MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 9, 1996
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QUIZ 2

USEFUL INFORMATION:

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $\quad R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$
$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$
$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda}\left\{ g_{ij} \frac{dx^j}{d\lambda} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{d\lambda} \frac{dx^\ell}{d\lambda}$$

or: $$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

Each of the following questions is worth 5 points.

a) The geometry of a closed universe resembles that of ____________________ (*the surface of a sphere* **OR** *a saddle shaped surface*). For the space described by your choice for filling in the first blank, every triangle has ______ (*more* **OR** *less*) than 180°, and the circumference of a circle is ______ (*more* **OR** *less*) than 2π times its radius.

b) Einstein originally proposed a static cosmological model in which a cosmological constant was introduced to create a repulsive force to prevent the universe from collapsing under the normal force of gravity. Qualitatively, what would happen to this universe if the cosmological constant were slightly higher than the value proposed by Einstein?

(c) A theory of big bang nucleosynthesis was first worked out in the late 1940's by George Gamow, Ralph Alpher, and Robert Herman. This theory differed from the currently accepted theory in at least four significant ways. Name one.

d) Silk emphasizes that there are two light elements or isotopes for which "there appear to be no other plausible astrophysical sources" besides big bang nucleosynthesis. What are these elements or isotopes?

e) According to estimates quoted in Weinberg's book, the synthesis of light chemical elements in the big bang occurred mainly at which of the following times: 7 seconds; 3 minutes; $3\frac{3}{4}$ minutes; 17 minutes; 1.5 years; or 200,000 years?

PROBLEM 2: GEODESICS *(25 points)*

The following was Problem 13 of the Quiz 2 Review Problems:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 \ .$$

(a) *(15 points)* Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) *(10 points)* Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$

$$y = r\sin\theta \ .$$

Re-express the line $y = 1$ in polar coordinates, and then use your answer to (a) to show that it is a geodesic curve. You may find any of the following derivatives useful:

$$\frac{d}{du}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}; \quad \frac{d}{du}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}}; \quad \frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2} \ .$$

PROBLEM 3: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE *(50 points)*

The spacetime metric for a homogeneous, isotropic, open universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}\ ,$$

where I have taken $k = -1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sinh\psi\ .$$

Then

$$\frac{dr}{\sqrt{1+r^2}} = d\psi\ ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}\ .$$

The form of $R(t)$ depends on the nature of the matter in the universe, but for this problem you should consider $R(t)$ to be an arbitrary function. You should simplify your answers as far as it is possible without knowing the function $R(t)$.

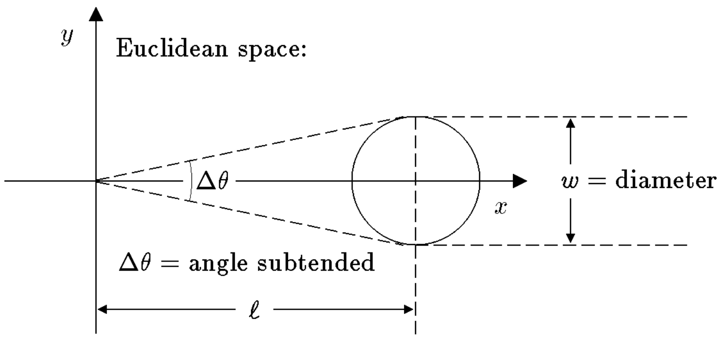
a) *(7 points)* Suppose that the Earth is at the origin of the coordinate system ($\psi = 0$), and that at the present time, t_0, we receive a light pulse from a distant galaxy G, located at $\psi = \psi_G$. Write down an equation which determines the time t_G at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that $d\tau = 0$ for any segment of it. Since you don't know $R(t)$ you cannot solve this equation, so please do not try.)

b) *(3 points)* What is the redshift Z_G of the light from galaxy G? (Your answer may depend on t_G, as well as ψ_G or any property of the function $R(t)$.)

c) *(10 points)* To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy G. (You may leave your answer in the form of a definite integral, which may be expressed in terms of ψ_G, t_G, Z_G, or the function $R(t)$.)

d) *(10 points)* There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called ***proper distance***, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the distant galaxy. The rulers have different velocities, because each is at rest with respect to the matter in its own vicinity. They are arranged so that, at the present instant of time, each ruler just touches its neighbors on either side. Write down an expression for the proper distance ℓ_{prop} of galaxy G.

e) *(10 points)* Another common definition of distance is ***angular size distance***, determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter w at a distance ℓ will subtend an angle $\Delta\theta = w/\ell$:

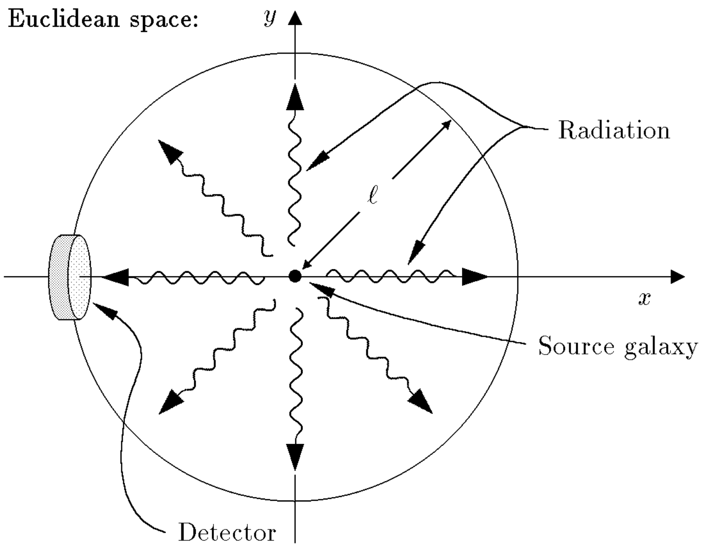


Motivated by this relation, cosmologists define the angular size distance ℓ_{ang} of an object by

$$\ell_{\text{ang}} \equiv \frac{w}{\Delta\theta} \ .$$

What is the angular size distance ℓ_{ang} of galaxy G?

f) *(10 points)* A third common definition of distance is called ***luminosity distance***, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux J received from a source of power P at a distance ℓ is given by $J = P/(4\pi\ell^2)$:



Cosmologists therefore define the luminosity distance by

$$\ell_{\rm lum} \equiv \sqrt{\frac{P}{4\pi J}} \quad .$$

Find the luminosity distance $\ell_{\rm lum}$ of galaxy G. (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy G is at the origin.)