8.286 Lecture 8
October 1, 2013

DYNAMICS OF HOMOGENEOUS EXPANSION, PART IV
Summary: Mathematical Model

\[ t_i \equiv \text{time of initial picture} \]
\[ R_{\text{max},i} \equiv \text{initial maximum radius} \]
\[ \rho_i \equiv \text{initial mass density} \]
\[ \vec{v}_i = H_i \vec{r} \]
**Summary: Equations**

**Want:** \( r(r_i, t) \equiv \text{radius at } t \text{ of shell initially at } r_i \)

**Find:** \[ r(r_i, t) = a(t)r_i \], where

\[
\begin{align*}
\text{Friedmann Equations} & \quad \ddot{a} = -\frac{4\pi}{3} G \rho(t) a \\
& \quad H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{k c^2}{a^2} \quad \text{(Friedmann Eq.)}
\end{align*}
\]

and

\[
\rho(t) \propto \frac{1}{a^3(t)} \quad \text{or} \quad \rho(t) = \left[ \frac{a(t_1)}{a(t)} \right]^3 \rho(t_1) \text{ for any } t_1.
\]

Units: \([r] = \text{meter}, [r_i] = \text{notch}, [a(t)] = \text{m/notch}, [k] = 1/\text{notch}^2\).
Summary: Conventions

Us: Notch is arbitrary (free to be redefined each time we use it).
[Our construction used \( a(t_i) = 1 \text{ m/notch}, \) but we can interpret this equation as the definition of \( t_i \). But we can forget the definition of \( t_i \) if we don’t intend to use it.]

Ryden: \( a(t_0) = 1 \) (where \( t_0 = \text{now} \)). For us, \( 1 \rightarrow 1 \text{ m/notch} \).

Many Other Books: if \( k \neq 0 \), then \( k = \pm 1 \).
(For us, \( \pm 1 \rightarrow \pm 1 \text{ m/notch} \).)
Summary: Types of Solutions

\[ \dot{a}^2 = \frac{8\pi G}{3} \frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2. \]

For intuition, remember that \( k \propto -E \), where \( E \) is a measure of the energy of the system.

Types of Solutions:

1) \( k < 0 \) (\( E > 0 \)): unbound system. \( \dot{a}^2 > (-kc^2) > 0 \), so the universe expands forever. **Open Universe.**

2) \( k > 0 \) (\( E < 0 \)): bound system. \( \dot{a}^2 \geq 0 \) \( \implies \)

\[ a_{\text{max}} = \frac{8\pi G}{3} \frac{\rho(t_1)a^3(t_1)}{kc^2}. \]

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**
3) $k = 0 \ (E = 0)$: critical mass density.

$$H^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} \quad \Rightarrow \quad \rho \equiv \rho_c = \frac{3H^2}{8\pi G}.$$ 

**Flat Universe.**

Summary: $\rho > \rho_c \iff$ closed, $\rho < \rho_c \iff$ open, $\rho = \rho_c \iff$ flat.

Numerical value: For $H = 67.3$ km-s$^{-1}$-Mpc$^{-1}$ (Planck 2013 plus other experiments),

$$\rho_c = 8.4 \times 10^{-30} \text{ g/cm}^3 \approx 5 \text{ proton masses per m}^3.$$ 

Definition: $\Omega \equiv \frac{\rho}{\rho_c}$. 
Summary: Evolution of a Flat Universe

If \( k = 0 \), then

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho = \frac{\text{const}}{a^3} \implies \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}
\]

\[
\implies a^{1/2} a \, da = \text{const} \, dt \implies \frac{2}{3} a^{3/2} = (\text{const})t + c'.
\]

Choose the zero of time to make \( c' = 0 \), and then

\[
a(t) \propto t^{2/3}.
\]