}{4}\,\frac{\pi^2}{30}\,\frac{(kT_\nu)^4}{(\hbar c)^3}\ .$$

Combining these two expressions and using $T_\nu = (4/11)^{1/3}\,T_\gamma$, one has

$$u = u_\gamma + u_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{(\hbar c)^3}\ ,$$

so finally

$$\boxed{g_1 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\ .}$$

(c) The Friedmann equation tells us that, for a flat universe,

$$H^2 = \frac{8\pi}{3}G\rho\ ,$$

where in this case $H = 1/(2t)$ and

$$\rho = \frac{u}{c^2} = g_1\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{\hbar^3 c^5}\ .$$

Thus

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3}\,g_1\,\frac{\pi^2}{30}\,\frac{(kT_\gamma)^4}{\hbar^3 c^5}\ .$$

Solving for T_γ,

$$\boxed{T_\gamma = \frac{1}{k}\left(\frac{45\hbar^3 c^5}{16\pi^3 g_1 G}\right)^{1/4}\,\frac{1}{\sqrt{t}}\ .}$$

(d) The condition for Hubble crossing is

$$\lambda(t) = cH^{-1}(t)\ ,$$

and the first Hubble crossing always occurs during the inflationary era. Thus any Hubble crossing during the radiation-dominated era must be the second Hubble crossing.

If λ is the present physical wavelength of the density perturbations under discussion, the wavelength at time t is scaled by the scale factor $a(t)$:

$$\lambda(t) = \frac{a(t)}{a(t_0)}\,\lambda\ .$$

Between the second Hubble crossing and now, there have been no freeze-outs of particle species. Today the entropy of the universe is still dominated by photons and neutrinos, so the conservation of entropy implies that aT_γ has remained essentially constant between then and now. Thus,

$$\lambda(t) = \frac{T_{\gamma,0}}{T_\gamma(t)}\,\lambda\ .$$

Using the previous results for $cH^{-1}(t)$ and for $T_\gamma(t)$, the condition $\lambda(t) = cH^{-1}(t)$ can be rewritten as

$$kT_{\gamma,0}\left(\frac{16\pi^3 g_1 G}{45\hbar^3 c^5}\right)^{1/4}\sqrt{t}\,\lambda = 2ct\ .$$

Solving for t, the time of second Hubble crossing is found to be

$$\boxed{t_{H2}(\lambda) = (kT_{\gamma,0}\lambda)^2\left(\frac{\pi^3 g_1 G}{45\hbar^3 c^9}\right)^{1/2}\ .}$$

Extension: You were not asked to insert numbers, but it is of course interesting to know where the above formula leads. If we take $\lambda = 10^6$ lt-yr, it gives

$$t_{H2}(10^6\ \text{lt-yr}) = 1.04\times10^7\ \text{s} = 0.330\ \text{year}\ .$$

For $\lambda = 1$ Mpc,

$$t_{H2}(1\ \text{Mpc}) = 1.11\times10^8\ \text{s} = 3.51\ \text{year}\ .$$

Taking $\lambda = 2.5\times10^6$ lt-yr, the distance to Andromeda, the nearest spiral galaxy,

$$t_{H2}(2.5\times10^6\ \text{lt-yr}) = 6.50\times10^7\ \text{sec} = 2.06\ \text{year}\ .$$

PROBLEM 18: EVOLUTION OF FLATNESS *(15 points)*

(a) We start with the Friedmann equation from the formula sheet on the quiz:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} .$$

The critical density is the value of ρ corresponding to $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c .$$

Using this expression to replace H^2 on the left-hand side of the Friedmann equation, and then dividing by $8\pi G/3$, one finds

$$\rho_c = \rho - \frac{3kc^2}{8\pi G a^2} .$$

Rearranging,

$$\frac{\rho - \rho_c}{\rho} = \frac{3kc^2}{8\pi G a^2 \rho} .$$

On the left-hand side we can divide the numerator and denominator by ρ_c, and then use the definition $\Omega \equiv \rho/\rho_c$ to obtain

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^2}{8\pi G a^2 \rho} . \tag{S18.1}$$

For a matter-dominated universe we know that $\rho \propto 1/a^3(t)$, and so

$$\frac{\Omega - 1}{\Omega} \propto a(t) .$$

If the universe is nearly flat we know that $a(t) \propto t^{2/3}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t^{2/3} .}$$

(b) Eq. (1) above is still true, so our only task is to re-evaluate the right-hand side. For a radiation-dominated universe we know that $\rho \propto 1/a^4(t)$, so

$$\frac{\Omega - 1}{\Omega} \propto a^2(t) .$$

If the universe is nearly flat then $a(t) \propto t^{1/2}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t .}$$

PROBLEM 19: THE EVENT HORIZON FOR OUR UNIVERSE *(25 points)*

(a) In a spherical pulse each light ray is moving radially outward, so $\mathrm{d}\theta = \mathrm{d}\phi = 0$. A light ray travels along a null trajectory, meaning that $\mathrm{d}s^2 = 0$, so we have

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}r^2 = 0\ . \tag{S19.1}$$

from which it follows that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{c}{a(t)}\ . \tag{S19.2}$$

We are interested in a radial pulse that starts at $r = 0$ at time $t = t_0$, so the limiting value of r is given by

$$\boxed{r_{\rm max} = \int_{t_0}^{\infty} \frac{c}{a(t)}\,\mathrm{d}t\ .} \tag{S19.3}$$

(b) Changing variables of integration to

$$x = \frac{a(t)}{a(t_0)}\ , \tag{S19.4}$$

the integral becomes

$$r_{\rm max} = \int_1^{\infty} \frac{c}{a(t)}\,\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{c}{a(t_0)}\int_1^{\infty}\frac{1}{x}\,\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x\ , \tag{S19.5}$$

where we used the fact that $t = t_0$ corresponds to $x = a(t_0)/a(t_0) = 1$. As given to us on the formula sheet, the first-order Friedmann equation can be written as

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2}\ . \tag{S19.6}$$

Using this substitution,

$$\boxed{r_{\rm max} = \frac{c}{a(t_0)H_0}\int_1^{\infty}\frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4}}\ ,} \tag{S19.7}$$

where we have used $\Omega_{k,0} = 0$, since the universe is taken to be flat.

(c) To find the value of the redshift for the light that we are presently receiving from coordinate distance $r_{\rm max}$, we can begin by noticing that the time of emission t_e can be determined by the equation which implies that the coordinate distance traveled by a light pulse between times t_e and t_0 must equal $r_{\max}$. Using Eq. (S19.2) for the coordinate velocity of light, this equation reads

$$\int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t = r_{\max} \; . \tag{S19.8}$$

The "half-credit" answer to the quiz problem would include the above equation, followed by the statement that the redshift z_{eh} can be determined from

$$z = \frac{a(t_0)}{a(t_e)} - 1 \; . \tag{S19.9}$$

The "full-credit" answer is obtained by changing the variable of integration as in part (b), so Eq. (S19.8) becomes

$$\begin{aligned} r_{\max} &= \int_{x_e}^{1} \frac{c}{a(t)} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x \\ &= \frac{c}{a(t_0)} \int_{x_e}^{1} \frac{1}{x} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x \; , \end{aligned} \tag{S19.10}$$

where x_e is the value of x corresponding to $t = t_e$. Then using Eq. (S19.6) with $\Omega_{k,0} = 0$, we find

$$\boxed{r_{\max} = \frac{c}{a(t_0)H_0} \int_{x_e}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}} \; .} \tag{S19.11}$$

To complete the answer in this language, we use

$$\boxed{z = \frac{1}{x_e} - 1 \; .} \tag{S19.12}$$

Eqs. (S19.11) and (S19.12) constitute a full answer to the question, but one could go further and replace $r_{\max}$ using Eq. (S19.7), finding

$$\boxed{\begin{aligned} &\int_{1}^{\infty} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}} \\ &\qquad = \int_{x_e}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4}} \; . \end{aligned}} \tag{S19.13}$$

In this form the answer depends only on the values of $\Omega_{X,0}$.

You were of course not asked to evaluate this formula numerically, but you might be interested in knowing that the Planck 2013 values $\Omega_{m,0} = 0.315$, $\Omega_{\text{vac},0} = 0.685$, and $\Omega_{\text{rad},0} = 9.2 \times 10^{-5}$ lead to $z_{\text{eh}} = 1.87$. Thus, no event that is happening now (i.e., at the same value of the cosmic time) in a galaxy at redshift larger than 1.87 will ever be visible to us or our descendants, even in principle.

PROBLEM 20: A MESSAGE FROM A DISTANT GALAXY *(35 points)*

(a) For these questions the formulas on the Formula Sheets under GENERALIZED COSMOLOGICAL EVOLUTION are relevant, but with $\Omega_{k,0} \equiv 0$. Since $t_{\text{look-back}}(z)$ is defined as the amount of cosmic time by which the time of emission is earlier than the present time, one way to answer this question is to write

$$\boxed{t_e = t_0 - t_{\text{look-back}}(z_G) \ ,} \tag{S20.1}$$

where each quantity on the right-hand side can be written in terms of given quantities by using formulas on the formula sheet:

$$\begin{aligned} t_0 &= \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3+\Omega_{r,0}(1+z)^4+\Omega_{v,0}}} \ , \\ t_{\text{look-back}}(z_G) &= \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3+\Omega_{r,0}(1+z)^4+\Omega_{v,0}}} \ . \end{aligned} \tag{S20.2}$$

The two terms can be combined to give

$$\boxed{t_e = \frac{1}{H_0}\int_{z_G}^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3+\Omega_{r,0}(1+z)^4+\Omega_{v,0}}} \ .} \tag{S20.3}$$

Alternatively, one can go back to basics and start with the first order Friedmann equation, which for a flat universe is given by

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho \ . \tag{S20.4}$$

Defining $x(t) \equiv a(t)/a(t_0)$, this can be rewritten as

$$\left(\frac{\dot x}{x}\right)^2 = \frac{8\pi}{3}G\rho \ . \tag{S20.5}$$

Expressing $\rho = \rho_m + \rho_r + \rho_v$, we can express each component as a fraction of $\rho_c = 3H^2/(8\pi G)$ (as given in the Formula Sheet):

$$\rho(t_0) = \frac{3H_0^2}{8\pi G}\left(\Omega_{m,0}+\Omega_{r,0}+\Omega_{v,0}\right) \ . \tag{S20.6}$$

Knowing that $\rho_m \propto 1/a^3$, $\rho_r \propto 1/a^4$, and $\rho_v \propto a^0$, we can write

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0}\right] , \tag{S20.7}$$

which can be used in Eq. (S20.5) to give

$$\frac{\dot{x}}{x} = H_0\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0}} , \tag{S20.8}$$

which we can multiply by x^2 just to avoid the awkward denominators:

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = H_0 F(x) , \tag{S20.9}$$

where

$$F(x) \equiv \sqrt{\Omega_{m,0}\, x + \Omega_{r,0} + \Omega_{v,0}x^4} . \tag{S20.10}$$

Eqs. (S20.9) and (S20.10) match the first equation in GENERALIZED COSMOLOGICAL EVOLUTION in the formula sheet, which could have also been taken as a starting point. The equation can be written as a relation between time intervals and intervals in x:

$$\mathrm{d}t = \frac{1}{H_0}\,\frac{x\,\mathrm{d}x}{F(x)}. \tag{S20.11}$$

x is related to the redshift z by $x = 1/(1+z)$, so the value of x at time t_e is

$$x_G = \frac{1}{1+z_G} . \tag{S20.12}$$

Eq. (S20.11) can then be integrated. As t varies from 0 to t_e, x varies from 0 to x_G, so

$$\boxed{t_e = \frac{1}{H_0}\int_0^{x_G} \frac{x\,\mathrm{d}x}{F(x)} ,} \tag{S20.13}$$

where x_G and $F(x)$ are given by Eqs. (S20.12) and (S20.10). By a redefinition of the variable of integration, this formula can be seen to be equivalent to Eq. (S20.3).

(b) The coordinate velocity of light is given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{a(t)} . \tag{S20.14}$$

The total coordinate distance between us and G is the coordinate distance that light can travel during the interval from t_e to t_0, so

$$\ell_c = \int_{t_e}^{t_0} \frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}t \; . \tag{S20.15}$$

Rewriting as an integral over x,

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t = \frac{c}{a(t_0)} \int_{t_e}^{t_0} \frac{1}{x}\,\mathrm{d}t = \frac{c}{a(t_0)H_0} \int_{x_G}^{1} \frac{\mathrm{d}x}{F(x)} \; , \tag{S20.16}$$

where I used Eq. (S20.11) to express $\mathrm{d}t$ in terms of $\mathrm{d}x$, and I replaced the limits of integration, t_e and t_0, by the corresponding values of x, x_G and 1. The physical distance to the galaxy today is then

$$\boxed{\ell_{p,0} = a(t_0)\ell_c = \frac{c}{H_0} \int_{x_G}^{1} \frac{\mathrm{d}x}{F(x)} \; ,} \tag{S20.17}$$

where again x_G and $F(x)$ are given by Eqs. (S20.12) and (S20.10).

(c) Eq. (S20.16) tells us that light can travel a coordinate distance ℓ_c during the time interval in which x changes from x_G and 1. The light signal from G must travel the same coordinate distance during the time interval in which x changes from 1 to $x_r \equiv a(t_r)/a(t_0)$, which is given by an analogous formula:

$$\ell_c = \frac{c}{a(t_0)H_0} \int_{1}^{x_r} \frac{\mathrm{d}x}{F(x)} \; . \tag{S20.18}$$

Comparing Eqs. (S20.16) and (S20.18), we see that

$$\boxed{a(t_r) = x_r\, a(t_0) \; ,} \tag{S20.19}$$

where x_r is determined by the equation

$$\boxed{\int_{1}^{x_r} \frac{\mathrm{d}x}{F(x)} = \int_{x_G}^{1} \frac{\mathrm{d}x}{F(x)} \; ,} \tag{S20.20}$$

where $F(x)$ is defined by Eq. (S20.10).

(d) The time of receipt would be the present time plus the transit time, so

$$\begin{aligned} t_r &= t_0 + \int_{t_0}^{t_r} \mathrm{d}t \\ &= \boxed{\; t_0 + \frac{1}{H_0} \int_1^{x_r} \frac{x\,\mathrm{d}x}{F(x)} \;,\;} \end{aligned} \qquad \text{(S20.21)}$$

where I used Eq. (S20.11) to express $\mathrm{d}t$ in terms of $\mathrm{d}x$, and where x_r is determined by Eq. (S20.20).

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 7, 2022
Prof. Alan Guth

QUIZ 3

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins,
and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	20	
3	30	
4	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(2 points)* At the time of recombination, the temperature is

(i) above the proton and electron masses

(ii) below the proton mass but above the electron mass

(iii) below the proton and electron masses

(b) *(2 points)* At the time of recombination, the universe is

(i) matter-dominated

(ii) radiation-dominated

(iii) vacuum-energy dominated

(c) *(2 points)* The reaction that produces the most deuterium in the early universe is

(i) proton-proton fusion ($p + p \to D+$ maybe other elements)

(ii) proton-neutron fusion ($p + n \to D+$ maybe other elements)

(iii) neutron-neutron fusion ($n + n \to D+$ maybe other elements)

(d) *(2 points)* For the microcanonical ensemble (describing an isolated system in thermal equilibrium), the probability that the system is in a state i with energy E_i is proportional to

(i) $\Omega_{\text{bath}}(E_{\text{tot}} - E_i)$, where E_{tot} is the energy of the heat bath and the small system and Ω_{bath} is the density of states of the bath

(ii) $\exp(-E_i/kT)$

(iii) $1/M$ if $E < E_i < E + \delta E$, where M is the total number of states in this energy range

(e) *(2 points)* In the early universe, two important reactions occurred: deuterium production and hydrogen production from proton-electron recombination. How do the binding energies of deuterium and hydrogen compare?

(i) the binding energy of deuterium is larger

(ii) the binding energies of deuterium and hydrogen are (almost) exactly the same

(iii) the binding energy of hydrogen is larger

— Problem 1 continues on the next page. —

(f) *(2 points)* The universe can be approximated as

(i) homogeneous and isotropic at all scales

(ii) homogeneous and isotropic on small scales ($\lesssim$ 100 Mpc)

(iii) homogeneous and isotropic on large scales ($\gtrsim$ 100 Mpc)

(iv) inhomogeneous and anisotropic at all scales

(g) *(2 points)* In order to analyze the ionization and recombination of hydrogen as a function of temperature in the early universe, we studied the ratio of the number density of hydrogen atoms n_H to the product of the number densities of the free elements that react to create them: protons, with number density n_p, and electrons, with number density n_e. For 2 points credit, you can EITHER state the name that is given to the equation that describes this ratio, or you can write the equation. (If you try to write the equation, be sure to notice the relevant formulas on the formula sheet.)

(h) *(2 points)* Gravity tends to make small density perturbations grow rapidly with time. What keeps small density perturbations from creating instabilities in the matter density and leading to collapse? (For example, what stops the air in this room from undergoing gravitational collapse.)

(i) *(2 points)* Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon-baryon flui d. The speed of sound in the baryonic gas is $\underset{\text{Circle one}}{\text{[larger than / equal to / smaller than]}}$ the speed of sound in the photon gas.

(j) *(2 points)* Baryon acoustic oscillations are due to the interaction of baryons with

(i) photons

(ii) dark matter

(iii) neutrinos

— *End of Problem 1.* —

PROBLEM 2: SHORT ANSWER QUESTIONS: GRAND UNIFIED THEORIES, MAGNETIC MONOPOLES, AND INFLATION *(20 points)*

(a) *(2 points)* A quark is specified by its flavor [u(p), d(own), c(harmed), s(trange), t(op), b(ottom)], its spin [up or down, along any chosen z axis], whether it is a quark or antiquark, and its color. **T** or **F** .

(b) *(2 points)* Any isolated system of quarks must be a color singlet. A single red quark is an example of a color singlet, since it has only one color. **T** or **F** .

(c) *(3 points)* Write the definition of the group SU(3). Assume that the reader does not know the meaning of “unitary” or “special”, but that they do know the standard mathematical notation of matrix multiplication AB, matrix inversion A^{-1}, matrix transpose A^T, matrix adjoint $A^\dagger$, etc.

— Problem 2 continues on the next page. —

In electromagnetism, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be written in terms of the scalar and vector potentials ϕ and $\vec{A}$ as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} , \quad \vec{B} = \vec{\nabla} \times \vec{A} .$$

For any function $\Lambda(\vec{x}, t)$, one can define a gauge transformation on ϕ and $\vec{A}$ by

$$\phi'(\vec{x}, t) = \phi(\vec{x}, t) - \frac{\partial \Lambda(\vec{x}, t)}{\partial t} , \quad \vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla}\Lambda(\vec{x}, t) .$$

(d) *(2 points)* If $\vec{E}'(\vec{x}, t)$ and $\vec{B}'(\vec{x}, t)$ are the new electric and magnetic fields, calculated from $\phi'(\vec{x}, t)$ and $\vec{A}'(\vec{x}, t)$, how are they related to the original $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$? [If you know the answer, you can write it without any derivation.]

$$\vec{E}'(\vec{x}, t) = \vec{E}(\vec{x}, t) \ + \ \underline{\hspace{4cm}}$$

$$\vec{B}'(\vec{x}, t) = \vec{B}(\vec{x}, t) \ + \ \underline{\hspace{4cm}}$$

(e) *(3 points)* Consider the line integral

$$W = \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} ,$$

where P is a closed loop in (three-dimensional) space. If $\vec{A}(\vec{x}, t)$ is replaced by its gauge transform $\vec{A}'(\vec{x}, t)$, how is the resulting W' related to the original W? [Here you should show a short derivation.]

— *Problem 2 continues on the next page.* —

(f) *(2 points)* The SU(5) grand unified theory relies on the fact that the gauge group of the standard model, SU(3)×SU(2)×U(1), is a subgroup of SU(5). Is SU(3)×SU(2)×U(1) also a subgroup of SU(6)? **Yes** or **No** .

(g) *(2 points)* Inflation explains the homogeneity of the universe by modifying the nature of the initial singularity, so the universe is created in a homogeneous state. **T** or **F** .

(h) *(2 points)* Inflation proposes that the early universe went through a period of nearly exponential expansion driven by the repulsive gravity caused by a material with a negative pressure. **T** or **F** .

(i) *(2 points)* In inflationary models, the enormous expansion tends to smooth out any nonuniformities that may have been present before inflation. Nonetheless, there remain faint ripples in the cosmic microwave background, because there was not enough inflation to smooth the universe completely. **T** or **F** .

— End of Problem 2 —

PROBLEM 3: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X *(30 points)*

The following problem was Problem 3 of Quiz 3, 2016, and Problem 10 of the Review Problems for Quiz 3, 2022.

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, X^+, X^-, and X^0, all with masses 0.511 MeV/c^2, exactly the same as the electron. The X^- is the antiparticle of the X^+, and the X^0 is its own antiparticle. Since the X's are spin-1 particles with nonzero mass, each particle has three spin states.

The X's do not interact with neutrinos any more strongly than the electrons and positrons do, so when the X's freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total energy density of the X^+, X^-, and X^0 particles?

(b) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total number density of the X^+, X^-, and X^0 particles?

(c) *(15 points)* The X particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as kT decreases from values that are large compared to 0.511 MeV/c^2 to values that are small compared to it. If the X's, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio T_ν/T_γ, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

(d) *(5 points)* If the mass of the X's was, for example, 0.100 MeV/c^2, so that the electron-positron pairs froze out first, and then the X's froze out, would the final ratio T_ν/T_γ be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

— End of Problem 3 —

PROBLEM 4: SIZE OF THE HORIZON ON THE SURFACE OF LAST SCATTERING *(30 points)*

In lecture we discussed an approximate calculation of the horizon problem as it relates to the cosmic microwave background, crudely estimating that the radius of the surface of last scattering was about 23 times larger than the horizon distance at the time of last scattering (which is also called the time of decoupling). Here you are asked to write the equations that determine this number accurately, taking into account all the known contributions to the energy density of the universe.

You should assume that the universe is flat, and that you are given the present values of the contributions to Ω of the known components: $\Omega_{\rm vac,0}$ for vacuum energy, $\Omega_{m,0}$ for nonrelativistic matter (dark matter and baryons), and $\Omega_{\rm rad,0}$ for radiation, with

$$\Omega_{\rm vac,0} + \Omega_{m,0} + \Omega_{\rm rad,0} = 1 \ . \tag{P4.1}$$

You are also given the present value H_0 of the Hubble expansion rate, the redshift $z_{\rm ls}$ of the surface of last scattering, and of course the speed of light c. Following Ryden's conventions, you should take $a(t_0) \equiv 1$.

(a) *(15 points)* What is the coordinate radius $\ell_{{\rm ls},c}$ of the surface of last scattering? Your answer should take the form of a definite integral, with explicit limits of integration. (Note that the function $a(t)$ is not given, so it should not appear in your final answer.)

(b) *(10 points)* What is the coordinate size $h_{{\rm ls},c}$ of the horizon distance at the time of last scattering? Your answer should again take the form of an integral, as in part (a).

(c) *(5 points)* If two points on the surface of last scattering were separated from each other by a horizon distance at the time of last scattering, what will be the angular separation between these two points as seen on Earth today? You may use small angle approximations. Your answer can include any of the given variables, and also the answers $\ell_{{\rm ls},c}$ and $h_{{\rm ls},c}$ to the previous two parts, whether or not you answered those parts.

Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 24, 2022
Prof. Alan Guth

QUIZ 3 SOLUTIONS

Quiz Date: December 7, 2022

Please answer all questions in this stapled booklet.
Ask if you need extra pages, but we expect that you will not.
Closed book, no calculators, no internet.
Formula Sheet will be handed out separately.
We intend to scan the quizzes, so please write inside the margins,
and if you use a pencil, write darkly. Thanks!

Problem	**Maximum**	**Score**
1	20	
2	20	
3	30	
4	30	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(2 points)* At the time of recombination, the temperature is

(i) above the proton and electron masses

(ii) below the proton mass but above the electron mass

(iii) $\boxed{\text{below the proton and electron masses}}$

Comment: The temperature of recombination was estimated in Lecture Notes 6 as about 4,000 K, and was calculated by Ryden after Eq. (8.37) as $T_{\text{rec}} = 3720\,\text{K}$. Ryden gives the equivalent energy in Eq. (8.37) as $kT_{\text{rec}} = 0.324$ eV, which is far below the proton mass ($m_p c^2 = 938$ MeV) or the electron mass ($m_e c^2 = 0.511$ MeV).

(b) *(2 points)* At the time of recombination, the universe is

(i) $\boxed{\text{matter-dominated}}$

(ii) radiation-dominated

(iii) vacuum-energy dominated

Comment: In Table 8.1 (p. 157) Ryden gives the time of recombination as 250,000 years, and the time of matter-radiation equality as 50,000 years. (For completeness, we mention that the same table gives the time of photon decoupling as 370,000 years.) In Table 5.2 (p. 96), Ryden gives the time of matter-lambda equality (i.e., the equality of matter and vacuum energy) as 10.2 Gyr. So the time of recombination is unambiguously in the matter-dominated era.

(c) *(2 points)* The reaction that produces the most deuterium in the early universe is

(i) proton-proton fusion ($p + p \to D+$ maybe other elements)

(ii) $\boxed{\text{proton-neutron fusion } (p + n \to D+ \text{ maybe other elements})}$

(iii) neutron-neutron fusion ($n + n \to D+$ maybe other elements)

Comment: This is discussed by Ryden on pp. 172–173.

(d) *(2 points)* For the microcanonical ensemble (describing an isolated system in thermal equilibrium), the probability that the system is in a state i with energy E_i is proportional to

(i) $\Omega_{\rm bath}(E_{\rm tot} - E_i)$, where $E_{\rm tot}$ is the energy of the heat bath and the small system and $\Omega_{\rm bath}$ is the density of states of the bath

(ii) $\exp(-E_i/kT)$

(iii) $\boxed{1/M \text{ if } E < E_i < E + \delta E, \text{ where } M \text{ is the total number of states in this energy range}}$

Comment: This was Eq. (1) in *Notes on Thermal Equilibrium.*

(e) *(2 points)* In the early universe, two important reactions occurred: deuterium production and hydrogen production from proton-electron recombination. How do the binding energies of deuterium and hydrogen compare?

(i) $\boxed{\text{the binding energy of deuterium is larger}}$

(ii) the binding energies of deuterium and hydrogen are (almost) exactly the same

(iii) the binding energy of hydrogen is larger

Comment: On p. 153, Ryden gives the binding energy of hydrogen as 13.6 eV, and on p. 174 she gives the binding energy of deuterium as 2.22 MeV. So the binding energy of deuterium is about 163,000 times larger.

(f) *(2 points)* The universe can be approximated as

(i) homogeneous and isotropic at all scales

(ii) homogeneous and isotropic on small scales ($\lesssim$ 100 Mpc)

(iii) $\boxed{\text{homogeneous and isotropic on large scales } (\gtrsim 100 \text{ Mpc})}$

(iv) inhomogeneous and anisotropic at all scales

Comment: see the first sentence of Ryden's Chaper 11.

(g) *(2 points)* In order to analyze the ionization and recombination of hydrogen as a function of temperature in the early universe, we studied the ratio of the number density of hydrogen atoms n_H to the product of the number densities of the free elements that react to create them: protons, with number density n_p, and electrons, with number density n_e. For 2 points credit, you can EITHER state the name that is given to the equation that describes this ratio, or you can write the equation. (If you try to write the equation, be sure to notice the relevant formulas on the formula sheet.)

This is the *Saha equation*, as given by Ryden as Eq. (8.29) on p. 153:

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right) ,$$

where $Q = (m_p + m_e - m_H)c^2$.

(h) *(2 points)* Gravity tends to make small density perturbations grow rapidly with time. What keeps small density perturbations from creating instabilities in the matter density and leading to collapse? (For example, what stops the air in this room from undergoing gravitational collapse.)

It is *pressure* that opposes gravitational collapse, as Ryden discusses in Sec. 11.2, *The Jeans Length.*

(i) *(2 points)* Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon-baryon fluid. The speed of sound in the baryonic gas is [larger than / equal to / $\boxed{\text{smaller}}$ than] the speed of sound in the photon gas.
Circle one

Comment: On p. 212, Ryden explains that the sound speed before decouping was $c/\sqrt{3}$, while after decoupling it plummets to 5 kilometer per second.

(j) *(2 points)* Baryon acoustic oscillations are due to the interaction of baryons with

(i) $\boxed{\text{photons}}$

(ii) dark matter

(iii) neutrinos

This is discussed by Ryden at the beginning of Sec. 11.6, *Baryon Acoustic Oscillations.*

PROBLEM 2: SHORT ANSWER QUESTIONS: GRAND UNIFIED THEORIES, MAGNETIC MONOPOLES, AND INFLATION *(20 points)*

(a) *(2 points)* A quark is specified by its flavor [u(p), d(own), c(harmed), s(trange), t(op), b(ottom)], its spin [up or down, along any chosen z axis], whether it is a quark or antiquark, and its color. $\boxed{\textbf{T}}$ or **F** .

(b) *(2 points)* Any isolated system of quarks must be a color singlet. A single red quark is an example of a color singlet, since it has only one color. **T** or $\boxed{\textbf{F}}$.

Comment: As discussed in Class 23, 11/28/22, the simplest color singlets — and the only ones known to exist in nature — are 3-quark states (baryons), with equal parts of red, blue, and green, and quark-antiquark states (mesons), with equal parts of each color and its anticolor. A singlet state is one that is invariant under $SU(3)$ gauge transformations.

(c) *(3 points)* Write the definition of the group SU(3). Assume that the reader does not know the meaning of "unitary" or "special", but that they do know the standard mathematical notation of matrix multiplication AB, matrix inversion A^{-1}, matrix transpose A^T, matrix adjoint $A^\dagger$, etc.

$SU(3)$ is the group of 3×3 complex matrices U which are "special," meaning that $\det U = 1$, and also "unitary," meaning that $U^\dagger U = 1$.

In electromagnetism, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be written in terms of the scalar and vector potentials ϕ and $\vec{A}$ as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} , \quad \vec{B} = \vec{\nabla} \times \vec{A} .$$

For any function $\Lambda(\vec{x}, t)$, one can define a gauge transformation on ϕ and $\vec{A}$ by

$$\phi'(\vec{x}, t) = \phi(\vec{x}, t) - \frac{\partial \Lambda(\vec{x}, t)}{\partial t} , \quad \vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla}\Lambda(\vec{x}, t) .$$

(d) *(2 points)* If $\vec{E}'(\vec{x}, t)$ and $\vec{B}'(\vec{x}, t)$ are the new electric and magnetic fields, calculated from $\phi'(\vec{x}, t)$ and $\vec{A}'(\vec{x}, t)$, how are they related to the original $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$? [If you know the answer, you can write it without any derivation.]

$$\vec{E}'(\vec{x}, t) = \vec{E}(\vec{x}, t) + \vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Lambda = \boxed{\vec{E}(\vec{x}, t) ,}$$

$$\vec{B}'(\vec{x}, t) = \vec{B}(\vec{x}, t) + \vec{\nabla} \times \vec{\nabla}\Lambda = \boxed{\vec{B}(\vec{x}, t) .}$$

(e) *(3 points)* Consider the line integral

$$W = \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} ,$$

where P is a closed loop in (three-dimensional) space. If $\vec{A}(\vec{x}, t)$ is replaced by its gauge transform $\vec{A}'(\vec{x}, t)$, how is the resulting W' related to the original W? [Here you should show a short derivation.]

$$\begin{aligned} W' &\equiv \oint_P \vec{A}'(\vec{x}, t) \cdot d\vec{x} \\ &= \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} + \oint_P \vec{\nabla}\Lambda \cdot d\vec{x} \\ &= \oint_P \vec{A}(\vec{x}, t) \cdot d\vec{x} = \boxed{W ,} \end{aligned}$$

since the line integral of a gradient around a closed loop always vanishes.

(f) *(2 points)* The SU(5) grand unified theory relies on the fact that the gauge group of the standard model, SU(3)×SU(2)×U(1), is a subgroup of SU(5). Is SU(3)×SU(2)×U(1) also a subgroup of SU(6)? $\boxed{\textbf{Yes}}$ or **No** .

Comment: SU(5) is clearly a subgroup of SU(6). To see this, note that an SU(6) matrix $U^{(6)}$ corresponding to an element $V^{(5)}$ of an SU(5) subgroup can be constructed explicitly as

$$U^{(6)}_{ij} = \begin{cases} V^{(5)}_{ij} & \text{if } i,\, j \in \{1,2,3,4,5\} \\ 1 & \text{if } i = j = 6 \\ 0 & \text{otherwise,} \end{cases}$$

or more pictorially as

$$\boldsymbol{U^{(6)}} = \left(\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & \boldsymbol{V^{(5)}} & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) .$$

If SU(3)×SU(2)×U(1) is a subgroup of SU(5), and SU(5) is a subgroup of SU(6), it follows that SU(3)×SU(2)×U(1) is a subgroup of SU(6).

(g) *(2 points)* Inflation explains the homogeneity of the universe by modifying the nature of the initial singularity, so the universe is created in a homogeneous state. **T** or $\boxed{\textbf{F}}$.

Comment: Inflation explains the homogeneity of the universe by inserting a period of colossal expansion into the scenario of the early universe, so that the presently observed universe can evolve from a tiny region in which there had been sufficient time for uniformity to have been established by standard thermal equilibrium processes. Inflation in no way affects the initial singularity, if there was an initial singularity.

(h) *(2 points)* Inflation proposes that the early universe went through a period of nearly exponential expansion driven by the repulsive gravity caused by a material with a negative pressure. $\boxed{\textbf{T}}$ or **F** .

(i) *(2 points)* In inflationary models, the enormous expansion tends to smooth out any nonuniformities that may have been present before inflation. Nonetheless, there remain faint ripples in the cosmic microwave background, because there was not enough inflation to smooth the universe completely. **T** or $\boxed{\textbf{F}}$.

Comment: The faint ripples in the cosmic microwave background are believed to have been caused by quantum fluctuations in the inflaton field as inflation was ending. Thus these fluctuations would be present no matter how much inflation took place.

PROBLEM 3: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X *(30 points)*

The following problem was Problem 3 of Quiz 3, 2016, and Problem 10 of the Review Problems for Quiz 3, 2022.

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, X^+, X^-, and X^0, all with masses 0.511 MeV/c^2, exactly the same as the electron. The X^- is the antiparticle of the X^+, and the X^0 is its own antiparticle. Since the X's are spin-1 particles with nonzero mass, each particle has three spin states.

The X's do not interact with neutrinos any more strongly than the electrons and positrons do, so when the X's freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total energy density of the X^+, X^-, and X^0 particles?

(b) *(5 points)* In thermal equilibrium when $kT \gg 0.511$ MeV/c^2, what is the total number density of the X^+, X^-, and X^0 particles?

(c) *(15 points)* The X particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as kT decreases from values that are large compared to 0.511 MeV/c^2 to values that are small compared to it. If the X's, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio T_ν/T_γ, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

(d) *(5 points)* If the mass of the X's was, for example, 0.100 MeV/c^2, so that the electron-positron pairs froze out first, and then the X's froze out, would the final ratio T_ν/T_γ be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

Answers:

(a) *(5 points)* The formula sheet tells us that the energy density of black-body radiation is

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ , \tag{S3.1}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases} . \tag{S3.2}$$

Since the X is spin-1, and 1 is an integer, the X particles are bosons and $g = 1$ per spin state. There are 3 species, X^+, X^-, and X^0, and each species we are told has three spin states, so there are a total of 9 spin states, so $g = 9$. Thus,

$$\boxed{u = 9\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3}\ .} \tag{S3.3}$$

Alternatively, one could count the X^+ and X^- as one species with a distinct particle and antiparticle, so $g_{X^+X^-}$ is given by

$$g_{X^+X^-} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 6\ . \tag{S3.4}$$

The X^0 is its own antiparticle, which means that the particle/antiparticle factor is one, so

$$g_{X^0} = \underbrace{1}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{1}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{3}_{\text{Spin states}} = 3\ , \tag{S3.5}$$

so the total g for X^+, X^-, and X^0 is again equal to 9.

(b) *(5 points)* The formula sheet tells us that the number density of particles in black-body radiation is

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ , \tag{S3.6}$$

where

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ . \end{cases} \tag{S3.7}$$

For bosons $g^* = g$, so g^* for the X particles is 9. Then

$$\boxed{n_X = 9\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .} \tag{S3.8}$$

(c) *(15 points)* We are told that, when the X particles freeze out, all of their energy and entropy is given to the photons. We use entropy rather than energy to determine the final temperature of the photons, because the entropy in a comoving volume is simply conserved, while the energy density varies as

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) . \tag{S3.9}$$

Thus, to track the energy, we need to know exactly how p behaves, and the behavior of p during freeze-out is complicated, and we have not calculated it in this course.

The formula sheet tells us that the entropy density of a constituent of black-body radiation is given by

$$s = g\frac{2\pi^2}{45}\;\frac{k^4T^3}{(\hbar c)^3} . \tag{S3.10}$$

(At this point I will depart from the solution given in the Review Problems for Quiz 3, using an approach that seems simpler, but the solution in the Review Problems is also correct.)

The quantity s is the entropy per physical volume, so the entropy per *coordinate* volume is given by

$$S = a^3 s . \tag{S3.11}$$

The conservation of entropy means that the total entropy in a coordinate volume is conserved, and that implies that the entropy per coordinate volume is conserved. In this situation, we are told then when the e^+e^- pairs and the X's freeze out, all of their entropy is given to the photons, and none to the neutrinos. If we use the subscripts 1 to denote times before this freeze-out begins, and 2 to denote times after the freeze-out is complete, we can conclude that

$$\begin{aligned} S_{\gamma,2} &= S_{\gamma,1} + S_{e^+e^-,1} + S_{X,1} , \\ S_{\nu,2} &= S_{\nu,1} , \end{aligned} \tag{S3.12}$$

where S_X refers to the total entropy density of X^+, X^-, and X^0. It follows that the ratios of the left-hand sides must equal the ratios of the right-hand sides:

$$\frac{S_{\gamma,2}}{S_{\nu,2}} = \frac{S_{\gamma,1} + S_{e^+e^-,1} + S_{X,1}}{S_{\nu,1}} . \tag{S3.13}$$

But each $S = a^3 s$ is proportional to the value of gT^3 for the relevant particle, with all other factors being particle-independent. Furthermore, before the freeze-outs, all these particles were at the same temperature. Thus, the above equation implies that

$$\frac{g_\gamma T^3_{\gamma,2}}{g_\nu T^3_{\nu,2}} = \frac{g_\gamma + g_{e^+e^-} + g_X}{g_\nu} , \tag{S3.14}$$

which can be solved for the temperature ratio to give

$$\frac{T_{\nu,2}}{T_{\gamma,2}} = \left(\frac{g_\gamma}{g_\gamma + g_{e^+e^-} + g_X}\right)^{1/3} . \tag{S3.15}$$

But $g_\gamma = 2$ and $g_{e^+e^-} = 7/2$, which can be calculated or taken from the formula sheet, so

$$\frac{T_{\nu,2}}{T_{\gamma,2}} = \left(\frac{2}{2 + \frac{7}{2} + 9}\right)^{1/3} = \boxed{\left(\frac{4}{29}\right)^{1/3} .} \tag{S3.16}$$

(d) *(5 points)* The answer would be the same, since it was completely determined by the conservation equation (S3.12) in the above answer. Regardless of the order in which the freeze-outs occurred, the total entropy from the e^+e^- pairs and the X's would ultimately be given to the photons, so the amount of heating of the photons would be the same.

PROBLEM 4: SIZE OF THE HORIZON ON THE SURFACE OF LAST SCATTERING *(30 points)*

In lecture we discussed an approximate calculation of the horizon problem as it relates to the cosmic microwave background, crudely estimating that the radius of the surface of last scattering was about 23 times larger than the horizon distance at the time of last scattering (which is also called the time of decoupling). Here you are asked to write the equations that determine this number accurately, taking into account all the known contributions to the energy density of the universe.

You should assume that the universe is flat, and that you are given the present values of the contributions to Ω of the known components: $\Omega_{\rm vac,0}$ for vacuum energy, $\Omega_{m,0}$ for nonrelativistic matter (dark matter and baryons), and $\Omega_{\rm rad,0}$ for radiation, with

$$\Omega_{\rm vac,0} + \Omega_{m,0} + \Omega_{\rm rad,0} = 1 \ . \tag{P4.1}$$

You are also given the present value H_0 of the Hubble expansion rate, the redshift $z_{\rm ls}$ of the surface of last scattering, and of course the speed of light c. Following Ryden's conventions, you should take $a(t_0) \equiv 1$.

(a) *(15 points)* What is the coordinate radius $\ell_{{\rm ls},c}$ of the surface of last scattering? Your answer should take the form of a definite integral, with explicit limits of integration. (Note that the function $a(t)$ is not given, so it should not appear in your final answer.)

(b) *(10 points)* What is the coordinate size $h_{{\rm ls},c}$ of the horizon distance at the time of last scattering? Your answer should again take the form of an integral, as in part (a).

(c) *(5 points)* If two points on the surface of last scattering were separated from each other by a horizon distance at the time of last scattering, what will be the angular separation between these two points as seen on Earth today? You may use small angle approximations. Your answer can include any of the given variables, and also the answers $\ell_{{\rm ls},c}$ and $h_{{\rm ls},c}$ to the previous two parts, whether or not you answered those parts.

Answers:

(a) If we let $t_{\rm ls}$ denote the time at which the CMB photons left the surface of last scattering, then the coordinate distance from the surface of last scattering to us is given by

$$\ell_{{\rm ls},c} = \int_{t_{\rm ls}}^{t_0} \frac{c\,dt}{a(t)} \ , \tag{S4.1}$$

since $c/a(t)$ is the coordinate speed of light. This cannot be our final answer, however, since we don't know $a(t)$ or $t_{\rm ls}$. However, we can find $a(t_{\rm ls})$, since

$$1 + z_{\rm ls} = \frac{a(t_0)}{a(t_{\rm ls})} \quad \Longrightarrow \quad \boxed{a(t_{\rm ls}) = \frac{1}{1+z_{\rm ls}} \ ,} \tag{S4.2}$$

where $z_{\rm ls}$ is given. We can make use of this by changing the variable of integration in Eq. (S4.1) to a. To carry out this change of variable we need to know $\dot{a} \equiv da/dt$, which can be found from the first order Friedmann equation. Using

$$\Omega \equiv \frac{\rho}{\rho_c} \ , \ \text{and} \ \rho_c \equiv \frac{3H^2}{8\pi G} \ , \tag{S4.3}$$

we can write

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho(t) \ . \tag{S4.4}$$

Here $H(t)$ is the Hubble expansion rate and $\rho(t)$ is the total mass density. The mass density of the universe has 3 components: vacuum mass density $\rho_{\rm vac}$ which is independent of time, matter density $\rho_m(t)$ which falls off as $1/a^3$, and radiation density $\rho_{\rm rad}(t)$ which falls off as $1/a^4$. Thus

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3} G\left[\rho_{\rm vac} + \rho_m(t) + \rho_{\rm rad}(t)\right] \\ &= \frac{H_0^2}{\rho_{c,0}}\left[\rho_{\rm vac} + \frac{\rho_{m,0}}{a^3} + \frac{\rho_{\rm rad,0}}{a^4}\right] \\ &= H_0^2\left[\Omega_{\rm vac} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{\rm rad,0}}{a^4}\right] \ . \end{aligned} \tag{S4.5}$$

Thus, the coordinate distance between us and the surface of last scattering is then

$$\begin{aligned} \ell_{{\rm ls},c} &= \int_{t_{\rm ls}}^{t_0} \frac{c\,dt}{a(t)} = \int_{t_{\rm ls}}^{t_0} \frac{c\,a\,\dot{a}\,dt}{a^2\,\dot{a}} = \int_{a(t_{\rm ls})}^{1} \frac{c\,da}{Ha^2} \\ &= \frac{c}{H_0}\int_{a(t_{\rm ls})}^{1} \frac{da}{a^2\sqrt{\frac{\Omega_{\rm rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\rm vac,0}}} \\ &= \boxed{\frac{c}{H_0}\int_{a(t_{\rm ls})}^{1} \frac{da}{\sqrt{\Omega_{\rm rad,0} + \Omega_{m,0}\,a + \Omega_{\rm vac,0}\,a^4}} \ ,} \end{aligned} \tag{S4.6}$$

where $a(t_{\rm ls})$ is given in Eq. (S4.2).

Alternatively, one can use z as the variable of integration, where

$$a = \frac{1}{1+z} \quad \Longrightarrow \quad da = -\frac{dz}{(1+z)^2} \ , \tag{S4.7}$$

which leads to

$$\boxed{\ell_{\mathrm{ls},c} = \frac{c}{H_0} \int_0^{z_{\mathrm{ls}}} \frac{dz}{\sqrt{\Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\mathrm{vac},0}}} \ .} \tag{S4.8}$$

Note that the minus sign in Eq. (S4.7) implies that the limits of integration of Eq. (S4.8) are reversed relative to Eq. (S4.6). The lower limit in Eq. (S4.6), $a = a(t_{\mathrm{ls}})$, corresponds to the upper limit, $z = z_{\mathrm{ls}}$, in Eq. (S4.8), while the upper limit in Eq. (S4.6), $a = 1$, corresponds to the lower limit in Eq. (S4.8), $z = 0$.

(b) $h_{\mathrm{ls},c}$ is the total coordinate distance that light could have traveled between $t = 0$ and $t = t_{\mathrm{ls}}$, so the answer is the same as in part (a), except for the limits of integration. So

$$\boxed{h_{\mathrm{ls},c} = \frac{c}{H_0} \int_0^{a(t_{\mathrm{ls}})} \frac{da}{\sqrt{\Omega_{\mathrm{rad},0} + \Omega_{m,0}\, a + \Omega_{\mathrm{vac},0}\, a^4}} \ .} \tag{S4.9}$$

If one prefers to use z as the integration variable, this would be written as

$$\boxed{\ell_{\mathrm{ls},c} = \frac{c}{H_0} \int_{z_{\mathrm{ls}}}^{\infty} \frac{dz}{\sqrt{\Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\mathrm{vac},0}}} \ .} \tag{S4.10}$$

(c) Since the comoving diagram describes a flat (Euclidean) space, the angle subtended by a length $h_{\mathrm{ls},c}$ at a distance $\ell_{\mathrm{ls},c}$ is, in small angle approximation, given simply by the ratio

$$\boxed{\theta = \frac{h_{\mathrm{ls},c}}{\ell_{\mathrm{ls},c}} \ .} \tag{S4.11}$$

You were of course not asked to evaluate these quantities numerically, but you might be interested in the answers. Some of the answers were already given in Lecture Notes

8. We use the Planck 2018 parameters*

$$\begin{aligned} H_0 &= 67.36 \text{ km s}^{-1}\,\text{Mpc}^{-1} \\ \Omega_{m,0} &= 0.3153 \\ \Omega_{\text{vac},0} &= 0.6847 \\ z_{\text{ls}} &= 1089.92\ , \end{aligned} \tag{S4.12}$$

and $\Omega_{\text{rad},0}$ given by

$$\Omega_{\text{rad},0} = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right] \frac{\pi^2}{30} \frac{(kT_0)^4}{\hbar^3 c^5} \cdot \frac{8\pi G}{3H_0^2}\ , \tag{S4.13}$$

[See Eq. (6.65) of Lecture Notes 6 and Eq. (3.33) of Lecture Notes 3] with†

$$T_0 = 2.72548 \text{ K}\ . \tag{S4.14}$$

Eq. (S4.13) then gives

$$\Omega_{\text{rad},0} = 9.163 \times 10^{-5}\ . \tag{S4.15}$$

One then finds that the comoving distance (i.e., the coordinate distance with $a(t_0) \equiv 1$) to the surface of last scattering is

$$\ell_{\text{ls},c} = 45.23 \times 10^9 \text{ light-years}\ , \tag{S4.16}$$

and that the comoving horizon distance at the time of last scattering is

$$h_{\text{ls},c} = 0.9150 \times 10^9 \text{ light-years.} \tag{S4.17}$$

The sum of these two numbers is the current horizon distance,

$$\ell_{p,\text{horizon}}(t_0) = 46.15 \times 10^9 \text{ light-years}\ . \tag{S4.18}$$

One finds that the horizon problem for the CMB is actually worse than the very crude calculation that we did in class, by about a factor of 2. In class we found that the radius of the surface of last scattering was about 23 horizon distances, while these numbers tell

* Planck Collaboration, N. Aghanim et al., "Planck 2018 results, VI: Cosmological parameters," Astron. & Astrophys. 641, A6 (2020), arXiv:1807.06209v4 [astro-ph.CO], abstract and column 5 of Tables 1 or 2.

† D.J. Fixsen, "The Temperature of the Cosmic Microwave Background," Astrophys. J. 707, 916–920 (2009), arXiv:0911.1955v2 [astro-ph.CO].

us that the radius of the surface of last scattering was about 49.43 horizon distances. Thus, two points on the CMB sky separated by

$$\theta \approx \frac{1}{49.43} \text{ radian} \approx 1.159^\circ \tag{S4.19}$$

were 1 horizon distance apart on the surface of last scattering, at the time of decoupling. If one treats the quoted uncertainties in the various quantities as being independent, the final result can be written as

$$\theta = 1.159^\circ \pm 0.006^\circ \ . \tag{S4.20}$$

Thus, CMB photons that we receive from points separated by slightly more that 1.16° were last scattered from points that were separated by more than a horizon distance, so neither point could have received a signal from the other. They could, however, have both received a signal from some point in between the two. For 2 points on the CMB sky separated by slightly more than twice this angle, i.e., slightly more than 2.32°, then the two points of last scattering could not have even had any common influences.