MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 25, 1998
Prof. Alan Guth

QUIZ 2 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

(a) (3) The net electric charge of a closed universe must be exactly equal to zero.

(b) I apologize for this question, because I no longer think that it quite makes sense. Herschel certainly made one discovery that helped to confirm the validity of Newton's law of gravity: he found that the orbits of binary stars are consistent with Newton's law. He also discovered Uranus, and when I made up the question I was under the mistaken belief that Uranus was found by observing the gravitational perturbations that it induced in the orbits of the other planets. The truth, however, is that Herschel discovered Uranus in 1781 during a telescopic survey of the sky. (Sixty-five years later the planet Neptune was discovered through the effect that it had on the orbit of Uranus, so I was at least right that Uranus was involved in a story of this sort.) Herschel also discovered extragalactic nebulae, a discovery that was very important for cosmology, but did not say much about Newton's law of gravity. (I guess one could reasonably assume that the nebulae were held together by gravity, but there was certainly no way of knowing if the detailed form of Newton's law applied.) Since the question was flawed, we compensated by grading generously. Students who answered "Uranus" or "nebulae" were given the full 5 points, and students who answered "binary stars" were given 6 points.

(c) This subject is discussed in **The Big Bang**, by Joseph Silk, on p. 70. The answer is 3.9 billion years.

(d) Mainly photons, e^+-e^- pairs, and neutrino-antineutrino pairs. (The neutrinos and antineutrinos were each of three types: electron neutrinos, muon neutrinos, tau neutrinos, electron antineutrinos, muon antineutrinos, and tau antineutrinos—but there was no need to mention this.)

PROBLEM 2: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Using chain rule, the standard formula for the Hubble constant can be rewritten as

$$H(\theta) = \frac{1}{R}\frac{dR}{dt} = \frac{1}{R}\frac{dR}{d\theta}\frac{d\theta}{dt} \; .$$

By differentiating the parametric equations for R and t, one finds

$$\frac{dR}{d\theta} = \alpha\sqrt{k}\sin\theta \; ,$$

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1-\cos\theta) \; .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{k}\alpha(1-\cos\theta)}\right]\left[\alpha\sqrt{k}\sin\theta\right]\left[\frac{c}{\alpha(1-\cos\theta)}\right]$$

$$= \boxed{\frac{c\sin\theta}{\alpha(1-\cos\theta)^2}\ .}$$

(b) The evolution equation for a homogeneous isotropic universe can be written as

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}\ .$$

Then, solving for ρ gives,

$$\rho = \frac{3}{8\pi G}\left(H^2 + \frac{kc^2}{R^2}\right)\ .$$

Using the answer from (a), and the parametric expression for $R/\sqrt{k}$, one has

$$\rho = \frac{3}{8\pi G}\left[\frac{c^2\sin^2\theta}{\alpha^2(1-\cos\theta)^4} + \frac{c^2}{\alpha^2(1-\cos\theta)^2}\right]$$

$$= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{\sin^2\theta}{(1-\cos\theta)^2} + 1\right]$$

This expression is greatly simplified by using the following trigonometric identity

$$\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$$

Using this in our expression for ρ we have

$$\rho = \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1-\cos\theta)} + 1\right]$$

$$= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{1+\cos\theta}{1-\cos\theta} + 1\right]$$

$$= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{2}{1-\cos\theta}\right]$$

$$= \boxed{\frac{3c^2}{4\pi G\alpha^2(1-\cos\theta)^3}\ .}$$

(c) Using the answer from (a) and the standard expression for ρ_c, one has

$$\rho_c = \frac{3H^2}{8\pi G} = \frac{3c^2 \sin^2\theta}{8\pi G\alpha^2(1-\cos\theta)^4} \ .$$

Then

$$\Omega \equiv \rho/\rho_c = \frac{2(1-\cos\theta)}{\sin^2\theta} = \boxed{\frac{2}{1+\cos\theta} \ .}$$

PROBLEM 3: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\,dt^2 + R^2(t)\,d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{R^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{R(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\text{hor}} = \int_0^t \frac{c}{R(t')}\,dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\text{hor}}$, which according to the metric is given by

$$\ell_{\text{phys}}(t) = \int_{\psi=0}^{\psi=\psi_{\text{hor}}} ds = \int_0^{\psi_{\text{hor}}} R(t)\,d\psi = \boxed{R(t)\int_0^t \frac{c}{R(t')}\,dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{R(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{R(t)} \ .$$

Then using $R = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $R = \alpha(1 - \cos\theta)$, one can see that R returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 6. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

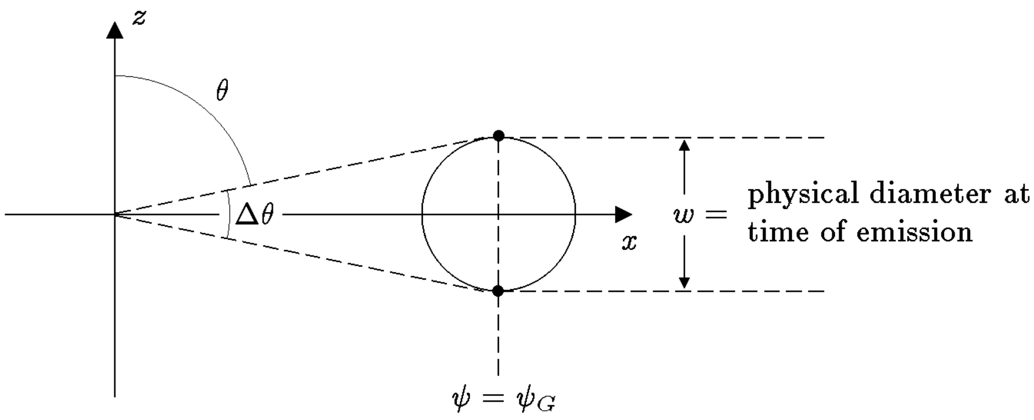
$$x^2 + y^2 + z^2 + w^2 = a^2 \ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost $1/2$, and it would approach $1/2$ as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 4: BRIGHTNESS AND ANGULAR SIZE IN A CLOSED UNIVERSE

(a) If we choose the axis shown vertically in the diagram to be the z-axis, then the angle labeled $\Delta\theta$ will represent an increment of the Robertson-Walker coordinate θ, as the label $\Delta\theta$ suggests:



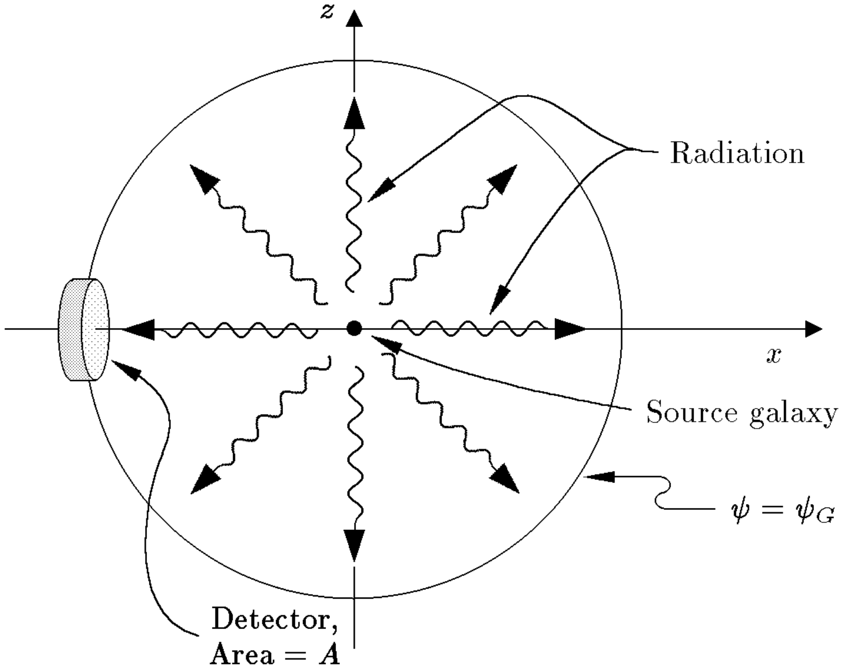
At time t_G the distance between the two edges of the galaxy is given, according to the metric, by

$$ds = R(t_G) \sin\psi_G \, \Delta\theta \ ,$$

where I have assumed that $\Delta\theta \ll 1$. But the problem tells us that this distance is w, so

$$w = R(t_G) \sin\psi_G \, \Delta\theta \quad \Longrightarrow \quad \boxed{\Delta\theta = \frac{w}{R(t_G)\sin\psi_G}\ .}$$

(b) Following the hint, we draw Robertson-Walker coordinates with the galaxy G in the center. The radial coordinate of the detector, on Earth, will be ψ_G. The diagram also shows a sphere at the same radial coordinate, ψ_G:



Since the speed of light is independent of angle, all the photons that left the galaxy G at time t_G are arriving at the $\psi = \psi_G$ sphere at the present time, t_0. To calculate the power received by the detector, we need to know what fraction of those photons hit the detector. The fraction is simply the area of the detector divided by the area of the sphere. The area of the sphere can be calculated by restricting the metric to the case $dt = d\psi = 0$, $t = t_0$, $\psi = \psi_G$:

$$ds^2 = R^2(t_0)\sin^2\psi_G\,(d\theta^2 + \sin^2\theta\,d\phi^2)\ .$$

This expression is identical to the metric of the surface of a sphere of radius $r = R(t_0)\sin\psi_G$. The area is therefore $A = 4\pi r^2 = 4\pi R^2(t_0)\sin^2\psi_G$. So,

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi R^2(t_0)\sin^2\psi_G}\ .$$

The power hitting the detector is further reduced by one factor of $(1+z) = R(t_0)/R(t_G)$ because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of $(1+z)$ because the rate of arrival of photons is reduced by this factor. Thus, if P is the power that the galaxy was emitting at time t_G, then the power received by the detector today is

$$\begin{aligned} P_{\text{received}} &= P\frac{A}{4\pi R^2(t_0)\sin^2\psi_G}\left[\frac{R(t_G)}{R(t_0)}\right]^2 \\ &= P\frac{AR^2(t_G)}{4\pi R^4(t_0)\sin^2\psi_G}\ . \end{aligned}$$

The flux is given by

$$\boxed{J = \frac{P_{\text{received}}}{A} = P \frac{R^2(t_G)}{4\pi R^4(t_0) \sin^2 \psi_G} \; .}$$

c) To evaluate the solid angle subtended by the galaxy, imagine surrounding the observer by a small sphere of arbitrary radius r. The galaxy would appear on this sphere as a disk with a angular radius $\Delta\theta/2$, which implies a radius of $r\,\Delta\theta/2$, and an area $A = \pi r^2\,\Delta\theta^2/4$. The solid angle is given by

$$\Delta\Omega \equiv \frac{A}{r^2} = \frac{\pi\,\Delta\theta^2}{4} \; .$$

Using the answers from the previous two parts, the surface brightness is given by

$$\begin{aligned} \sigma = \frac{J}{\Delta\Omega} = \frac{4J}{\pi\,\Delta\theta^2} &= \frac{4JR^2(t_G)\sin^2\psi_G}{\pi w^2} \\ &= \frac{PR^4(t_G)}{\pi^2 w^2 R^4(t_0)} = \boxed{\frac{P}{\pi^2 w^2}\frac{1}{(1+z)^4} \; .} \end{aligned}$$

While we derived this formula for a closed universe, we would have found the same result in an open or flat universe.

Note that this result implies that for $z \ll 1$, the surface brightness is independent of distance. This result is consistent with Euclidean geometry, which says that both the energy flux and the solid angle are inversely proportional to the square of the distance, so the surface brightness is independent of distance.

Amazingly, astronomers have within the past few months (*Science*, March 13, 1998, p. 1627) detected a galaxy with a record-breaking redshift of $z = 5.34$. According to this result, the surface brightness of such a distant galaxy is suppressed by the factor $(1+z)^4 \approx 1600$.