MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 23, 1996
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QUIZ 2 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) the surface of a sphere *(1 point)*; more than 180 degrees *(2 points)*; less than 2π times its radius *(2 points)*.

b) The repulsive force would be too strong, so the universe would expand *(3 points)*. Since the mass density would go down, while the cosmological constant would remain constant, the expansion would continue forever *(2 points)*.

c) (This was Problem 12(d) on the Review Problems for Quiz 2.) The four answers that I had in mind were:

(1) They assumed that the universe began in a state of all neutrons, rather the thermal equilibrium mix assumed in modern calculations.

(2) They took into account the conversion of neutrons to protons only by free decay of the neutrons. They ignored the reactions

$$n + e^{+} \longleftrightarrow p + \bar{\nu}_e$$
$$n + \nu_e \longleftrightarrow p + e^{-} ,$$

which play a very important role in modern calculations.

(3) They attempted (unsuccessfully) to account for all of nucleosynthesis — they did not realize that the nucleosynthesis of heavier elements takes place primarily in the interior of stars.

(4) They used fewer than the presently accepted number of neutrinos.

A few other answers were also given full credit.

d) Silk mentioned helium and deuterium. Silk was apparently talking about He^4, but full credit was given for "helium," "He^3," or "He^4."

e) The synthesis of light chemical elements in the big bang occurred mainly at about $3\frac{3}{4}$ minutes.

PROBLEM 2: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\frac{dx^\ell}{d\lambda} .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2\, d\theta^2 ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 , \qquad g_{r\theta} = g_{\theta r} = 0 .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{g_{rr}\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r g_{\theta\theta}\right)\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r r^2\right)\left(\frac{d\theta}{d\lambda}\right)^2 ,$$

or

$$\boxed{\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i,j). So

$$\boxed{\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 .}$$

(b) The first step is to parametrize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parametrized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parametrization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ ,$$

$$\theta(\lambda) = \tan^{-1}\frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ .$$

Calculating the needed derivatives,*

$$\frac{dr}{d\lambda} = \frac{\lambda}{\sqrt{\lambda^2+1}}$$

$$\frac{d^2r}{d\lambda^2} = \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3}$$

$$\frac{d\theta}{d\lambda} = -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2}\frac{1}{\lambda^2} = -\frac{1}{r^2} \ .$$

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* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi/\cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = (1 + \tan^2\phi)d\phi \ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2} \ .$$

Note that the formula for $d\tan^{-1}u/du$ shown on the quiz has the wrong sign, as was announced during the quiz.

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \Longleftrightarrow \frac{1}{r^3} = r\left(-\frac{1}{r^2}\right)^2 \ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \Longleftrightarrow \frac{d}{d\lambda}\left\{r^2\left(-\frac{1}{r^2}\right)\right\} = 0 \ ,$$

which also checks.

PROBLEM 3: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE

a) The geodesic is along a radial line, so $d\theta = d\phi = 0$. Then $d\tau = 0$, which is always true for a light pulse traveling in a vacuum, implies that

$$-c^2\,dt^2 + R^2(t)\,d\psi^2 = 0 \ , \tag{1}$$

or

$$\frac{d\psi}{dt} = -\frac{c}{R(t)} \ .$$

Note that Eq. (1) has two roots, $d\psi/dt = \pm c/R(t)$, but the negative sign is right for this problem because the value of ψ for the light pulse starts at ψ_G (which is always positive) and ***decreases*** to 0. Integrating,

$$d\psi = -\frac{c}{R(t)}\,dt$$

$$\int_{\psi_G}^{0} d\psi = -\int_{t_G}^{t_0}\frac{c}{R(t)}\,dt$$

$$\boxed{\psi_G = \int_{t_G}^{t_0}\frac{c}{R(t)}\,dt \ .}$$

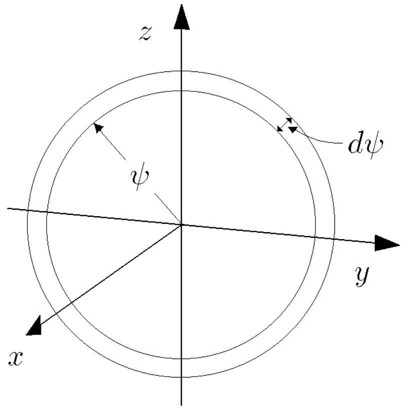
b) As stated on the front of the exam, the redshift in general is given by

$$1+Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} .$$

Since $t_{\text{observed}} = t_0$ and $t_{\text{emitted}} = t_G$, it follows that

$$\boxed{Z_G = \frac{R(t_0)}{R(t_G)} - 1 .}$$

c) To find the volume of space with redshifts smaller than that of galaxy G, the first step is to recognize that the redshift increases monotonically with ψ_G. (If you doubt this statement, note that the answer to (a) implies that t_G decreases monotonically with ψ_G. Assuming that $R(t)$ is monotonically increasing, the answer to (b) then implies that Z_G increases monotonically with ψ_G.) Thus, the region with Z smaller than that of galaxy G is the region with $0 < \psi < \psi_G$. To integrate the volume of this region, divide space into concentric shells, with radial coordinate ψ and coordinate thickness $d\psi$, as shown:



The area of the spherical shell is determined by the metric on the surface, which can be obtained from the full metric by treating t and ψ as fixed:

$$ds^2 = R^2(t)\sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2) .$$

This expression is identical to the metric of the surface of a sphere of radius $r = R(t)\sinh\psi$. The area is therefore $A = 4\pi r^2 = 4\pi R^2(t)\sinh^2\psi$. Looking again at the metric, one sees that the physical thickness of the shell is $ds = R(t)\,d\psi$. The volume of the shell is then

$$dV = A\,R(t)\,d\psi = 4\pi R^3(t)\sinh^2\psi\,d\psi ,$$

and the total volume is found by integration:

$$\boxed{V = 4\pi R^3(t)\int_0^{\psi_G} \sinh^2\psi\,d\psi .}$$

Extension: You were not asked to evaluate the integral, but it can be done as follows:

$$\begin{aligned}
\int_0^{\psi_G} \sinh^2\psi\,d\psi &= \int_0^{\psi_G}\left[\frac{e^{\psi}-e^{-\psi}}{2}\right]^2 d\psi \\
&= \frac{1}{4}\int_0^{\psi_G}\left[e^{2\psi}+e^{-2\psi}-2\right] d\psi \\
&= \frac{1}{4}\left[\frac{1}{2}e^{2\psi}-\frac{1}{2}e^{-2\psi}-2\psi\right]\Bigg|_0^{\psi_G} \\
&= \frac{1}{4}\left[\frac{1}{2}\left(e^{2\psi_G}-e^{-2\psi_G}\right)-2\psi_G\right] \\
&= \frac{1}{4}\left[\sinh(2\psi_G)-2\psi_G\right] .
\end{aligned}$$

The volume is then

$$\boxed{V = \pi R^3(t)\left[\sinh(2\psi_G)-2\psi_G\right] .}$$

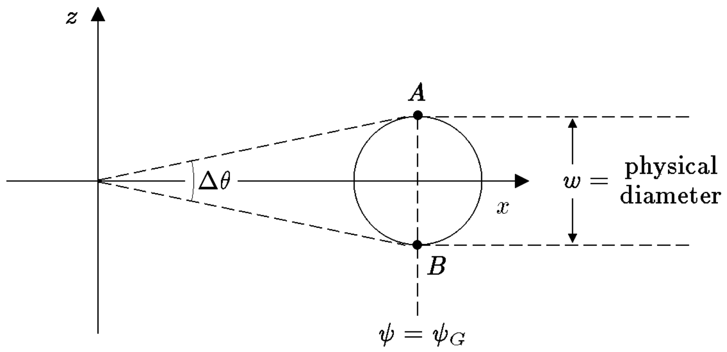
d) The proper distance between (ψ, θ, ϕ) and $(\psi + d\psi, \theta, \phi)$ is what is measured by a ruler at rest in this coordinate system, but that is exactly the meaning of the ds that appears in the expression for the metric. Since t, θ, and ϕ are constant along the radial line between Earth and the galaxy G, the metric at $t = t_0$ reduces to

$$ds = R(t_0)\, d\psi \ .$$

Integrating,

$$\boxed{\ell_{\rm prop} = \int ds = R(t_0)\, \psi_G \ .}$$

e) The calculation of the angular size distance is similar to the angular size calculation in Problem 5 of Problem Set 1, but it is not quite identical. There we were talking about a flat universe, but this time we are interested in a curved universe. The basic method is the same, however, so long as we remember that all distances have to be determined via the metric. Placing the galaxy for convenience along the x axis ($\theta = 0$), we draw it at the time of emission, t_G:



We draw the picture at the time of emission, because the photons that we receive today arrive on trajectories that were determined solely by the position of the galaxy at that time. Using the metric, we can express

the physical diameter of the galaxy at the time of emission. The only coordinate that changes between the points A and B is θ, so

$$w = ds = R(t_G)\, \sinh \psi_G\, \Delta\theta \ .$$

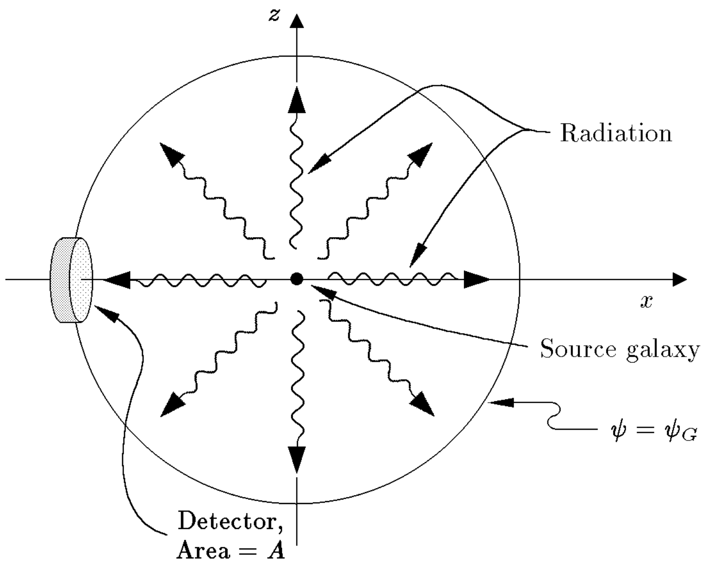
The angular size distance is then

$$\boxed{\ell_{\rm ang} \equiv \frac{w}{\Delta\theta} = R(t_G)\, \sinh \psi_G \ .}$$

Subtlety: To be sure that the above solution is correct, one must know that the coordinate separation $\Delta\theta$ between the points A and B, at the time of emission, is equal to the angular size that we observe today. This equality can be justified by using the fact that we are located at the origin of this coordinate system, and therefore the photons that we detect arrive along radial lines. (One can verify that trajectories that move along radial lines at the speed of light are geodesics, but I will not try to do that here.) The photons that left point A, at $\theta = \frac{1}{2}\Delta\theta$, will arrive today along the radial line at $\theta = \frac{1}{2}\Delta\theta$. Similarly, the photons that left point B, at $\theta = -\frac{1}{2}\Delta\theta$, will arrive today along the radial line at $\theta = -\frac{1}{2}\Delta\theta$. Thus, the angular size that we observe, the angular separation between these two radial lines, is $\Delta\theta$.

f) Following the hint, we draw Robertson-Walker coordinates with the galaxy G in the center. The radial coordinate of the detector, on Earth, will be

ψ_G. The diagram also shows a sphere at the same radial coordinate, ψ_G:



Since the speed of light is independent of angle, all the photons that left the galaxy G at time t_G are arriving at the $\psi = \psi_G$ sphere at the present time, t_0. To calculate the power received by the detector, we need to know what fraction of those photons hit the detector. The fraction is simply the area of the detector divided by the area of the sphere, or

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi R^2(t_0)\sinh^2\psi_G} .$$

(The formula for the area was discussed in the answer to (c).) The power hitting the detector is further reduced by one factor of $(1+Z) = R(t_0)/R(t_G)$ because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of $(1+Z)$ because the rate of arrival of photons is reduced by this factor. Thus, if P is the power that the galaxy was emitting at time t_G,

then the power received by the detector today is

$$\begin{aligned} P_{\text{received}} &= P\frac{A}{4\pi R^2(t_0)\sinh^2\psi_G}\left[\frac{R(t_G)}{R(t_0)}\right]^2 \\ &= P\frac{AR^2(t_G)}{4\pi R^4(t_0)\sinh^2\psi_G} . \end{aligned}$$

The flux is given by

$$J = \frac{P_{\text{received}}}{A} = P\frac{R^2(t_G)}{4\pi R^4(t_0)\sinh^2\psi_G} .$$

From the definition of luminosity distance,

$$\boxed{\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4\pi J}} = \frac{R^2(t_0)\sinh\psi_G}{R(t_G)} .}$$