MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe May 7, 1996
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QUIZ 3 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

(a) (i) is true. (ii) is false. The possibility of a limiting temperature, introduced by R. Hagedorn of CERN, was taken very seriously in the years before the quark model was accepted. Today, however, no one considers it plausible. (iii) is true. Weinberg did not name the experimenters, but the team was led by Jerome Friedman and Henry Kendall of MIT, and Richard Taylor of SLAC (Stanford Linear Accelerator Center). They were awarded the Nobel Prize for this work in 1990. (iv) is false. At "several million million degrees Kelvin," as Weinberg puts it, protons and neutrons are believed to break apart into their constituent quarks, which then behave as a free gas of quarks. This temperature corresponds to kT on the order of several hundred MeV. Despite the apparent weakness of the interquark force at these high temperatures, the force between quarks is believed to be so strong that an isolated quark is absolutely impossible. The forces are weak at short distances, however, and when kT exceeds several MeV quark-antiquark pairs are produced as a blackbody radiation that is so dense that the interquark distances are short, and the forces are weak.

(b) It is called the Planck era.

(c) This was Problem 1(a) on the Review Problems for Quiz 3. The subject is discussed in Joseph Silk's book, on pp. 126–27. Statement (A) is true. A cosmic string can cause a double image, with one image visible on each side of the string. If the string crosses the line of sight of several distant galaxies, then these double images will form a line. Statement (B) is also true: a moving cosmic string actually causes a temperature differential between the radiation passing on either side of the string. Statement (C) is false: the galaxies that would be formed by cosmic string perturbations would look like normal galaxies, so it is conceivable that all the galaxies that we observe were in fact formed by cosmic strings. Statement (D) is true: cosmic strings actually dissipate most of their energy in gravitational radiation. Today these waves would distort the space through which the pulsar's radio waves are traveling, resulting in a nonuniformity in the time intervals between signals.

PROBLEM 2: FREEZE-OUT OF MUONS

(a) The factors contributing to g from the muons are the following:

2 since there are two particles, the muon and the antimuon

2 since there are two spin states for each particle

$\frac{7}{8}$ since the μ^- and the μ^+ are fermions

Thus

$$\boxed{g_\mu = 2 \times 2 \times \frac{7}{8} = \frac{7}{2} \ .}$$

(b) Besides the muons, the particles in thermal equilibrium when kT is just above 106 MeV are photons, neutrinos and electrons. As found in class

$$\begin{aligned} g_\gamma &= 2 \\ g_\nu &= 3 \times 2 \times \frac{7}{8} = \frac{21}{4} \\ g_{e^-} &= 2 \times 2 \times \frac{7}{8} = \frac{7}{2} \ . \end{aligned}$$

So, for kT just above 106 MeV, g is the sum of all of these contributions:

$$\boxed{g = g_\mu + g_\gamma + g_\nu + g_{e^-} = \frac{57}{4} = 14.25 \ .}$$

(c) We know that entropy S is conserved and that it can be written as $S = R^3 \times s$ where s is the entropy density. The expression for the entropy density is given on the cover of the exam. It is

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \ .$$

Therefore $S = R^3 \times s$ is given by

$$S = C \times g(T) R^3 T^3 \quad \text{where} \quad C = \text{constant}.$$

Let T_a denote the temperature of the universe when kT_a is just *above* 106 MeV. Let T_b denote the temperature of the universe when kT_b is just *below* 106 MeV. Since the entropy is constant $S(T_a) = S(T_b)$. Using the above expression for S we find

$$C \times g(T_a)(RT_a)^3 = C \times g(T_b)(RT_b)^3 \qquad \Longrightarrow$$

$$\frac{RT_b}{RT_a} = \left[\frac{g(T_a)}{g(T_b)}\right]^{1/3} \ .$$

We found $g(T_a)$ in part (b), $g(T_a) = 14.25$. After the muons disappear from the black body radiation they no longer contribute to the g in the expression for the entropy. Thus at temperatures below T_b, $g(T_b) = g_\gamma + g_\nu + g_{e^-} = 2 + \frac{21}{4} + \frac{7}{2} = \frac{43}{4} = 10.75$. Using these values in the expression above we obtain the increase in RT,

$$RT_b = \left(\frac{14.25}{10.75}\right)^{1/3} RT_a = \left(\frac{57}{43}\right)^{1/3} RT_a \qquad \Longrightarrow$$

$$\boxed{RT_b \approx (1.10) RT_a \ .}$$

PROBLEM 3: CHEMICAL EQUILIBRIUM FOR HELIUM PRODUCTION

(a) The number density of each of the three species can be expressed in terms of its chemical potential by using the formula on the front of the exam:

$$n_n = g_n \frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = g_p \frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_p - m_p c^2)/kT}$$

$$n_\alpha = g_\alpha \frac{(2\pi m_\alpha kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_\alpha - m_\alpha c^2)/kT} .$$

The proton and neutron are each spin-$\frac{1}{2}$ particles of nonzero rest mass, so they each have $2s+1$ spin states, with $s = \frac{1}{2}$, so $g_n = g_p = 2$. The α particle is spinless, so it has **one** possible spin state — the state of zero angular momentum. Alternatively, one can use the formula $g = 2s+1$, so $s = 0$ implies one spin state, or $g_\alpha = 1$. The ratio is then given by

$$\frac{n_n^2 n_p^2}{n_\alpha} = \frac{m_n^3 m_p^3}{\sqrt{2}\hbar^9 m_\alpha^{3/2}} \left(\frac{kT}{\pi}\right)^{9/2} \exp\left\{-\left[(2m_n + 2m_p - m_\alpha)c^2 + \right.\right.$$
$$\left.\left. + \ (\mu_\alpha - 2\mu_n - 2\mu_p)\right]/kT\right\} .$$

The binding energy B is defined as the energy that is released when an alpha particle is formed, so

$$2m_n c^2 + 2m_p c^2 = m_\alpha c^2 + B .$$

Since the sum of the chemical potentials on the left-hand side of the reaction equation must equal the sum of the chemical potentials on the right-hand side, we have

$$2\mu_n + 2\mu_p = \mu_\alpha ,$$

where I used the fact that the chemical potential of the photon is necessarily zero, since the photon carries no conserved quantities. Using the two relations above, the expression for the ratio simplifies to

$$\boxed{\frac{n_n^2 n_p^2}{n_\alpha} = \frac{m_n^3 m_p^3}{\sqrt{2}\hbar^9 m_\alpha^{3/2}} \left(\frac{kT}{\pi}\right)^{9/2} e^{-B/kT} .}$$

It is a good approximation to set $m_n = m_p$ and $m_\alpha = 4m_p$ in the prefactor, which gives the simpler expression

$$\boxed{\frac{n_n^2 n_p^2}{n_\alpha} = \frac{1}{8\sqrt{2}\hbar^9}\left(\frac{m_p kT}{\pi}\right)^{9/2} e^{-B/kT} \ .}$$

Either of the boxed answers is completely acceptable.

(b) For clarity, I will number the equations that we are given:

$$n_B = n_n^{\rm tot} + n_p^{\rm tot} \tag{1}$$

$$f = \frac{n_n^{\rm tot}}{n_B} \tag{2}$$

$$x = \frac{n_n}{n_n^{\rm tot}} \ . \tag{3}$$

Since there are two neutrons and two protons inside each alpha particle, one also has

$$n_n^{\rm tot} = n_n + 2n_\alpha \tag{4}$$

$$n_p^{\rm tot} = n_p + 2n_\alpha \ . \tag{5}$$

Then, using Eqs. (3) and then (2),

$$\boxed{n_n = x n_n^{\rm tot} = x f n_B \ .} \tag{6}$$

Using Eqs. (4) and then (2) and (6),

$$\begin{aligned} n_\alpha &= \frac{1}{2}\left[n_n^{\rm tot} - n_n\right] \\ &= \frac{1}{2}\left[f n_B - x f n_B\right] \\ &= \boxed{\frac{1}{2}(1-x) f\, n_B \ .} \end{aligned} \tag{7}$$

Finally, using Eq. (5), followed by (1), (7), and (2), one finds

$$\begin{aligned} n_p &= n_p^{\rm tot} - 2n_\alpha \\ &= \left(n_B - n_n^{\rm tot}\right) - 2\left[\frac{1}{2}(1-x) f\, n_B\right] \\ &= \left(n_B - f\, n_B\right) - 2\left[\frac{1}{2}(1-x) f\, n_B\right] \\ &= \boxed{(1 - 2f + xf)\, n_B \ .} \end{aligned} \tag{8}$$

(c) By combining the answers to the previous part, one has

$$\frac{n_n^2 n_p^2}{n_\alpha} = \frac{2fx^2(1-2f+fx)^2}{1-x}\, n_B^3 \ .$$

To eliminate n_B, use $\eta \equiv n_B/n_\gamma$, and then use the formula on the front of the exam for the number density of photons in blackbody radiation. For photons $g^* = 2$, so

$$n_B = \eta\, n_\gamma = \eta \frac{2\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ .$$

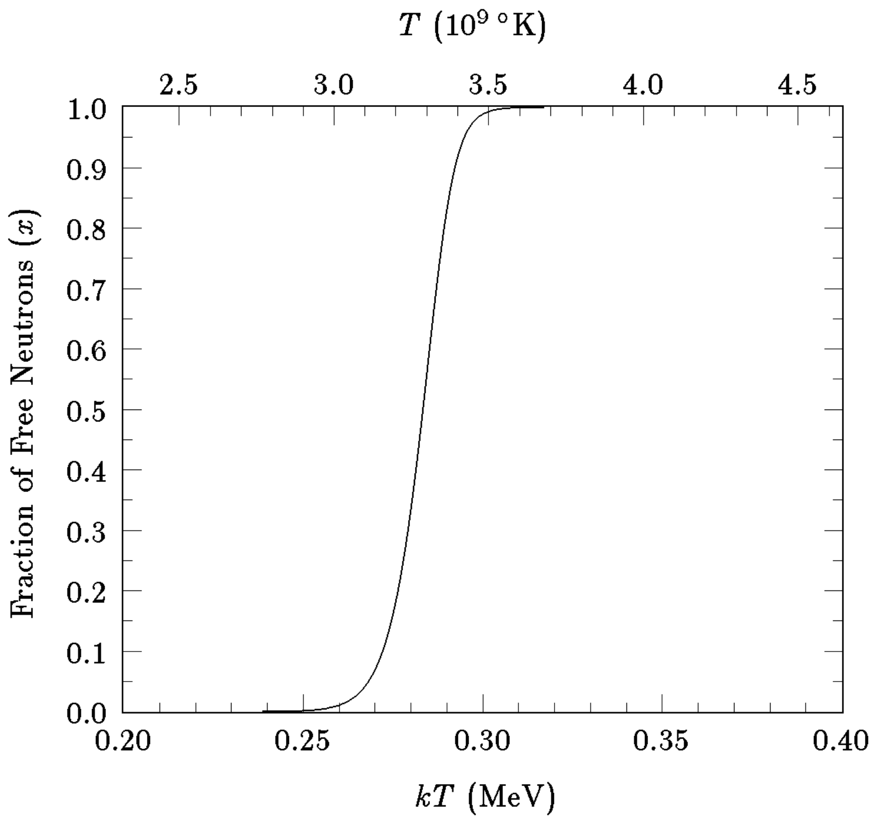
Inserting into the expression above,

$$\frac{n_n^2 n_p^2}{n_\alpha} = \frac{2fx^2(1-2f+fx)^2}{1-x} \left[\eta \frac{2\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3}\right]^3 \ .$$

By setting this expression equal to the expression obtained in part (a) for $n_n^2 n_p^2/n_\alpha$, we have the desired result. By using the second (approximate) answer from part (a), and by bringing the quantity in square brackets above to the right-hand side, one obtains

$$\boxed{\frac{2fx^2(1-2f+fx)^2}{1-x} = \frac{\pi^{3/2}}{64\sqrt{2}\,\zeta^3(3)\,\eta^3} \left(\frac{m_p c^2}{kT}\right)^{9/2} e^{-B/kT} \ .}$$

On the quiz you were expected to stop here, but it is interesting to see a graph of this relationship. Taking $\eta = 5 \times 10^{-10}$, $f = 1/8$, $B = 28.3$ MeV, $m_p c^2 = 938.3$ MeV, one finds:



Comparing this graph to the graph for deuterium abundance, one sees that He^4 is stable at temperatures about four times higher than the stability point for deuterium.

Remember that this does not mean that He^4 forms before deuterium in the early universe. The graph above shows what would happen if the universe expanded slowly enough to keep the helium abundance in equilibrium. But the processes that form He^4 begin with deuterium, so when the reaction rates are taken into account, one finds that essentially no helium is formed until the temperatures fall enough for deuterium to be stable. This delay in helium production is usually called the "deuterium bottleneck."