MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 5, 1998
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QUIZ 3 SOLUTIONS

ERRATUM: In the equations at the start of the quiz, the pressure of black-body radiation was written with the wrong sign. The correct formula is $p = \frac{1}{3}u$.

PROBLEM 1: DID YOU DO THE READING?

(a) The photino is related to the photon by $\boxed{\text{supersymmetry.}}$

(b) Silk mentions that the decay of photinos might be detectable by the production of $\boxed{\text{antiprotons}}$ or $\boxed{\text{gamma rays.}}$ Since a gamma ray is a kind of photon (one with an energy between 10 KeV and 10 MeV), the word $\boxed{\text{photon}}$ was also an acceptable answer.

(c) The authors were $\boxed{\text{Fred Hoyle}}$ and $\boxed{\text{Roger J. Tayler,}}$ both at Cambridge University. A number of students incorrectly answered George Gamow, a prominent big bang theorist who emphatically opposed the steady-state theory.

(d) The correct choices are (ii), (v), and (vi). *[Note: Statement (i) has no basis in fact, while statement (iii) is contradicted by Weinberg, who states that it was technologically possible to detect a 3° K background during the 1950s, and maybe even during the 1940s. As for statement (iv), Weinberg hardly mentions the steady state theory. Presumably the fact that one could not definitively choose between the steady state and big bang models contributed to the situation summarized by statement (v).]*

PROBLEM 2: A REVISED THERMAL HISTORY OF THE UNIVERSE

(a) In the standard model, the black-body radiation at $kT \approx 200$ MeV contains the following contributions:

$$\left.\begin{array}{ll} \text{Photons:} & g = 2 \\ e^+e^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \nu_e, \nu_\mu, \nu_\tau: & g = 6 \times \frac{7}{8} = 5\frac{1}{4} \\ \mu^+\mu^-: & g = 4 \times \frac{7}{8} = 3\frac{1}{2} \\ \pi^+\pi^-\pi^0 & g = 3 \end{array}\right\} g_{\text{TOT}} = 17\frac{1}{4}$$

$$\rho = \frac{u}{c^2} = g_{\text{TOT}} \frac{\pi^2}{30} \frac{(kT)^4}{\hbar^3 c^5}$$

$$= \left(17\frac{1}{4}\right) \frac{\pi^2}{30} \frac{\left[200 \times 10^6 \text{ eV} \times \dfrac{1.602 \times 10^{-12} \text{ erg}}{\text{eV}}\right]^4}{(1.055 \times 10^{-27} \text{ erg-sec})^3 \, (2.998 \times 10^{10} \text{ cm/sec})^5}$$

Units:

$$\frac{\text{erg}^4}{\text{erg}^3\text{-sec}^3\text{-cm}^5\text{-sec}^{-5}} = \frac{\text{erg-sec}^2}{\text{cm}^5}$$

$$= \frac{\left(\text{gm-cm}^2\text{-sec}^{-2}\right)\text{sec}^2}{\text{cm}^5} = \text{gm/cm}^3$$

$$\boxed{\rho = \left(17\frac{1}{4}\right) \frac{\pi^2}{30} \frac{\left[200 \times 10^6 \times 1.602 \times 10^{-12}\right]^4}{(1.055 \times 10^{-27})^3 \, (2.998 \times 10^{10})^5} \frac{\text{gm}}{\text{cm}^3} .}$$

With a calculator one finds that

$$\rho = 2.10 \times 10^{15} \text{ gm/cm}^3 .$$

Note: In the past many students have left out the conversion factor 1.602×10^{-12} erg/eV, and instead used $\hbar = 6.582 \times 10^{-16}$ eV-sec. Few of these students worked out the units of the answer. It turns out to be eV-sec^2/cm^5, which is a most peculiar set of units to measure a mass density.

> In the NTWI, the answer is the same. The neutrinos are just ceasing to interact with the rest of matter, but this does *not* cause them to disappear.

(b) As long as the universe is in thermal equilibrium, entropy is conserved. The entropy in a given volume of the comoving coordinate system is

$$R^3(t) s V_{\text{coord}} ,$$

where s is the entropy density and $R^3 V_{\rm coord}$ is the physical volume. So

$$R^3(t)s$$

is conserved. After the neutrinos decouple,

$$R^3 s_\nu \quad \text{and} \quad R^3 s_{\rm other}$$

are separately conserved ($s_{\rm other}$ = entropy of everything except ν).

Note that

$$s = gAT^3 \quad ,$$

where A is a constant. Before the disappearance of the e, μ, and π particles from the thermal equilibrium radiation,

$$s_\nu = \left(5\frac{1}{4}\right) AT^3$$

$$s_{\rm other} = 12AT^3 \quad .$$

So

$$\frac{s_\nu}{s_{\rm other}} = \frac{5\frac{1}{4}}{12} \quad .$$

If $R^3 s_\nu$ and $R^3 s_{\rm other}$ are conserved, then so is $s_\nu/s_{\rm other}$. By today, the entropy previously shared amoung the various particles still in equilibrium after neutrino decoupling has been transfered to the photons so that

$$s_{\rm other} = s_{\rm photons} = 2AT_\gamma^3 \quad .$$

The entropy in neutrinos, being separately conserved, is still

$$s_\nu = \left(5\frac{1}{4}\right) AT_\nu^3 \quad .$$

Since $s_\nu/s_{\rm other}$ is constant we know that

$$\frac{\left(5\frac{1}{4}\right) T_\nu^3}{2T_\gamma^2} = \frac{s_\nu}{s_{\rm other}} = \frac{\left(5\frac{1}{4}\right)}{12}$$

$$\Longrightarrow \quad \boxed{T_\nu = \left(\frac{1}{6}\right)^{1/3} T_\gamma} \quad .$$

(c) One can write

$$n = g^* BT^3 \quad ,$$

where B is a constant. Here $g_\gamma^* = 2$, and $g_\nu^* = 6 \times \frac{3}{4} = 4\frac{1}{4}$. In the standard model, one has today

$$\frac{n_\nu}{n_\gamma} = \frac{g_\nu^* T_\nu^3}{g_\gamma^* T_\gamma^3} = \frac{\left(4\frac{1}{2}\right)}{2}\frac{4}{11} = \boxed{\frac{9}{11}} \; .$$

In the NTWI,

$$\frac{n_\nu}{n_\gamma} = \frac{\left(4\frac{1}{2}\right)}{2}\frac{1}{6} = \boxed{\frac{3}{8}} \; .$$

(d) At $kT = 200$ MeV, the thermal equilibrium ratio of neutrons to protons is given by

$$\frac{n_{\rm n}}{n_{\rm p}} = e^{-1.29\,{\rm MeV}/200\,{\rm MeV}} \approx 1 \quad .$$

In the standard theory this ratio would decrease rapidly as kT fell below the p-n mass difference of 1.29 MeV, but in the NTWI the neutrons can disappear only through the slow process of free neutron decay. Thus, the NTWI predicts a higher neutron abundance than the standard model. At about $3\frac{3}{4}$ minutes all of the neutrons will become bound into He, exactly as in the standard model — this is a strong interaction, and will happen in the same way in either theory. The higher neutron abundance of the NTWI implies a

$$\boxed{\text{higher predicted He abundance.}}$$

To estimate the He abundance, note that in the absence of neutron decay, the neutron-proton ratio would be frozen at about one, and at $3\frac{3}{4}$ minutes essentially *all* the nucleons in the universe would collect into He. However, the neutron abundance is reduced by free decay by a factor

$$e^{-3\frac{3}{4}\,\mathrm{mins}/15\,\mathrm{mins}} \approx 0.78 \;,$$

so the He production is reduced by this factor. Thus,

$$\boxed{\text{Predicted He abundance by weight} \approx 80\% \;.}$$

PROBLEM 3: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad\Longrightarrow\quad \frac{d}{dt}\left(\rho c^2 R^3\right) = -p\frac{d}{dt}\left(R^3\right) \;. \tag{1}$$

Setting

$$\rho = \frac{\alpha}{R^5} \tag{2}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{R^2}\right) = -p\frac{d}{dt}\left(R^3\right) \;,$$

which implies

$$-2\frac{\alpha c^2}{R^3}\frac{dR}{dt} = -3pR^2\frac{dR}{dt} \;.$$

Thus

$$p = \frac{2}{3}\frac{\alpha c^2}{R^5} = \boxed{\frac{2}{3}\rho c^2 \;.}$$

For those students who could not reconstruct Eq. (1) or some equivalent equation from memory, the conservation of energy equation could be derived from the formulas for cosmological evolution on the front of the exam:

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \tag{3}$$

$$\ddot R = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R \;. \tag{4}$$

By rewriting Eq. (3) as

$$\dot R^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 \;,$$

the time derivative becomes

$$2\dot R\,\ddot R = \frac{8\pi}{3}G\dot\rho R^2 + \frac{16\pi}{3}G\rho R\dot R \;.$$

This equation can be solved for $\dot\rho$ to give

$$\dot\rho = \frac{3}{4\pi G}\frac{\dot R\,\ddot R}{R^2} - 2\frac{\dot R}{R}\rho \;.$$

Using Eq. (4) to replace $\ddot R$, one finds

$$\dot\rho = -\frac{\dot R}{R}\left(\rho + \frac{3p}{c^2}\right) - 2\frac{\dot R}{R}\rho = -3\frac{\dot R}{R}\left(\rho + \frac{p}{c^2}\right) \;. \tag{5}$$

It is easy to show that Eq. (5) is equivalent to Eq. (1), but it is not necessary to do so. The question can be answered directly from Eq. (5), by substituting Eq. (2) and manipulating.

(b) For a flat universe, Eq. (3) reduces to

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3}G\rho \;.$$

Using Eq. (2), this implies that

$$\dot{R} = \frac{\beta}{R^{3/2}} ,$$

for some constant β. Rewriting this as

$$R^{3/2}\, dR = \beta\, dt ,$$

we can integrate the equation to give

$$\frac{2}{5} R^{5/2} = \beta t + \text{const} ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $R = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{R \propto t^{2/5} .} \tag{6}$$

The arbitrary constant of proportionality in Eq. (6) is consistent with the wording of the problem, which states that "You should be able to determine the function $R(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α in Eq. (2), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then R is measured in meters per notch, and the constant of proportionality in Eq. (6) can be changed by changing the arbitrary definition of the notch.

(c) Combining Eq. (1) with $p = \frac{1}{6}\rho c^2$, one has

$$\frac{d}{dt}\left(\rho c^2 R^3\right) = -\frac{1}{6}\rho c^2 \frac{d}{dt}\left(R^3\right) ,$$

or equivalently

$$\frac{d}{dt}\left(\rho R^3\right) + \frac{1}{6}\rho\frac{d}{dt}\left(R^3\right) = 0 . \tag{7}$$

There are various ways to proceed from here. Since the problem told us that

$$\rho = \frac{\text{const}}{R^n} ,$$

the most straightforward approach would be to use this expression to replace ρ in Eq. (7), and then solve the equation for n. A cleverer approach would be to multiply Eq. (7) by $R^{1/2}$, and then rewrite it as

$$\frac{d}{dt}\left(\rho R^{7/2}\right) = 0 ,$$

from which one can see immediately that

$$\boxed{\rho(t) \propto \frac{1}{R^{7/2}(t)} ,}$$

and therefore

$$\boxed{n = 7/2 .}$$

PROBLEM 4: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)} kT .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\, kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\varsigma(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\varsigma(3)} k\ .}$$

Numerically, this gives $3.602\, k$, where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$E = \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\varsigma(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\varsigma(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\varsigma(3)} kT\ .}$$

Numerically, $E = 3.1514\, kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again $7/8$ and $3/4$ respectively, so

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\varsigma(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\varsigma(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{135\varsigma(3)} k\ .}$$

Numerically, this gives $S = 4.202\, k$.