

Ants on a Pringle



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18.10.2011

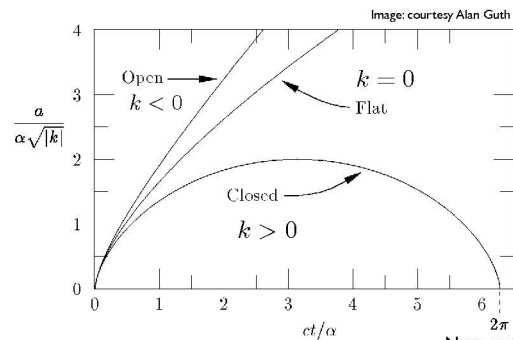
Introduction to non-Euclidean spaces



Recap



$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$



Note: matter dominated universes only!

k ?

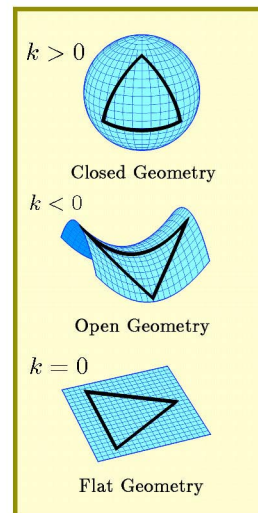


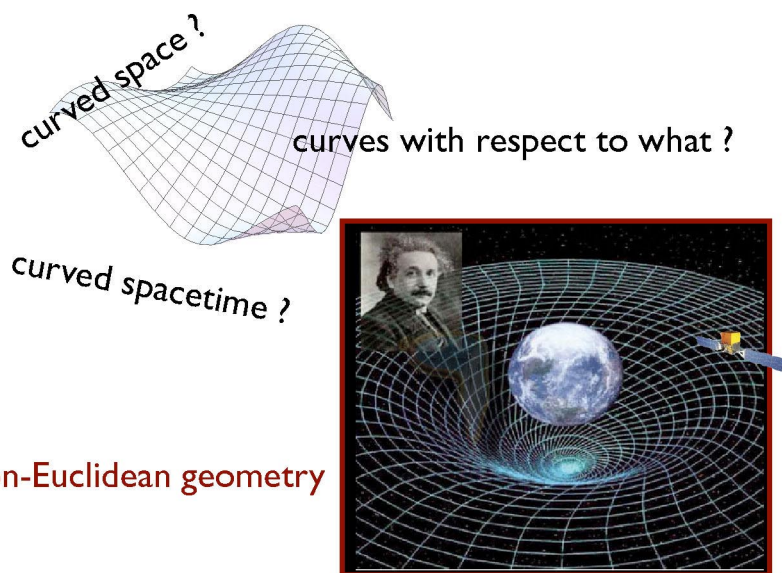
Image: courtesy Alan Guth

curved space ?

curves with respect to what ?

curved spacetime ?


non-Euclidean geometry



Note on image: I have added the "Dali" clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration


5

A bit of History ...



6

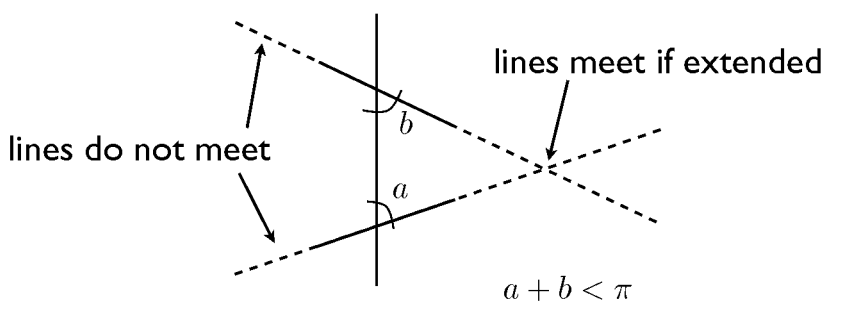
Euclid's *Postulates*



1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If a straight line falling on two straight lines makes the interior angles on the same side are less than the two right angles, the two straight lines if produced indefinitely meet on that line on which the angles are less than two right angles

7

5th Postulate



lines do not meet

lines meet if extended

$$a + b < \pi$$

5. If a straight line falling on two straight lines makes the interior angles on the same side are less than the two right angles, the two straight lines if produced indefinitely meet on that line on which the angles are less than two right angles

8

Many tried, but ...

The diagram illustrates the relationship between Euclidean geometry and non-Euclidean geometries. On the left, a triangle with sides a , b , and c is shown with the equation $a + b + c = \pi$. This is connected by a double-headed arrow to a set of three horizontal lines representing Euclidean geometry. Below this, a double-headed arrow connects the triangle to two star-shaped polygons representing elliptic geometry. To the right, another set of three horizontal lines represents hyperbolic geometry, with a dashed line intersecting them. A double-headed arrow connects this set to the star-shaped polygons.

(1667-1733)

Saccheri




What if ...

5th Postulate


↓

inconsistent ?

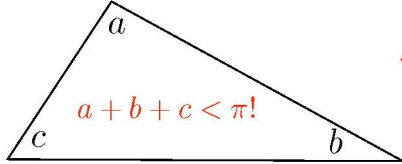
~1750-1850

- infinite
- constant negative curvature




5th Postulate




11


GBL geometry with Klein



1. constant negative curvature
2. infinite
3. ~~5th postulate~~




(x_1, y_1)



$x^2 + y^2 < 1$

(x_2, y_2)

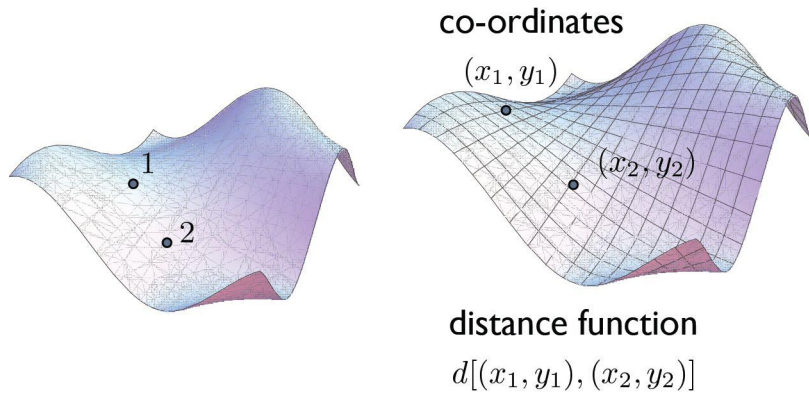


$$d(1,2) = a \cosh^{-1} \left[\frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2} \sqrt{1 - x_2^2 - y_2^2}} \right]$$

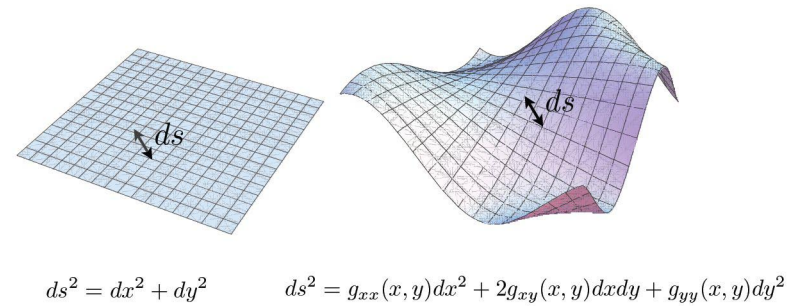
Note: no global embedding in 3D Euclidean space possible

12

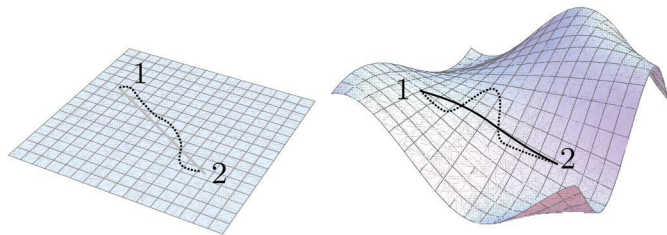
Geometry (after Klein)



tiny distances



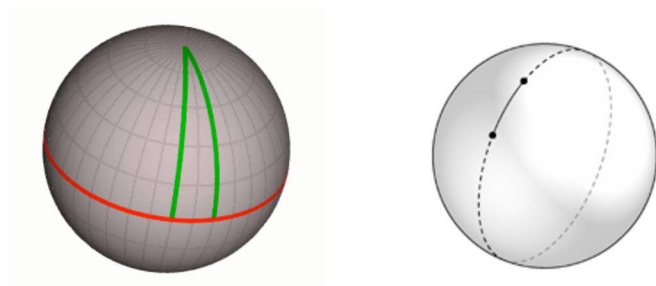
geodesics



a geodesic is a curve along which the distance between two given points is extremised.

note: important!

sphere: geodesics



longitudes: yes
 latitudes: no

quadratic form

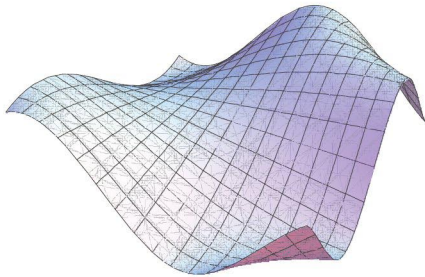
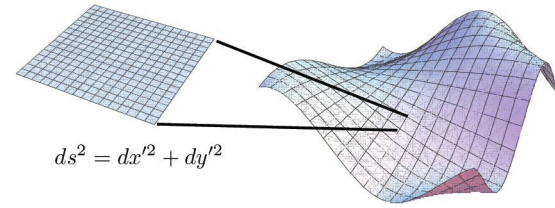


Image: www.esat.uct.ac.za/~caron/web/resources.htm

$$ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2$$

locally Euclidean



$$ds^2 = dx'^2 + dy'^2$$

$$ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$



Intrinsic Geometry

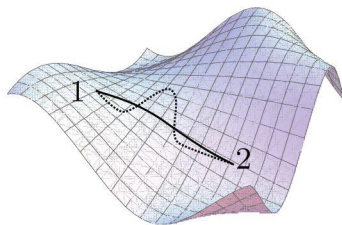
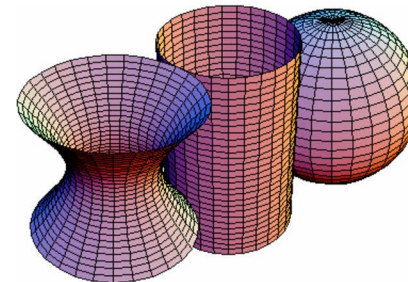


Image of ant from: <http://www.termitecity.com/>

curved ?

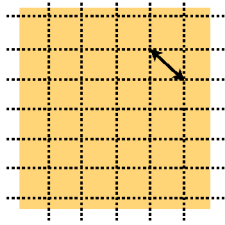


Above image: http://en.wikipedia.org/wiki/Gaussian_curvature

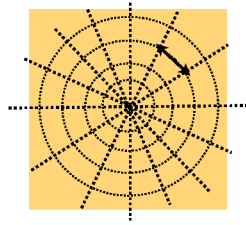
$$ds^2 = dx^2 + dy^2$$

$$ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2$$

metric and co-ordinates



$$ds^2 = dx^2 + dy^2$$



$$ds^2 = dr^2 + r^2 d\theta^2$$

21

Curved ?

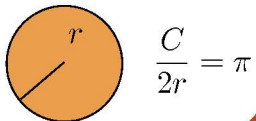


Image: www.earthshope.org/

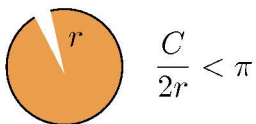
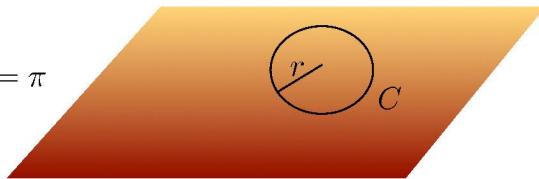
curved ?



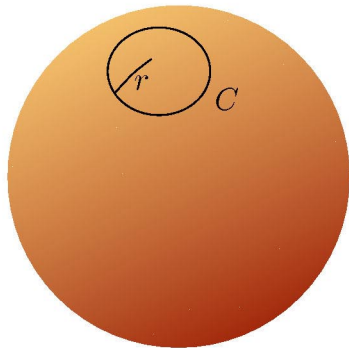
22



$$\frac{C}{2r} = \pi$$

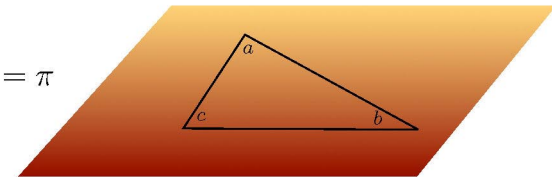


$$\frac{C}{2r} < \pi$$

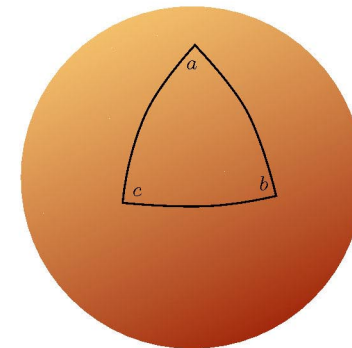


23

$$a + b + c = \pi$$



$$a + b + c > \pi$$



24



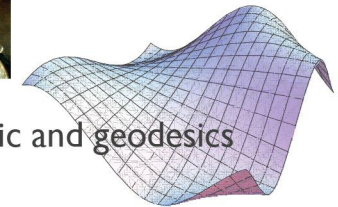
curved ?



25

SUMMARY so far ...

intrinsic/extrinsic properties



metric and geodesics

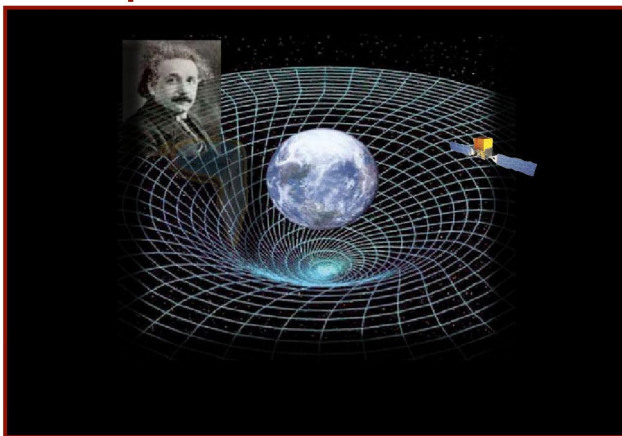


Riemannian
locally Euclidean
e.g. sphere, plane
curved or flat

pseudo-Riemannian
locally Lorentzian
e.g. spacetime
curved or flat



gravity:
spacetime curvature



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27

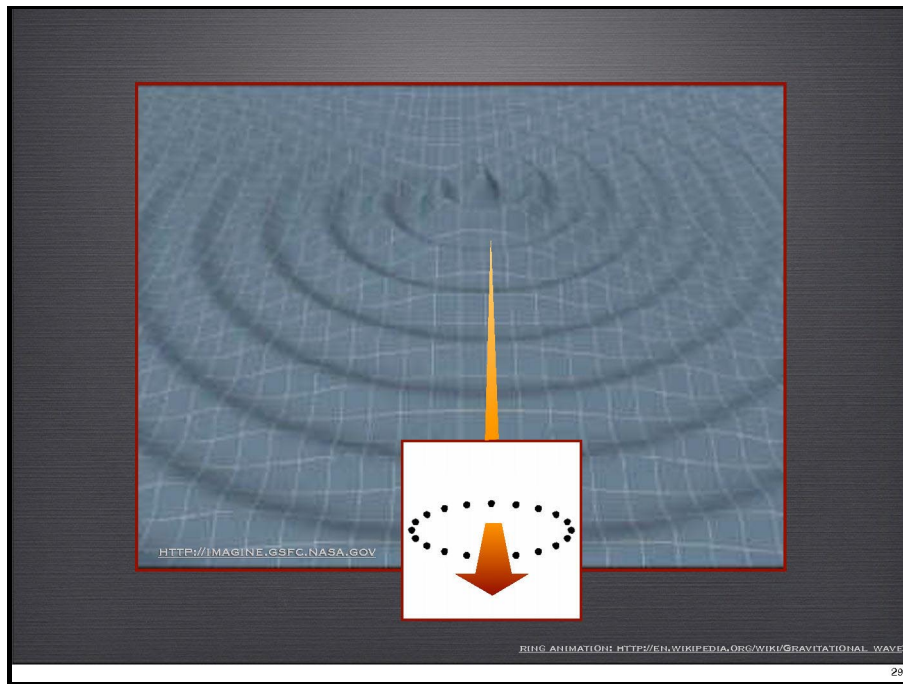


[HTTP://IMAGINE.GSFC.NASA.GOV](http://imagine.gsfc.nasa.gov)

PAPPALARDO SYMPOSIUM 2009

28

Mustafa Amin, *Introduction to non-Euclidean Spaces*, 8.286 Lecture, October 18, 2011, p. 8.



References

- Alan's notes
- Rindler's book on Gravitation
- Personal notes (from class taught for EPGY)