8.286 Lecture 10
October 10, 2013

INTRODUCTION TO NON-EUCLIDEAN SPACES
8.286 Lecture 10
October 10, 2013

INTRODUCTION TO NON-EUCLIDEAN SPACES

(After finishing dynamics of homogeneous expansion)
Evolution of a closed universe:

First order Friedmann equation \[ \Rightarrow \]

\[ d\tilde{t} = \frac{\ddot{a} d\tilde{a}}{\sqrt{2\alpha \ddot{a} - \dot{a}^2}} , \quad \text{where} \quad \ddot{a} \equiv \frac{a}{\sqrt{k}} , \quad \tilde{t} \equiv ct , \]

and \[ \alpha \equiv \frac{4\pi}{3} \frac{G\rho\ddot{a}^3}{c^2} . \]
Evolution of a closed universe:

First order Friedmann equation \( \Rightarrow \)

\[
\int_{0}^{\tilde{t}_f} d\tilde{t} = \int_{0}^{\tilde{a}_f} \frac{\tilde{a} d\tilde{a}}{\sqrt{2\alpha \tilde{a} - \tilde{a}^2}} , \quad \text{where} \quad \tilde{a} \equiv \frac{a}{\sqrt{k}} , \quad \tilde{t} \equiv ct ,
\]

and \( \alpha \equiv \frac{4\pi G \rho \tilde{a}^3}{3 \frac{c^2}{c^2}} \).

Summary of Lecture 9
Summary of Lecture 9

Evolution of a closed universe:

First order Friedmann equation \( \Rightarrow \)

\[
\int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a} \ d\tilde{a}}{\sqrt{2\alpha \tilde{a} - \tilde{a}^2}} , \quad \text{where} \quad \tilde{a} \equiv \frac{a}{\sqrt{k}} , \quad \tilde{t} \equiv c t ,
\]

and \( \alpha \equiv \frac{4\pi \ G \rho \tilde{a}^3}{3 \ \frac{c^2}{3}} \).

Substitute \( \tilde{a} - \alpha = -\alpha \cos \theta \) \( \Rightarrow \)

\[
\tilde{t}_f = \alpha (\theta_f - \sin \theta_f)
\]

\[
\tilde{a}_f = \alpha (1 - \cos \theta_f)
\]
Summary of Lecture 9

Evolution of a closed universe:

First order Friedmann equation \[ \int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a} d\tilde{a}}{\sqrt{2\alpha \tilde{a} - \tilde{a}^2}}, \quad \text{where} \quad \tilde{a} \equiv \frac{a}{\sqrt{k}}, \quad \tilde{t} \equiv ct, \]

and \[ \alpha \equiv \frac{4\pi}{3} \frac{G\rho \tilde{a}^3}{c^2}. \]

Substitute \( \tilde{a} - \alpha = -\alpha \cos \theta \) \[ \tilde{t}_f = \alpha(\theta_f - \sin \theta_f) \]

\[ \tilde{a}_f = \alpha(1 - \cos \theta_f) \]

\[ ct = \alpha(\theta - \sin \theta) \]

\[ \frac{a}{\sqrt{k}} = \alpha(1 - \cos \theta) \]
Age of a closed universe:
Age of a closed universe: (Want it in terms of $H$ and $\Omega$)
Age of a closed universe:

(Want it in terms of $H$ and $\Omega$)

$$H^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{a^2} \implies \ddot{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega} - 1}.$$
Age of a closed universe: (Want it in terms of $H$ and $\Omega$)

\[
H^2 = \frac{8\pi}{3} G\rho - \frac{k c^2}{a^2} \implies \tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H| \sqrt{\Omega} - 1}.
\]

\[
\alpha \equiv \frac{4\pi}{3} \frac{G \rho \tilde{a}^3}{c^2} \implies \alpha = \frac{c}{2 |H|} \frac{\Omega}{(\Omega - 1)^{3/2}}.
\]
Age of a closed universe:

\[
H^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{a^2} \quad \implies \quad \ddot{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H| \sqrt{\Omega - 1}}.
\]

\[
\alpha \equiv \frac{4\pi}{3} \frac{G \rho \ddot{a}^3}{c^2} \quad \implies \quad \alpha = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}}.
\]

\[
\frac{a}{\sqrt{k}} = \alpha (1 - \cos \theta) \quad \implies \quad \frac{c}{|H| \sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos \theta).
\]

\[
\implies \quad \sin \theta = \pm \frac{\sqrt{\Omega - 1}}{\Omega}.
\]
Age of a closed universe:

(Want it in terms of $H$ and $\Omega$)

$$H^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{a^2} \implies \ddot{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H| \sqrt{\Omega - 1}}.$$  

$$\alpha \equiv \frac{4\pi}{3} \frac{G \rho \tilde{a}^3}{c^2} \implies \alpha = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}}.$$  

$$\frac{a}{\sqrt{k}} = \alpha (1 - \cos \theta) \implies \frac{c}{|H| \sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos \theta).$$  

$$\implies \sin \theta = \pm \frac{\sqrt{\Omega - 1}}{\Omega}.$$  

Then $ct = \alpha(\theta - \sin \theta) \implies$

$$t = \frac{\Omega}{2|H| (\Omega - 1)^{3/2}} \left\{ \arcsin \left( \pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\}.$$  

--3--
\[ t = \frac{\Omega}{2 |H|(\Omega - 1)^{3/2}} \left\{ \arcsin \left( \pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} . \]

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Phase</th>
<th>( \Omega )</th>
<th>Sign Choice</th>
<th>( \theta = \sin^{-1}(\cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Expanding</td>
<td>1 to 2</td>
<td>Upper</td>
<td>0 to ( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>Expanding</td>
<td>2 to ( \infty )</td>
<td>Upper</td>
<td>( \frac{\pi}{2} ) to ( \pi )</td>
</tr>
<tr>
<td>3</td>
<td>Contracting</td>
<td>( \infty ) to 2</td>
<td>Lower</td>
<td>( \pi ) to ( \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>Contracting</td>
<td>2 to 1</td>
<td>Lower</td>
<td>( \frac{3\pi}{2} ) to 2( \pi )</td>
</tr>
</tbody>
</table>
Evolution of an Open Universe

The calculations are almost identical, except that one defines

\[ \tilde{a} \equiv \frac{a}{\sqrt{\kappa}} , \quad \text{where} \quad \kappa \equiv -k > 0 . \]

One finds hypergeometric functions instead of trigonometric functions, with

\[ ct = \alpha (\sinh \theta - \theta) \]
\[ \frac{a}{\sqrt{\kappa}} = \alpha (\cosh \theta - 1) \]

instead of

\[ ct = \alpha (\theta - \sin \theta) \]
\[ \frac{a}{\sqrt{\kappa}} = \alpha (1 - \cos \theta) . \]
The Age of a Matter-Dominated Universe

\[ |H| t = \begin{cases} 
\Omega \frac{2 \sqrt{1 - \Omega}}{2(1 - \Omega)^{3/2}} \left[ 2 \sqrt{1 - \Omega} \frac{\Omega}{\Omega} - \sinh^{-1} \left( \frac{2 \sqrt{1 - \Omega}}{\Omega} \right) \right] & \text{if } \Omega < 1 \\
2/3 & \text{if } \Omega = 1 \\
\Omega \frac{2 \sqrt{\Omega - 1}}{2(\Omega - 1)^{3/2}} \left[ \sin^{-1} \left( \pm \frac{2 \sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2 \sqrt{\Omega - 1}}{\Omega} \right] & \text{if } \Omega > 1 
\end{cases} \]
Age of Matter-Dominated Universe
Evolution of a Matter-Dominated Universe

\[ \frac{a}{\alpha \sqrt{|k|}} \]

\[ ct/\alpha \]
INTRODUCTION TO NON-EUCLIDEAN SPACES
curved space?

curves with respect to what?

curved spacetime?

non-Euclidean geometry

Note on image: I have added the “Dali” clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration.
Euclid’s Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

Corrected 10/10/13
5th Postulate

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles.

$\angle a + \angle b < \pi$
(a) “If a straight line intersects one of two parallels (i.e., lines which do not intersect however far they are extended), it will intersect the other also.”
(b) “There is one and only one line that passes through any given point and is parallel to a given line.”
Equivalent Statements of the 5th Postulate

(c) “Given any figure there exists a figure, similar to it, of any size.”
(Two polygons are similar if their corresponding angles are equal, and their corresponding sides are proportional.)
(d) “There is a triangle in which the sum of the three angles is equal to two right angles (i.e., 180°).”
In 1733, Saccheri, a Jesuit priest, published *Euclides ab omni naiveo vindicatus* (*Euclid Freed of Every Flaw*).

The book was a study of what geometry would be like if the 5th postulate were false.

He hoped to find an inconsistency, but failed.
Carl Friedrich Gauss (1777–1855)

German mathematician and physicist.

Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.
Carl Friedrich Gauss (1777–1855)

German mathematician and physicist.

Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.

His students included Richard Dedekind, Bernhard Riemann, Peter Gustav Lejeune Dirichlet, Gustav Kirchhoff, and August Ferdinand Möbius.
\( a + b + c < \pi! \)

- infinite
- constant negative curvature

\[\sim 1750-1850\]

Slide created by Mustafa Amin
GBL geometry with Klein

1. constant negative curvature
2. infinite
3. 5th postulate

\[(x_1, y_1) \quad x^2 + y^2 < 1 \quad (x_2, y_2)\]

\[d(1, 2) = \cosh^{-1} \left[ \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2} \sqrt{1 - x_2^2 - y_2^2}} \right]\]

Note: no global embedding in 3D Euclidean space possible
Geometry (after Klein)

co-ordinates

$(x_1, y_1)$

$(x_2, y_2)$

distance function

$d[(x_1, y_1), (x_2, y_2)]$
tiny distances

\[ ds^2 = dx^2 + dy^2 \quad \quad ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2 \]
quadratic form

\[ ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2 \]
locally Euclidean

\[ ds^2 = dx'^2 + dy'^2 \]

\[ ds^2 = g_{xx}(x, y)dx^2 + 2g_{xy}(x, y)dxdy + g_{yy}(x, y)dy^2 \]

\[ g_{xx}g_{yy} - g_{xy}^2 > 0 \]