MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 6, 1994
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QUIZ 2

USEFUL INFORMATION:

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $\quad R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following questions are worth 5 points each. Where two items are requested, you will receive 3 points for getting one right.

a) Weinberg emphasizes that most of the detailed properties of the early universe are determined by the assumption that it was in a state of thermal equilibrium. Thermal equilibrium, however, cannot change a conserved quantity, so each conserved quantity must be specified. Weinberg mentions three conserved quantities whose densities must be specified in the recipe for the early universe. One is electric charge (which is specified to be zero or negligibly small). What are the other two?

b) An important number in cosmology is the ratio of baryon number to the number of photons. Is this ratio approximately 10^{-9}, 10^{-3}, 1, or 10^{6}?

c) At three minutes after the big bang, when the processes of nucleosynthesis were nearing completion, the energy density of the universe was dominated by two types of particles from the following list: pions, protons, neutrons, photons, neutrinos, electrons, positrons, muons, quarks, and kaons. What were these two types of particles?

d) Calculations of big bang nucleosynthesis were carried out as early as the 1940's by George Gamow and his collaborators Ralph Alpher and Robert Herman. They tried unsuccessfully to explain the abundances of all species of nuclei in terms of synthesis during the big bang. In contrast, scientists today believe that a) all elements other than hydrogen were synthesized primarily in stars; b) all elements other than hydrogen, helium, and perhaps lithium were synthesized primarily in stars; c) all elements heavier than calcium were synthesized in stars, while those lighter than calcium were synthesized mainly in the big bang; or d) all elements heavier than iron were synthesized in stars, while those lighter than iron were synthesized mainly in the big bang. Which choice is correct?

e) At a temperature of about $3 \times 10^{15}\,{}^\circ\mathrm{K}$ ($kT \approx 300$ GeV $= 3 \times 10^{11}$ eV), it is believed that the matter in the early universe underwent a phase transition. The phase transition was marked by the change in behavior of the interactions of physics. In particular, there are two interactions that are distinct at low temperatures, but behave in a unified way at temperatures above this phase transition. What are these two interactions?

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE *(20 points)*

This problem appeared in the "Review Problems for Quiz 2," as Problem 8.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$R(t) = R_0 e^{\chi t} ,$$

where R_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift Z. Express the time t_r at which we receive the pulse in terms of Z, χ, and any relevant physical constants.

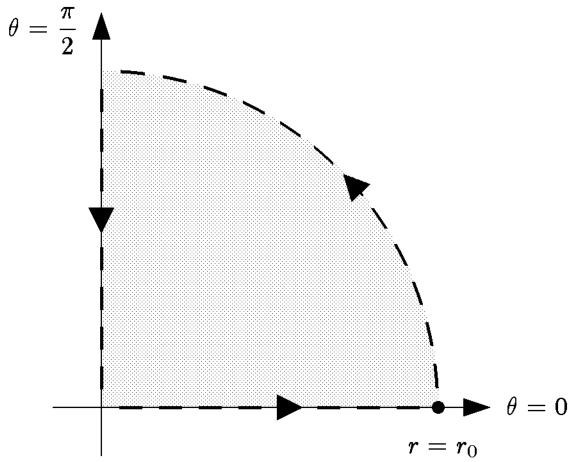
(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of Z, χ, and any relevant physical constants.

PROBLEM 3: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2 \, dr^2 + r^2 (1 + br)^2 \, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to $\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



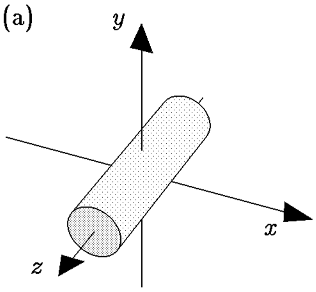
a) *(10 points)* Find the total length of this path.

b) *(15 points)* Find the area enclosed by this path.

PROBLEM 4: A CYLINDRICAL UNIVERSE *(30 points)*

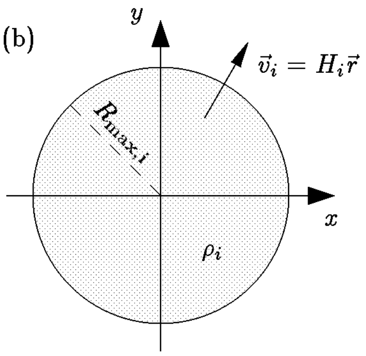
The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:

(a)



3D Perspective

(b)



2D Cross Section
(Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \; .$$

a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \; ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

c) *(8 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $R(t) \equiv u(r_i, t)$.

d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $R(t)$. Use this expression to obtain an expression for $\ddot{R}$ in terms of R, ρ, and any relevant constants.

e) *(7 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{R}^2 + V(R) .$$

What is $V(R)$? Will this universe expand forever, or will it collapse?