MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

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QUIZ 3

USEFUL INFORMATION:

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)} \rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{\kappa^{3/2} c^2} \ ,$$

$$\kappa \equiv -k \ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda} \left\{ g_{ij} \frac{dx^j}{d\lambda} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell} \right) \frac{dx^k}{d\lambda} \frac{dx^\ell}{d\lambda}$$

or: $$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma} \right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\, \text{erg}/^\circ\text{K}$$
$$= 8.617 \times 10^{-5}\, \text{eV}/^\circ\text{K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec}$$
$$= 6.582 \times 10^{-16} \text{ eV-sec} \ ,$$

$$c = 2.998 \times 10^{10} \text{ cm/sec}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} \ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

CHEMICAL EQUILIBRIUM:

$$n_i = g_i\frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3}\,e^{(\mu_i - m_i c^2)/kT}\ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question.

PROBLEM 1: DID YOU DO THE READING? *(18 points)*

(a) In Chapter 7 of *The First Three Minutes*, Steve Weinberg describes the theory of quarks, a theory that is better established today than it was at the time the book was written. State whether each of the following statements is true or false:

(i) *(2 points)* The original version of the quark theory is due to Murray Gell-Mann and (independently) George Zweig.

(ii) *(2 points)* According to the quark theory, there would be a maximum possible temperature, at which the energy density would become infinite. The value of this temperature is estimated at $2 \times 10^{12}\,^\circ$K.

(iii) *(2 points)* An experiment by an MIT-Stanford Linear Accelerator collaboration showed that the force between quarks seems to disappear when the quarks are very close to each other.

(iv) *(2 points)* The force with which quarks are bound into protons and neutrons is so strong that even at the highest temperatures imagined in the early universe, the protons and neutrons are believed to have been intact.

(b) *(5 points)* In Chapter 6 of *The Big Bang*, Joseph Silk discusses the first millisecond of the history of the universe. The earliest period, lasting only about 10^{-43} seconds, is the meeting place of quantum physics and cosmology. It is named for one of the founders of quantum theory. What name is given to this era?

(c) *(5 points)* Which of the following hypothetical observations would be considered evidence for the existence of cosmic strings? Indicate as many as apply.

(A) Double images of galaxies separated by a few arcseconds, especially if a line of such double images were observed.

(B) A linelike discontinuity in the temperature of the cosmic background radiation.

(C) Elongated galaxies with about 10^3 times more mass than ordinary galaxies.

(D) A background of gravitational waves, detectable through jitter in the time intervals of pulsar signals.

PROBLEM 2: FREEZE-OUT OF MUONS *(35 points)*

The following problem was Problem 3 of Review Problems for Quiz 3:

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) *(10 points)* The black-body radiation formula, as shown at the front of this quiz, is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) *(15 points)* When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) *(10 points)* As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting R denote the Robertson-Walker scale factor, by what factor does the quantity RT increase when the muons disappear?

PROBLEM 3: CHEMICAL EQUILIBRIUM FOR HELIUM PRODUCTION *(47 points)*

The process of helium formation in the early universe is delayed until approximately $3\frac{3}{4}$ minutes by the "deuterium bottleneck," the fact that appeciable amounts of helium cannot form until the temperature has fallen enough for deuterium to become stable. Helium itself, due to its large binding energy ($B = 28$ MeV per helium nucleus), would be stable at much higher temperatures. In this problem you will figure out how early helium could form, if all the nuclear reactions occurred quickly enough to maintain thermal equilibrium. In other words, you will compute the thermal equilibrium abundance of helium as a function of temperature.

Consider the reaction

$$n + n + p + p \longleftrightarrow \alpha + \text{photon} ,$$

where α refers to the nucleus of He^4, which is also called an alpha particle. Although this reaction almost never occurs in a single step, the reaction can nonetheless be used to compute the thermal equilibrium abundance of helium.

(a) *(17 points)* Given that an alpha particle is spinless, find an expression for

$$\frac{n_n^2\, n_p^2}{n_\alpha} ,$$

where n_n, n_p, and n_α refer to the number densities of free neutrons, free protons, and alpha particles, respectively. Your expression should depend only on particle masses, the binding energy B, the temperature, and fundamental constants. You need not evaluate anything numerically.

(b) *(15 points)* Let $n_n^{\rm tot}$ denote the total number density of neutrons, including both free neutrons and those bound in helium. Similarly, let $n_p^{\rm tot}$ denote the total number density of protons. Let n_B denote the total baryon number density, so $n_B = n_n^{\rm tot} + n_p^{\rm tot}$. Let f denote the neutron fraction,

$$f = \frac{n_n^{\rm tot}}{n_B} ,$$

and let x denote the fraction of neutrons that remain free,

$$x = \frac{n_n}{n_n^{\rm tot}} .$$

Assuming that no nuclei other than helium have formed, express the number densities n_n, n_p, and n_α in terms of f, x, and n_B. (Be sure to express each of the three quantities n_n, n_p, and n_α, and to use ***only*** f, x, and n_B in your answers.)

(c) *(15 points)* Assuming that the baryon to photon ratio $\eta \equiv n_B/n_\gamma$ and the neutron fraction f are known, find an expression that relates the fraction of free neutrons x to the temperature T. Your equation should involve only x, f, η, B, the particle masses, the temperature, and fundamental constants. You need not solve for x, or evaluate anything numerically.