MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — February 24, 1998
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Tuesday, March 3, 1998

COVERAGE: Lecture Notes 3; Problem Set 1; Weinberg, Chapters 1-3. **One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the homework assignment, from this set of Review Problems, or from Quiz 1 of 1996 (which will be handed out with these review problems).**

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, I am also circulating copies of Quiz 1 from 1996, the last time this course was offered. That quiz occurred slightly later in the term, however, so it covered more material than this year's Quiz 1 will cover. In particular, you should not feel responsible for answering questions 1(d) and 1(e), which were based on the reading from Silk's book, or for answering Problem 2, which is based on Lecture Notes 4. In addition, Problems 3(b) and 3(g) involve the concept of "horizon distance," which this year is beyond the cut-off for Quiz 1.

Two clarifications were issued at the time of the 1996 quiz. Both concerned parts of the quiz that are beyond the cut-off for Quiz 1 this year, but I mention them here for completeness. On Problem 3(b), it is the *physical* horizon distance, not the *coordinate* horizon distance, that is being sought. On Problem 3(g), I emphasized that this part does not refer to the galaxies A and B mentioned earlier. In particular, you should not assume that the light pulse left its source at time t_A. You are given only the time of reception, t, and the redshift Z, from which you should be able to answer the question. Problems 3(h) and 3(i), on the other hand, again refer to the same galaxies A and B discussed at the beginning of the question, so it is again true that the light pulse leaves galaxy A at time t_A, and arrives at galaxy B at time t_B.

INFORMATION TO BE GIVEN ON QUIZ:

Last year's quiz had a section of "useful information" at the beginning, but only one of those equations is relevant to this year's first quiz:

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{R(t_{\rm observed})}{R(t_{\rm emitted})}$$

For the sake of consistency, this formula will be included at the start of this year's quiz.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1994:

The following questions are worth 5 points each:

a) In 1750 the English instrument maker Thomas Wright published *Original Theory or New Hypothesis of the Universe.* In this book Wright described an astronomical object that is known today as the Crab Nebula, the solar system, the Milky Way, or the local supercluster?

b) In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens.* What new hypothesis was put forward in this book?

c) Estimate the diameter and thickness of the disk of the Milky Way galaxy. Any numbers within a factor of 2 of those given in Weinberg's book will be accepted.

d) The mathematical theory of an expanding universe was first published in 1922 by the Russian mathematician Alexandre Friedmann, the Dutch Astronomer Willem de Sitter, the American astronomer Edwin Hubble, or the Belgian cleric Georges Lemaître?

e) After discovering an inexplicable hiss coming from their radio telescope, Arno Penzias and Robert Wilson of Bell Laboratories learned that P.J.E. Peebles, a Princeton theorist, had calculated that the big bang would produce a background of cosmic radiation with a temperature today of 10° K. What MIT radio astronomer informed them of Peebles' work?

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE
(20 points)

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$R(t) = R_0 e^{\chi t} ,$$

where R_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift Z. Express the time t_r at which we receive the pulse in terms of Z, χ, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of Z, χ, and any relevant physical constants.

PROBLEM 3: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift? *[Note for 1998: this question is beyond the material for Quiz 1 of this year.]*

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, 10^4 light-years, 10^7 light years, or 10^{10} light-years? *[Note for 1998: this question is beyond the material for Quiz 1 of this year.]*

(i) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

(j) At the time of the discovery of the cosmic microwave background, an active but independent effort was taking place elsewhere. P.J.E. Peebles had estimated that the universe must contain background radiation with a temperature of at least 10°K, and Robert H. Dicke, P.G. Roll, and D.T. Wilkinson had mounted an experiment to look for it. At what institution were these people working?

PROBLEM 4: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(30 points)*

The following problem was Problem 4, Quiz 2, 1992:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{\gamma} ,$$

where b and γ are constants. *[This universe differs from the matter-dominated universe described in the lecture notes in that ρ is not proportional to $1/R^3(t)$. Such behavior is possible when pressures are large, because a gas expanding under pressure can loose energy (and hence mass) during the expansion.]* For the following questions, any of the answers may depend on γ, whether it is mentioned explicitly or not.

a) *(5 points)* Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by a distant object. In terms of these quantities, find the value of the redshift parameter Z with which the light is received.

b) *(5 points)* Find the "look-back" time as a function of Z and t_0. The look-back time is defined as the length of the interval in cosmic time between the emission and observation of the light.

c) *(6 points)* Express the present value of the physical distance to the object as a function of H_0, Z, and γ.

d) *(7 points)* Find the present, physical value of the horizon distance, $\ell_{p,\text{horizon}}$, for this model. *[Note for 1998: this question is beyond the material for Quiz 1 of this year.]*

e) *(7 points)* At the time of emission, the distant object had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this object at the earth today? Express your answer in terms of P, H_0, Z, and γ. *[Energy flux (which might be measured in erg-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow.]*

PROBLEM 5: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The following problem was Problem 3, Quiz 2, 1988:

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as

$$R(t) = bt^{1/3} .$$

Here b denotes a constant.

(a) If a light pulse is emitted at time t_e and observed at time t_o, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that t_e and t_o are given, find the observed redshift Z.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of c, t_o, and Z (i.e., without t_e appearing).

(d) At an arbitrary time t in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of c, t, and t_o.

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) Wright's book described the disk structure of the Milky Way.

b) Kant proposed that the faint nebulae seen in the sky are distant galaxies, similar to the Milky Way.

c) The Milky Way galaxy has a diameter of about 80,000 light-years, and a thickness of 6,000 light-years.

d) The mathematical theory of an expanding universe, in the context of general relativity, was invented by Alexandre Friedmann in 1922. (Actually the 1922 paper discussed only closed universes, but Friedmann published a second paper on open universes in 1924.) Willem de Sitter published his model of the universe in 1917. De Sitter's model was initially believed to be static, but it was later discovered that it appeared static only because it was written in peculiar coordinates— in fact it was also an expanding model. While Friedmann's equations described the general case of a homogeneous isotropic expanding universe, de Sitter's model was more specific: it was a model devoid of matter, with the expansion driven by a positive cosmological constant. The intended answer for this question was Friedmann, but full credit was given for either Friedmann or de Sitter.

e) It was Bernard Burke who told Arno Penzias about the prediction of radio noise from the big bang.

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{R}/R \ .$$

So

$$H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \boxed{\chi \ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{R_0} \int_0^t e^{-\chi t'} dt' \\ &= \frac{c}{R_0} \left[-\frac{1}{\chi} e^{-\chi t'} \right]_0^t \\ &= \boxed{\frac{c}{\chi R_0} \left[1 - e^{-\chi t} \right] \ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1 + Z = \frac{R(t_r)}{R(t_e)} \ .$$

Here $t_e = 0$, so

$$\begin{aligned} 1 + Z &= \frac{R_0 e^{\chi t_r}}{R_0} \\ &\Longrightarrow \quad e^{\chi t_r} = 1 + Z \\ &\Longrightarrow \quad \boxed{t_r = \frac{1}{\chi} \ln(1 + Z) \ .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1 + Z \ ,$$

and

$$x(t_r) = \frac{c}{\chi R_0}\left[1 - e^{-\chi t_r}\right] = \frac{c}{\chi R_0}\left[1 - \frac{1}{1+Z}\right] = \frac{cZ}{\chi R_0(1+Z)} .$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = R(t_r)x(t_r) = \frac{cZe^{\chi t_r}}{\chi(1+Z)} = \boxed{\frac{cZ}{\chi} .}$$

PROBLEM 3: "DID YOU DO THE READING?"

(a) The distinguishing quantity is $\Omega \equiv \rho/\rho_c$. The universe is open if $\Omega < 1$, flat if $\Omega = 1$, or closed if $\Omega > 1$.

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date* was made by the COBE satellite, which measured the temperature as 2.728 ± 0.004 Kelvin. The error here is quoted with a 95% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years.

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

* Astrophysical Journal, vol. **473**, p. 576 (1996): *The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data Sets*, D.J. Fixsen, E.S. Cheng, J.M. Gales, J.C. Mather, R.A. Shafer, and E.L. Wright.

(g) The absolute luminosity (*i.e.*, the total light output) of a Cepheid variable star appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to the Cepheid, by measuring the period and the apparent luminosity. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the $1/r^2$ law for the intensity of a point source to determine the distance r.

(h) 10^7 light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(j) Princeton University.

PROBLEM 4: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION

a) The cosmological redshift is given by the usual form,

$$1 + Z = \frac{R(t_0)}{R(t_e)} .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1 + Z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma} .$$

So,

$$\boxed{Z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1 .}$$

b) The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0\left(1 - \frac{t_e}{t_0}\right) .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of Z. From (a),

$$\frac{t_e}{t_0} = (1+Z)^{-1/\gamma} .$$

Therefore,

$$\boxed{t_0 - t_e = t_0[1 - (1+Z)^{-1/\gamma}] \quad .}$$

c) The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = R(t_0)\int_{t_e}^{t_0} \frac{c}{R(t)}dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{ct_0^\gamma}{1-\gamma}\left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Pulling $t_0^{\gamma-1}$ out of the parantheses gives

$$\ell_p(t_0) = \frac{ct_0}{1-\gamma}\left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of Z and H_0 using the result of part (a) as well as,

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1 - (1+Z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) The present physical value of the horizon distance is given by a similar integral to that in part (b) with different limits of integration,

$$\ell_{\rm horiz}(t_0) = R(t_0)\int_0^{t_0} \frac{c}{R(t)}dt \quad .$$

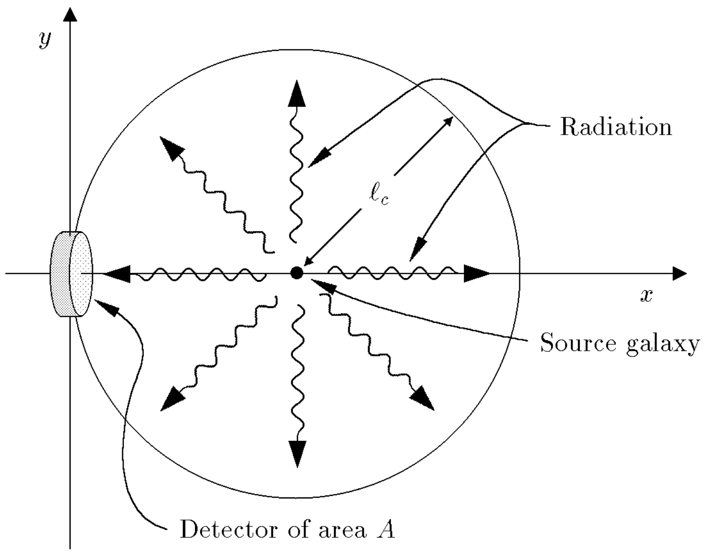
$$= \frac{ct_0}{1-\gamma} \quad .$$

Using H_0 from above, this can be reexpressed as,

$$\boxed{\ell_{\rm horiz}(t_0) = \frac{c\gamma}{1-\gamma}H_0^{-1} \quad .}$$

e) A nearly identical problem was worked through in Problem 6 of Problem Set 1. The discussion presented here briefly outlines the main elements of the solution and is intended as an indication of what is expected of students in an exam solution.

The energy of the observed photons will be redshifted by a factor of $(1+Z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+Z)$. Consequently, the observed power will be redshifted by two factors of $(1+Z)$ to $P/(1+Z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+Z)^2} .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+Z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, Z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+Z)^2\left[1-(1+Z)^{\frac{\gamma-1}{\gamma}}\right]^2}} .$$

PROBLEM 5: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The key to this problem is to work in comoving coordinates.

[Some students have asked me why one cannot use "physical" coordinates, for which the coordinates really measure the physical distances. In principle one can use any coordinate system on likes, but the comoving coordinates are the simplest. In any other system it is difficult to write down the trajectory of either a particle or a light-beam. In comoving coordinates it is easy to write the trajectory of either a light beam, or a particle which is moving with the expansion of the universe (and hence standing still in the comoving coordinates).

Note, by the way, that when one says that a particle is standing still in comoving coordinates, one has not really said very much about it's trajectory. One has said that it is moving with the matter which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor, $R(t)$.]

(a) The physical separation at t_o is given by the scale factor times the coordinate distance. The coordinate distance is found by integrating the coordinate velocity, so

$$\ell_p(t_o) = R(t_o)\int_{t_e}^{t_o}\frac{c\,dt'}{R(t')} = bt_o^{1/3}\int_{t_e}^{t_o}\frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct_o^{1/3}\left[t_o^{2/3}-t_e^{2/3}\right]$$

$$\boxed{= \frac{3}{2}ct_o\left[1-(t_e/t_o)^{2/3}\right]} .$$

(b) From the front of the exam,

$$1+Z = \frac{R(t_o)}{R(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/3}$$

$$\Longrightarrow \quad \boxed{Z = \left(\frac{t_o}{t_e}\right)^{1/3} - 1} .$$

(c) By combining the answers to (a) and (b), one has

$$\boxed{\ell_p(t_o) = \frac{3}{2}ct_o\left[1-\frac{1}{(1+Z)^2}\right]} .$$

(d) The physical distance of the light pulse at time t is equal to $R(t)$ times the coordinate distance. The coordinate distance at time t is equal to the starting coordinate distance, $\ell_c(t_e)$, minus the coordinate distance that the light pulse travels between time t_e and time t. Thus,

$$\begin{aligned}
\ell_p(t) &= R(t)\left[\ell_c(t_e) - \int_{t_e}^{t} \frac{c\,dt'}{R(t')}\right] \\
&= R(t)\left[\int_{t_e}^{t_o} \frac{c\,dt'}{R(t')} - \int_{t_e}^{t} \frac{c\,dt'}{R(t')}\right] \\
&= R(t)\int_{t}^{t_o} \frac{c\,dt'}{R(t')} \\
&= bt^{1/3}\int_{t}^{t_o} \frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct^{1/3}\left[t_o^{2/3} - t^{2/3}\right] \\
&= \boxed{\frac{3}{2}ct\left[\left(\frac{t_o}{t}\right)^{2/3} - 1\right]\ .}
\end{aligned}$$