Ants on a Pringle

Introduction to non-Euclidean spaces

Recap

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{k c^2}{a^2} \]

\( k \)?
Euclid’s Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If a straight line falling on two straight lines makes the interior angles on the same side are less than the two right angles, the two straight lines if produced indefinitely meet on that line on which the angles are less than two right angles.
Many tried, but ...  

Saccheri  

What if ...  

5th Postulate  
inconsistent ?  

GBL geometry with Klein  

1. constant negative curvature  
2. infinite  
3. 5th postulate  

\[(x_1, y_1) \quad x^2 + y^2 < 1\]  

\[d(1,2) = \cosh^{-1} \left[ \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2} \sqrt{1 - x_2^2 - y_2^2}} \right] \]  

\[(x_2, y_2) \]  

Note: no global embedding in 3D Euclidean space possible
Geometry (after Klein)

co-ordinates
\((x_1, y_1)\)

\((x_2, y_2)\)

distance function
\(d((x_1, y_1), (x_2, y_2))\)

tiny distances

\[ds^2 = dx^2 + dy^2\]

\[ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dx\,dy + g_{yy}(x, y)dy^2\]

geodesics

a geodesic is a curve along which the distance between two given points is extremised.

note: important!

sphere: geodesics

longitudes: yes
latitudes: no
quadratic form

\[ ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2 \]

locally Euclidean

\[ ds^2 = dx^2 + dy^2 \]
\[ ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2 \]
\[ g_{xx} g_{yy} - g_{xy}^2 > 0 \]

Intrinsic Geometry

drawn on a curved surface.

\[ ds^2 = dx^2 + dy^2 \]
\[ ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2 \]
metric and co-ordinates

\[ ds^2 = dx^2 + dy^2 \]
\[ ds^2 = dr^2 + r^2 d\theta^2 \]

Curved?

curved?

\[ \frac{C}{2r} = \pi \]

\[ \frac{C}{2r} < \pi \]

\[ a + b + c = \pi \]

\[ a + b + c > \pi \]
SUMMARY so far...

- intrinsic/extrinsic properties
- metric and geodesics
- Riemannian
  - locally Euclidean
  - e.g. sphere, plane
  - curved or flat
- pseudo-Riemannian
  - locally Lorentzian
  - e.g. spacetime
  - curved or flat

gravity:
spacetime curvature

Note: all images have been adapted and modified from the original source.
References

- Alan’s notes
- Rindler’s book on Gravitation
- Personal notes (from class taught for EPGY)