MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 16, 1996

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QUIZ 4

USEFUL INFORMATION:

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1+Z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{R(t_{\rm observed})}{R(t_{\rm emitted})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+\,r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\rm vac} = -\rho_{\rm vac}c^2 \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381\times10^{-16}\,\text{erg}/^\circ\text{K}$$
$$= 8.617\times10^{-5}\,\text{eV}/^\circ\text{K}\ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055\times10^{-27}\ \text{erg-sec}$$
$$= 6.582\times10^{-16}\ \text{eV-sec}\ ,$$

$$c = 2.998\times10^{10}\ \text{cm/sec}$$

$$1\ \text{eV} = 1.602\times10^{-12}\ \text{erg}\ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}$$

$$p = -\frac{1}{3}u \qquad \rho = u/c^2$$

$$n = g^*\frac{\varsigma(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\varsigma(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

CHEMICAL EQUILIBRIUM:

$$n_i = g_i\frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3}\,e^{(\mu_i - m_i c^2)/kT}\ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

PARTICLE PROPERTIES:

While working on this exam you may refer to any of the tables in Lecture Notes 10.

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question.

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) *(10 points)* In Wilczek's article, he states that the first indication that baryon number conservation cannot be exact came from a theoretical argument involving black holes. What is this argument?

(b) *(10 points)* Large voids, sometimes called "Hubble Bubbles," are found in redshift surveys. What is a redshift survey, and what is a void? A two to four sentence answer should suffice.

PROBLEM 2: BIG BANG NUCLEOSYNTHESIS *(25 points)*

The first four parts of the following problem come from Problem 8 of the Quiz 4 Review Problems.

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers. They are worth 5 points each.

(a) If the density of baryons were increased, would the calculated helium production go up or down?

(b) If the density of baryons were increased, would the calculated deuterium production go up or down?

(c) If the lifetime of the neutron were increased, would the calculated helium production go up or down?

(d) If the binding energy of deuterium were increased, would the calculated helium production go up or down?

(e) Suppose that the electron and positron (which have spin $\frac{1}{2}$) were replaced by particles which have spin 1, but otherwise behave the same as real electrons and positrons. Note that when the angular momentum of a spin-1 particle is measured along any arbitrary axis, there are three possible values that can be found: $\hbar$, 0, and $-\hbar$. Would the calculated helium production go up or down?

PROBLEM 3: QUARK DIAGRAM FOR $e^+ + n \rightarrow p +$NEUTRINO *(30 points)*

Consider the reaction

$$e^+ + n \rightarrow p + X \ ,$$

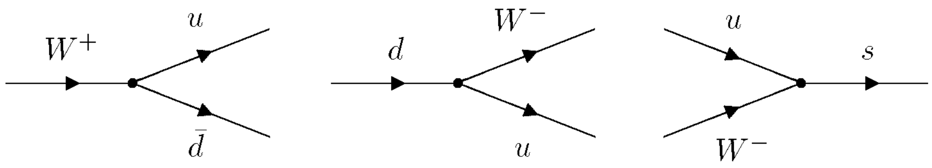
where X denotes a neutrino or an antineutrino.

(a) *(5 points)* Is the X a neutrino or an antineutrino, or could it be either? Explain in a sentence or two.

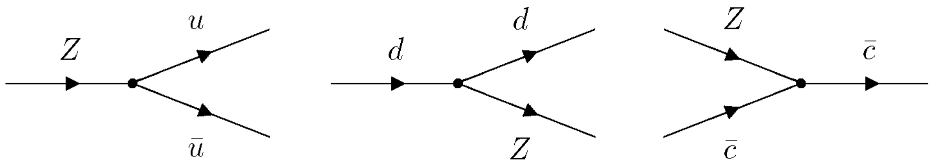
(b) *(5 points)* What type of neutrino (i.e., electron, muon, or tau) is the X, or is any type possible? Again explain in a sentence or two.

(c) *(20 points)* Draw a quark diagram for the process above.

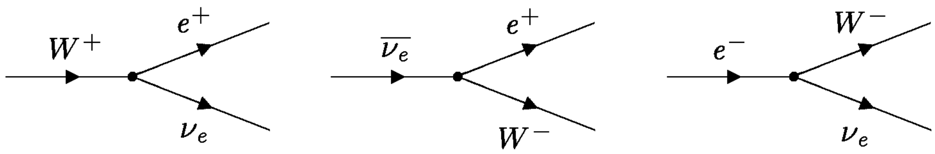
You may find the following information useful. The diagrams which describe the weak interactions of the quarks allow couplings between any two quarks and any one of the charged intermediate vector bosons (W^+ or W^-), provided that electric charge and baryon number are conserved. (Recall that each quark has a baryon number of 1/3, and each antiquark has a baryon number of -1/3. The W has baryon number zero.) For example, the following couplings are allowed:



The probability associated with these couplings is largest when the two quarks belong to the same generation, but diagrams which mix generations also exist. The neutral intermediate vector boson (Z^0) couples only to pairs of quarks with the same flavor, as in the diagrams below:



In addition to their interactions with quarks, the W bosons can undergo the following interactions with the leptons of the first generation:



— *CONTINUED* —

The complete set of diagrams to describe the interactions with the first generation of leptons can be obtained from those shown above by applying the following two rules:

(i) Given any diagram, the diagram obtained by interchanging the initial and final states is also an allowed diagram.

(ii) Given any diagram, the diagram obtained by replacing each particle with its antiparticle is also an allowed diagram.

PROBLEM 4: THE FLATNESS PROBLEM IN A UNIVERSE WITH $p = \frac{1}{2}u$ *(25 points)*

(a) *(10 points)* It was shown in Lecture Notes 9 that the quantity

$$\frac{\Omega - 1}{\Omega}$$

is equal to a function of the scale factor R, the mass density ρ, the constant k in the Robertson-Walker metric, and physical constants. What is this function? (To obtain full credit, you must show a derivation of this relation, rather than just state it from memory.)

(b) *(15 points)* Consider a hypothetical universe filled with a peculiar gas with pressure

$$p = \frac{1}{2}u$$

where u denotes the energy density. Assuming that such a universe is flat, you showed in Problem 1 of Problem Set 5 that the scale factor of such a universe behaves as

$$R(t) \propto t^{4/9} \ .$$

Now consider a hypothetical universe with $p = \frac{1}{2}u$ that is nearly flat, so that the above relation is a good approximation. Show that

$$\frac{\Omega - 1}{\Omega} \propto t^{\gamma} \ ,$$

for some power γ, and find the value of γ.