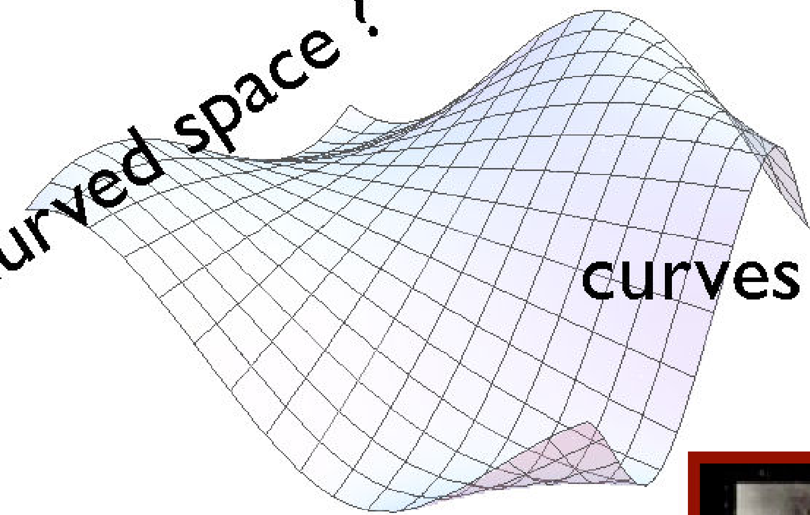
8.286 Lecture 11
October 17, 2013

NON-EUCLIDEAN SPACES: CLOSED UNIVERSES

REVIEW OF LECTURE 10:

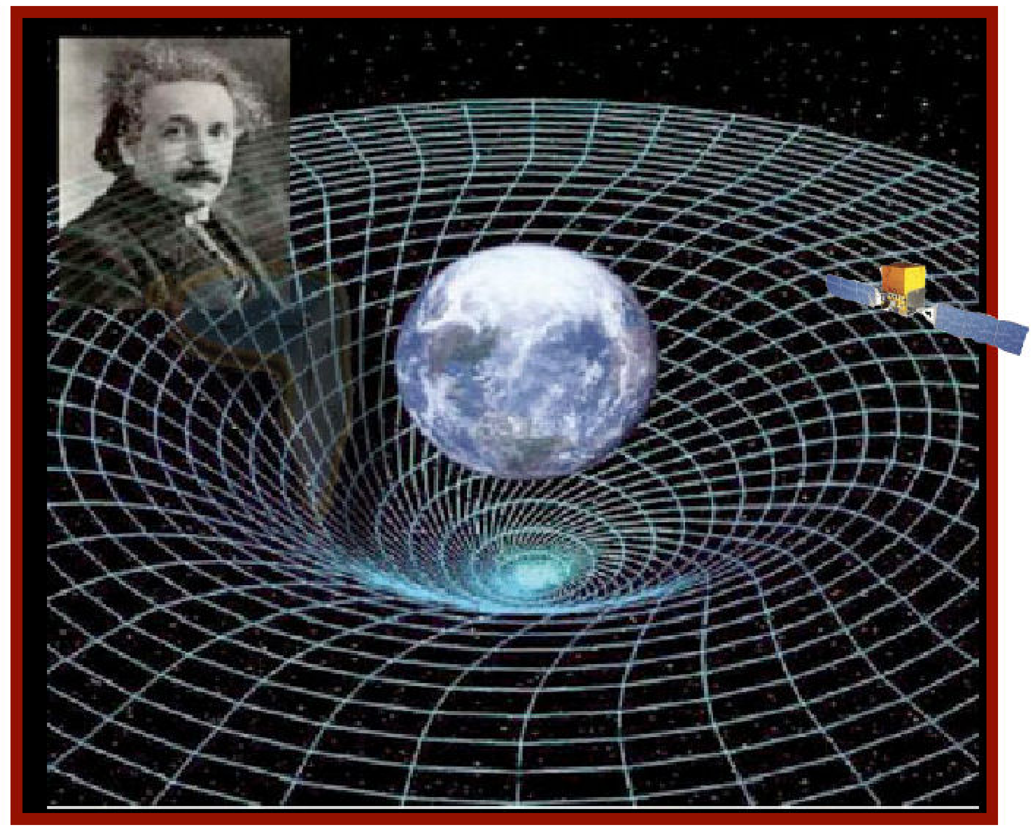
INTRODUCTION TO NON-EUCLIDEAN SPACES

curved space ?



curves with respect to what ?

curved spacetime ?



non-Euclidean geometry

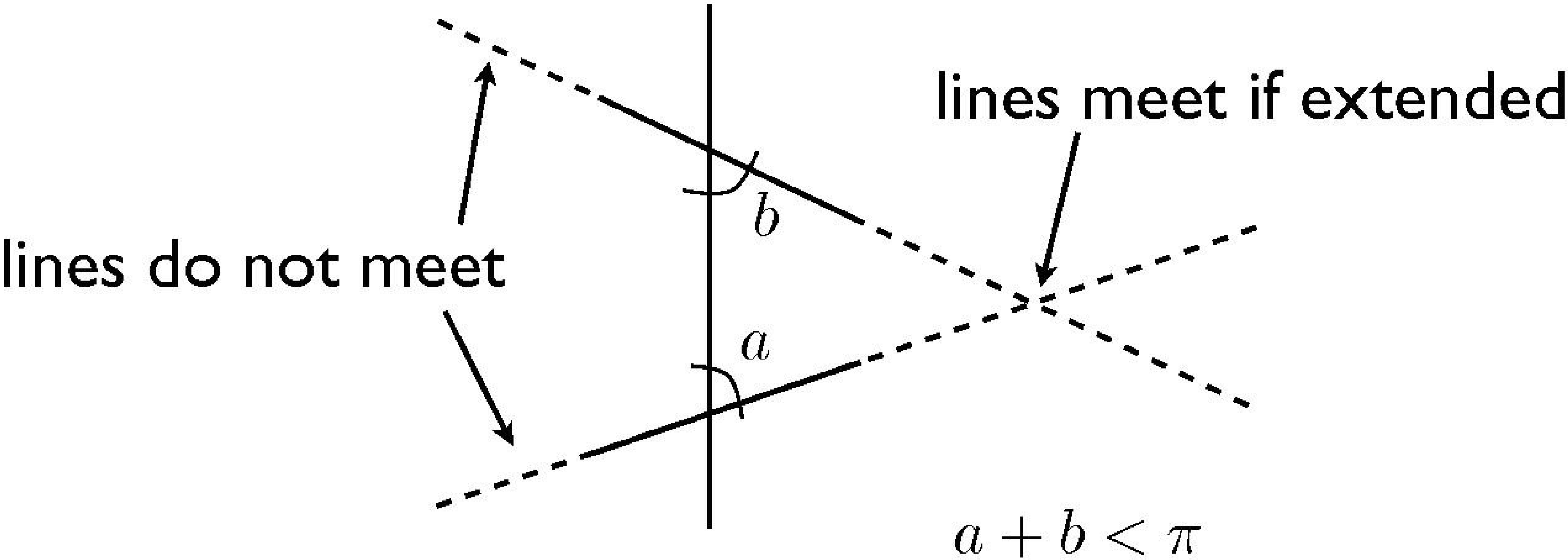
Note on image: I have added the "Dali" clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration

Euclid's *Postulates*

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment , a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

Corrected 10/10/13

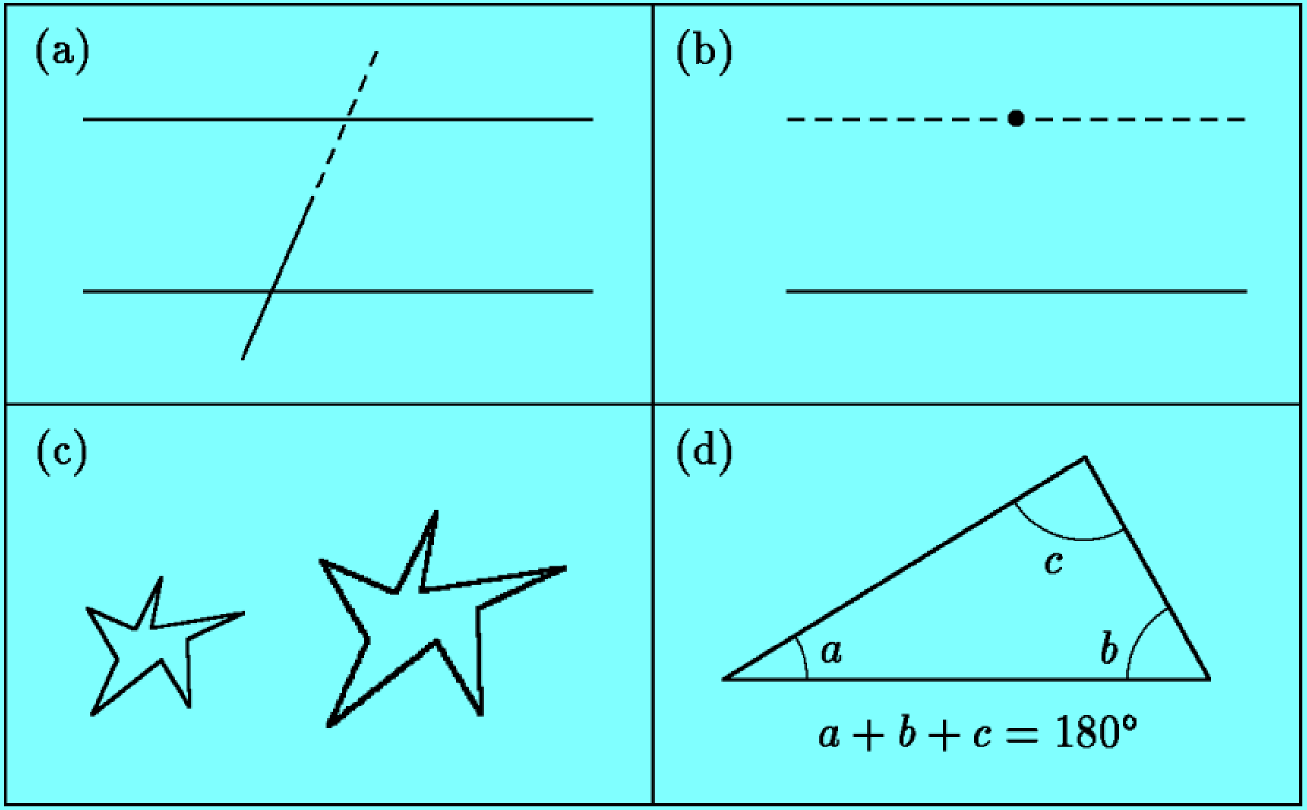
5th Postulate



5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

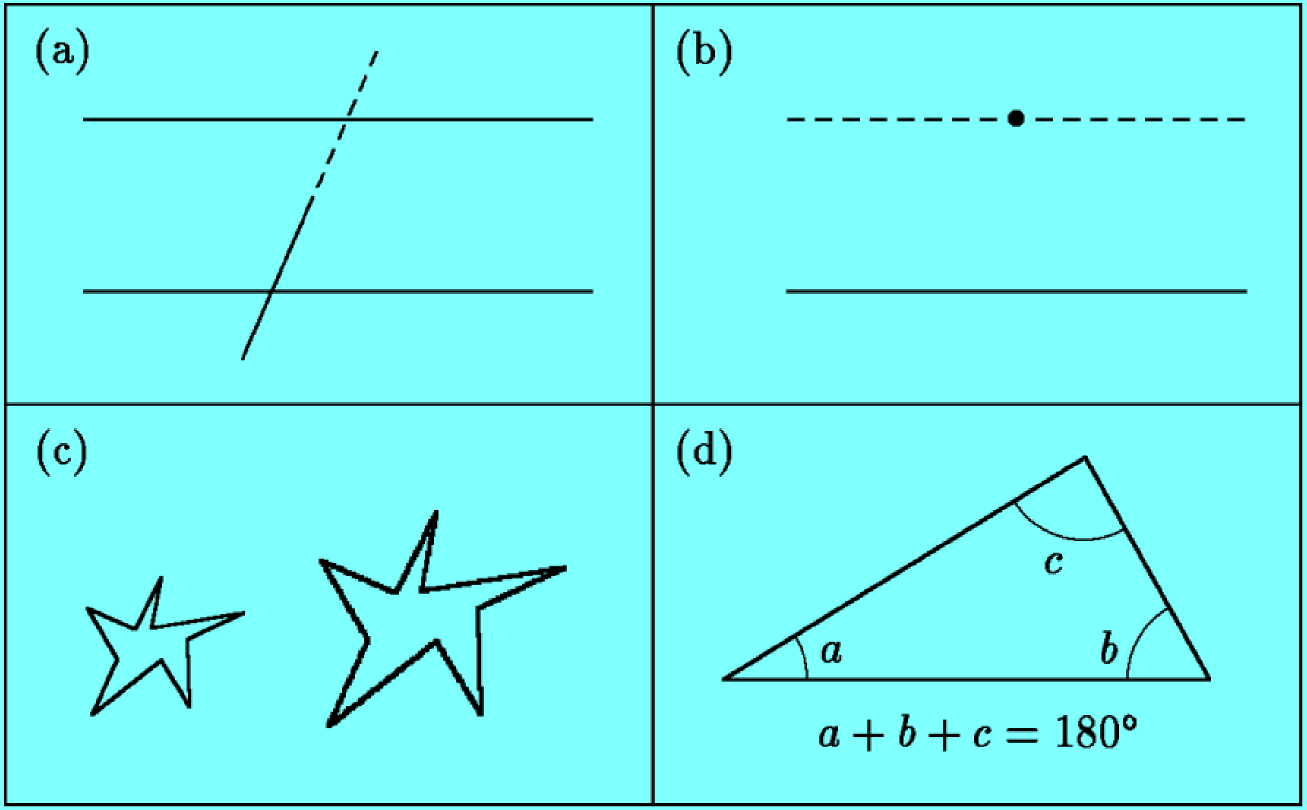
Corrected 10/10/13

Equivalent Statements of the 5th Postulate



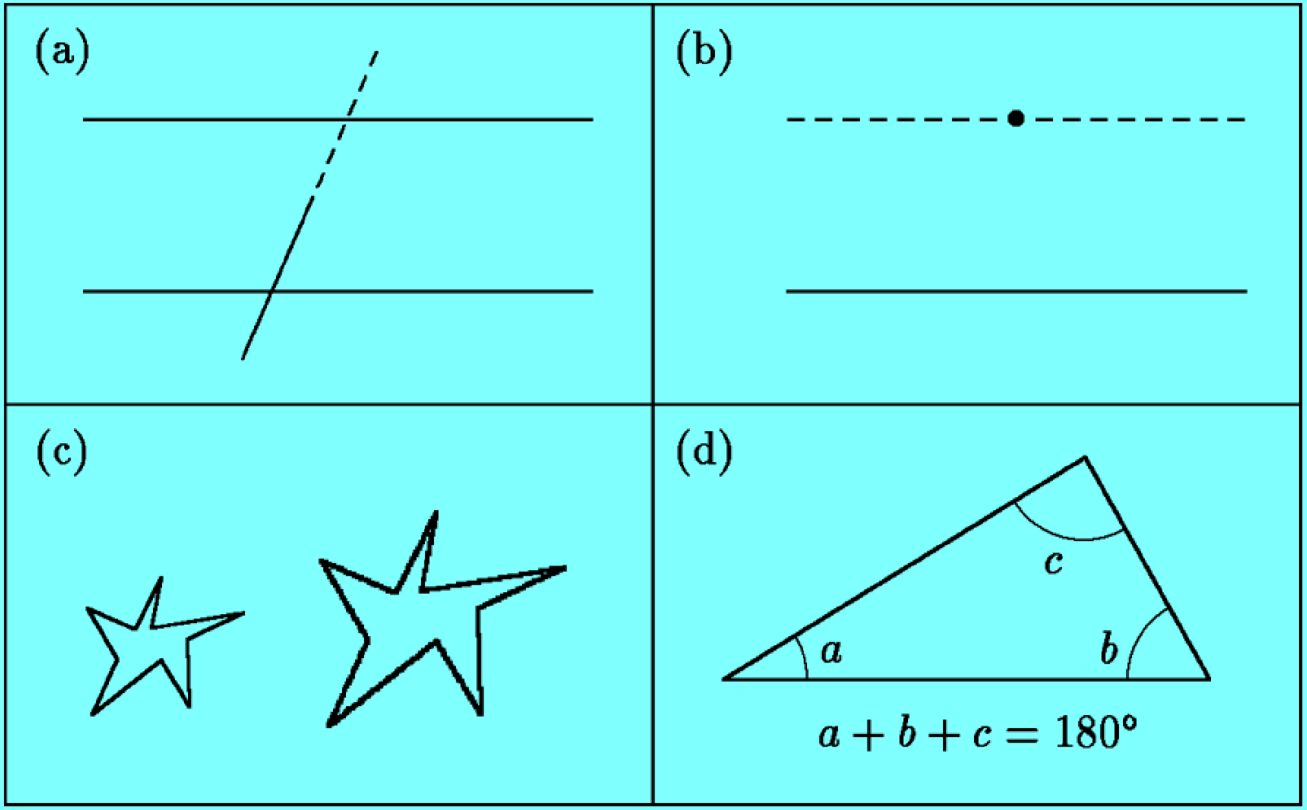
(a) "If a straight line intersects one of two parallels (i.e, lines which do not intersect however far they are extended), it will intersect the other also."

Equivalent Statements of the 5th Postulate



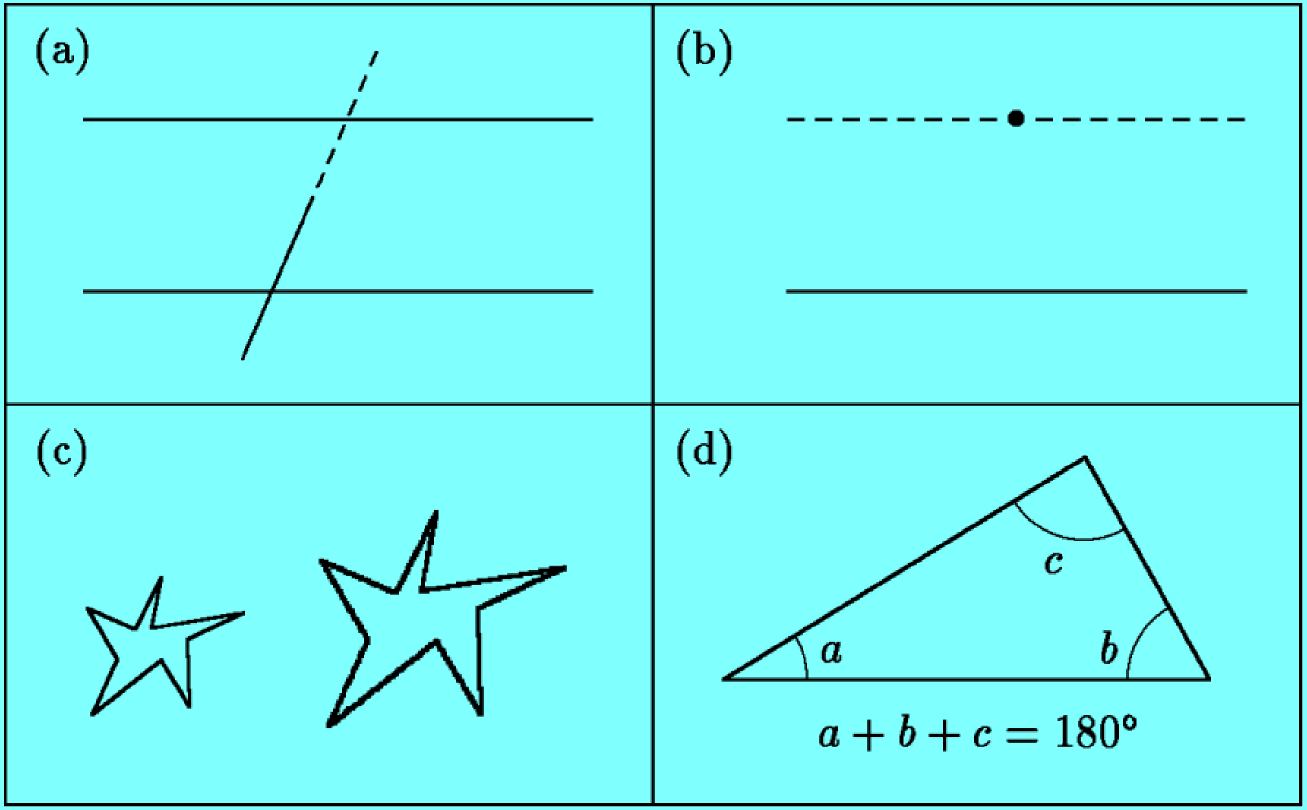
(b) "There is one and only one line that passes through any given point and is parallel to a given line."

Equivalent Statements of the 5th Postulate



(c) "Given any figure there exists a figure, similar to it, of any size." (Two polygons are similar if their corresponding angles are equal, and their corresponding sides are proportional.)

Equivalent Statements of the 5th Postulate



(d) "There is a triangle in which the sum of the three angles is equal to two right angles (i.e., 180°)."

Giovanni Geralamo Saccheri (1667–1733)

EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipſa univerſæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinenſi Univerſitate Matheſeos Profeſſore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permiſſu.

In 1733, Saccheri, a Jesuit priest, published *Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw)*.

The book was a study of what geometry would be like if the 5th postulate were false.

He hoped to find an inconsistency, but failed.

Carl Friedrich Gauss (1777–1855)

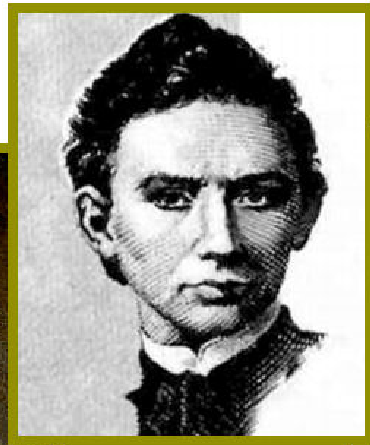
German mathematician and physicist.

Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.

His students included Richard Dedekind, Bernhard Riemann, Peter Gustav Lejeune Dirichlet, Gustav Kirchhoff, and August Ferinand Möbius.

~1750-1850

Gauss

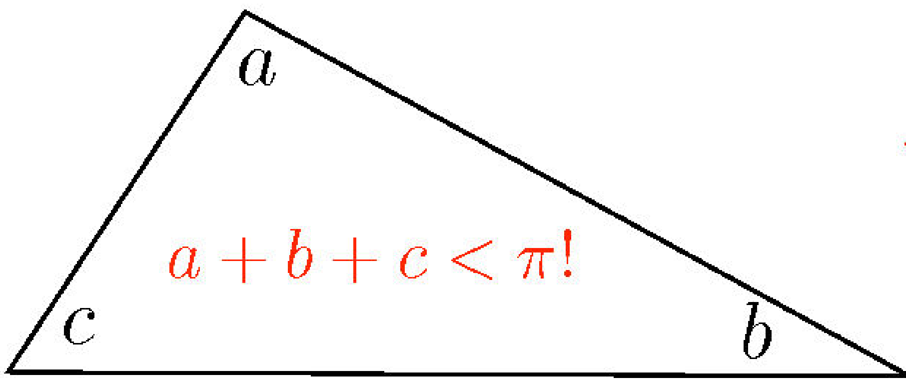


Bolyai

Lobachevski

- infinite
- constant negative curvature

~~5th Postulate~~



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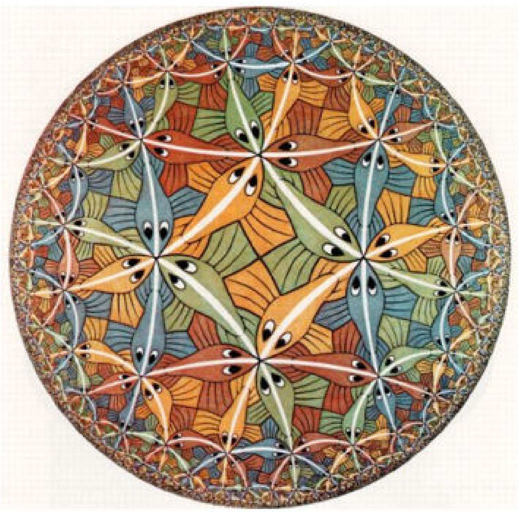
GBL geometry with Klein

1. constant negative curvature
2. infinite
3. ~~5th postulate~~



(x_1, y_1)

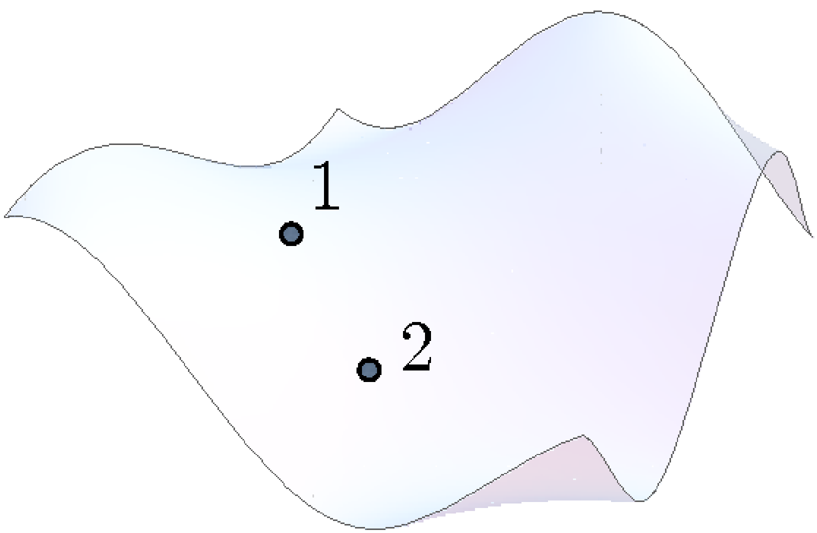
$x^2 + y^2 < 1$

$$d(1,2) = a\cosh^{-1}\left[\frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}}\right]$$

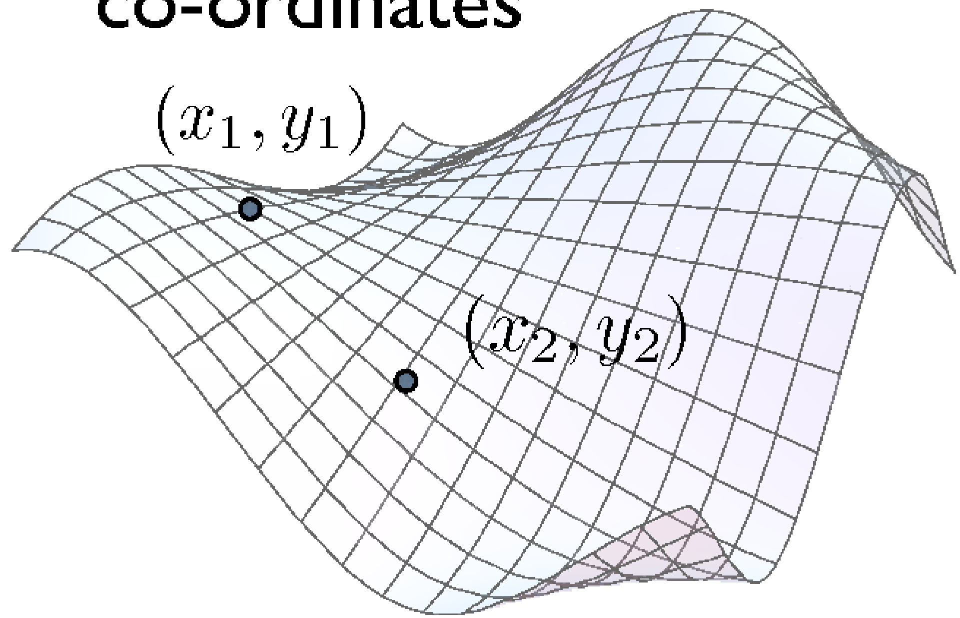
(x_2, y_2)

Note: no global embedding in 3D Euclidean space possible

Geometry (after Klein)



co-ordinates



distance function

$$d[(x_1, y_1), (x_2, y_2)]$$

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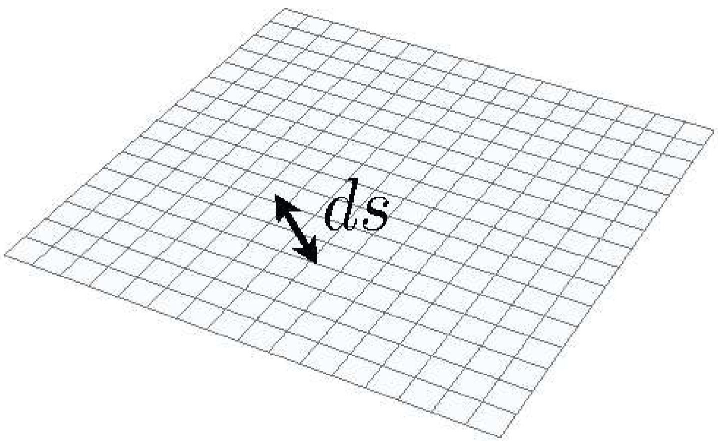
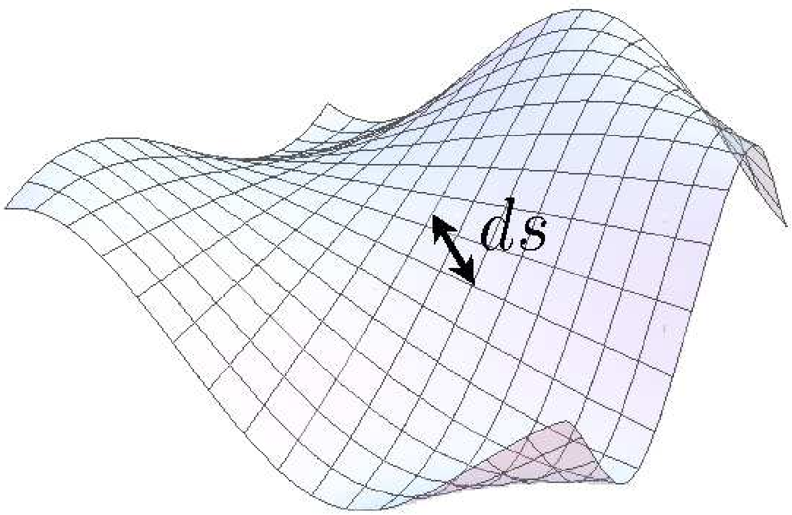
tiny distances



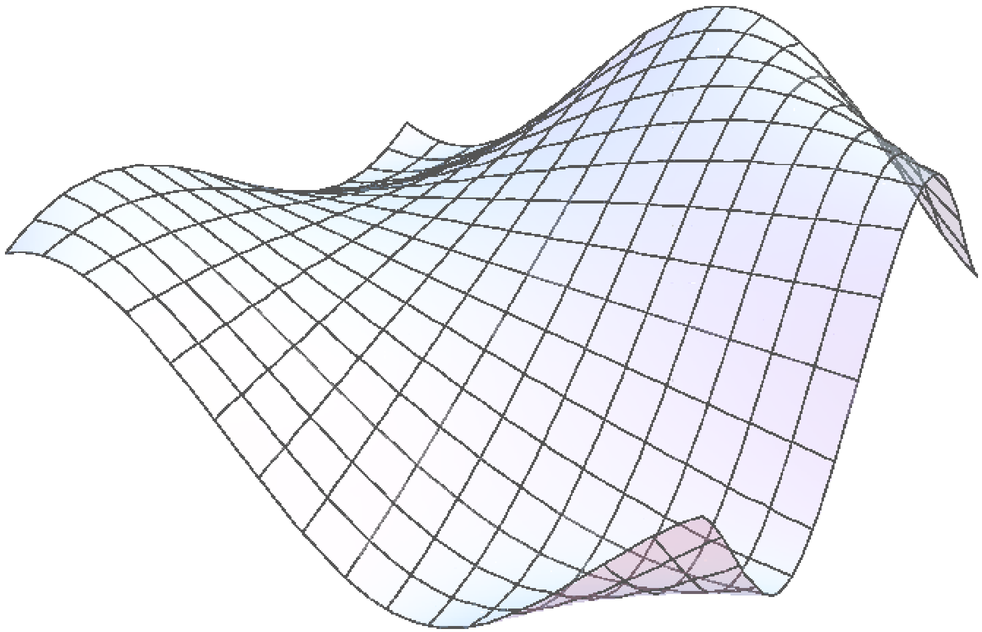


$$ds^2 = dx^2 + dy^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

quadratic form

Image:www.easternct.edu/career/webresources.htm



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

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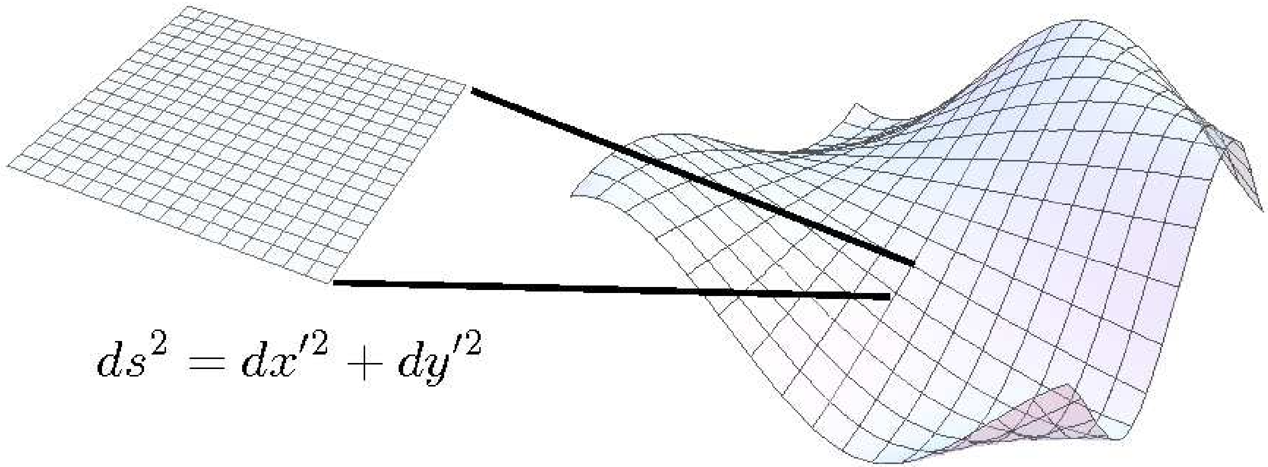
locally Euclidean

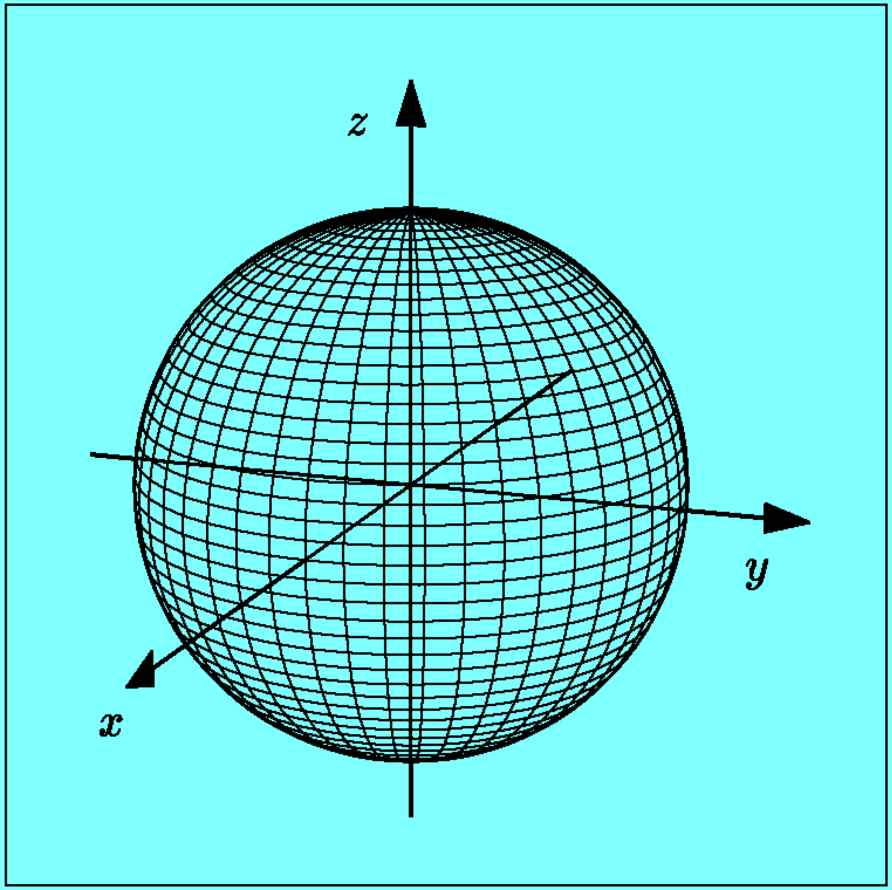


$$ds^2 = dx'^2 + dy'^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

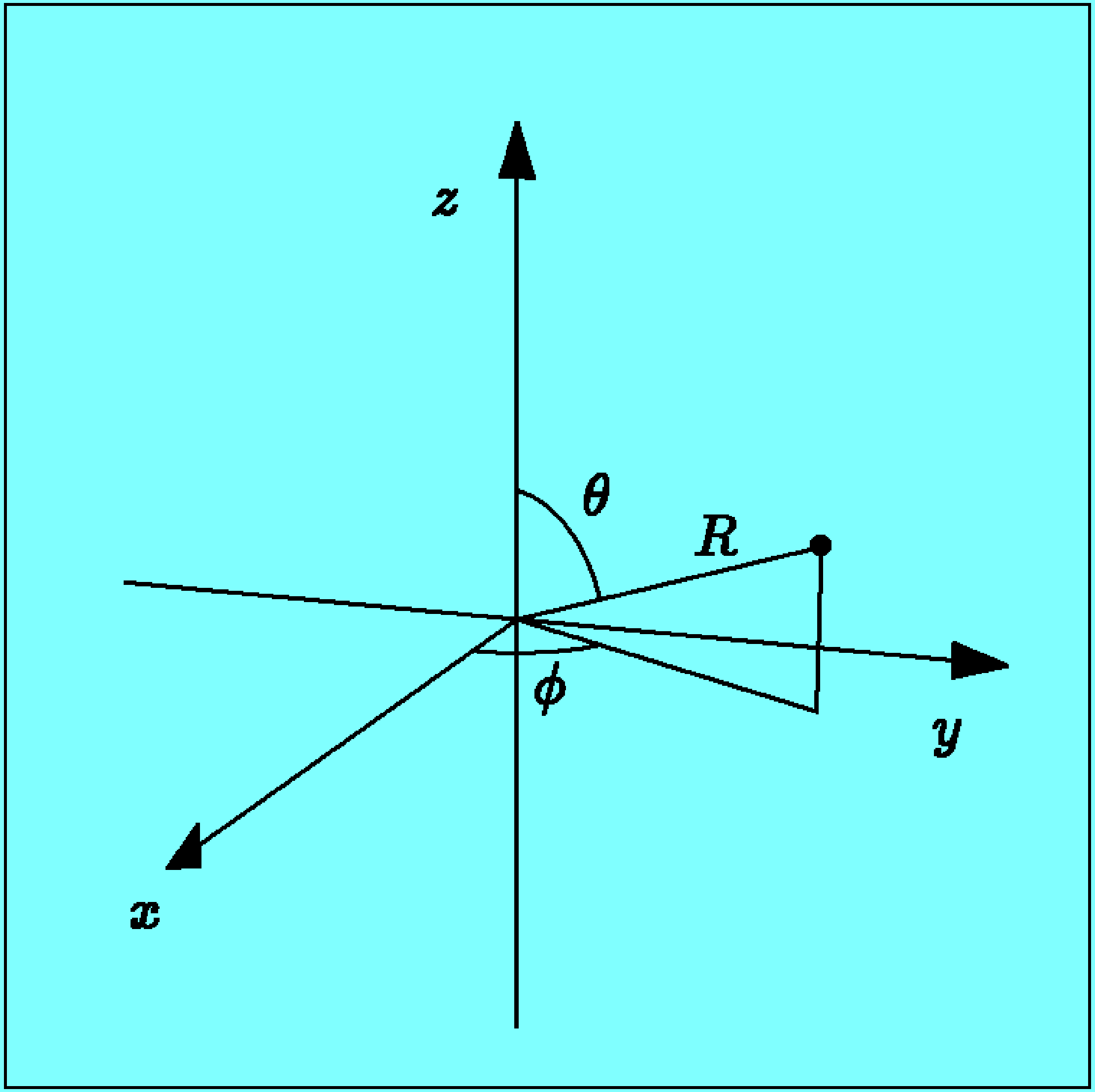
$$g_{xx}g_{yy} - g_{xy}^2 > 0$$

Non-Euclidean Geometry: The Surface of a Sphere

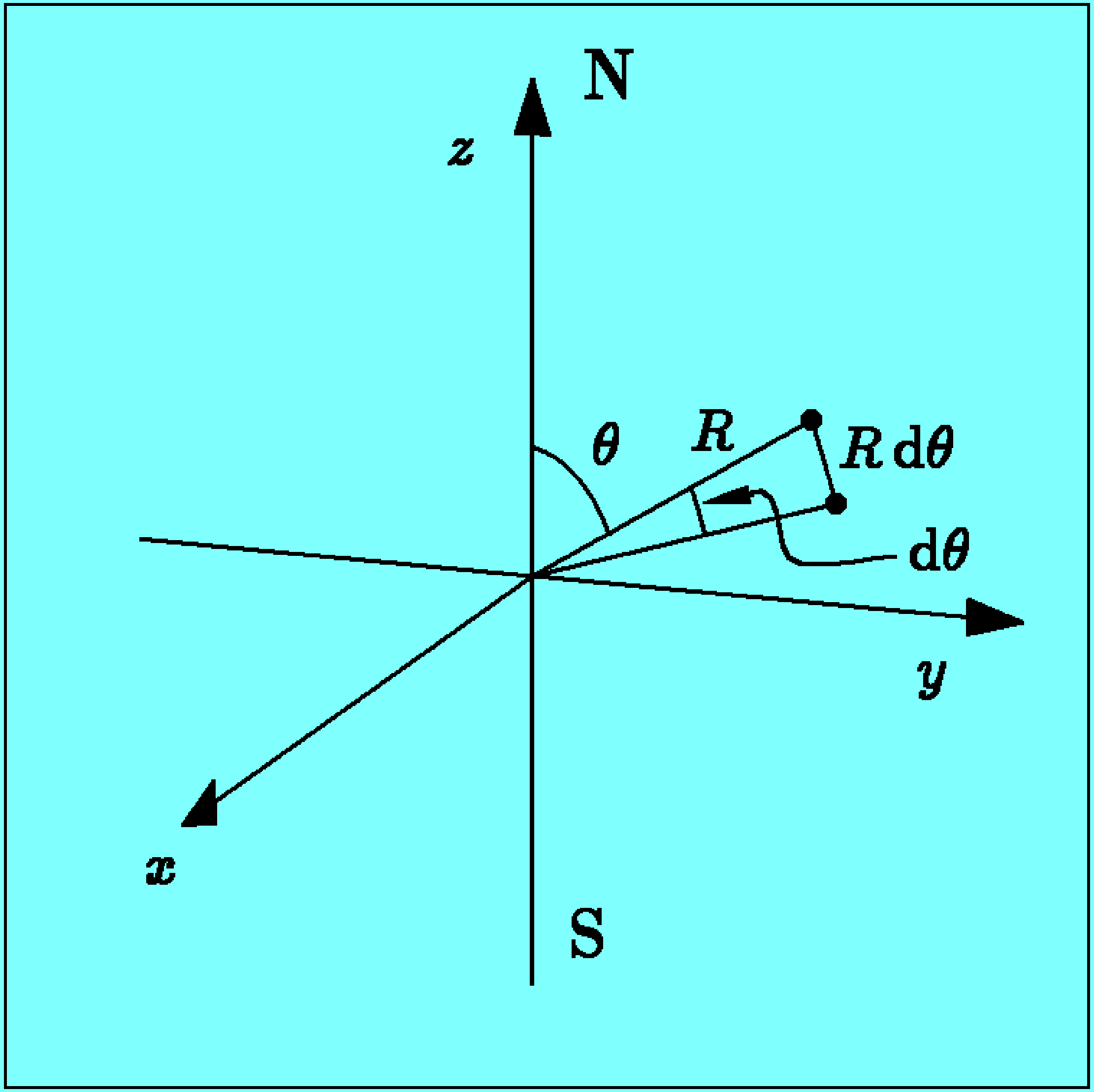


$$x^2 + y^2 + z^2 = R^2 \ .$$

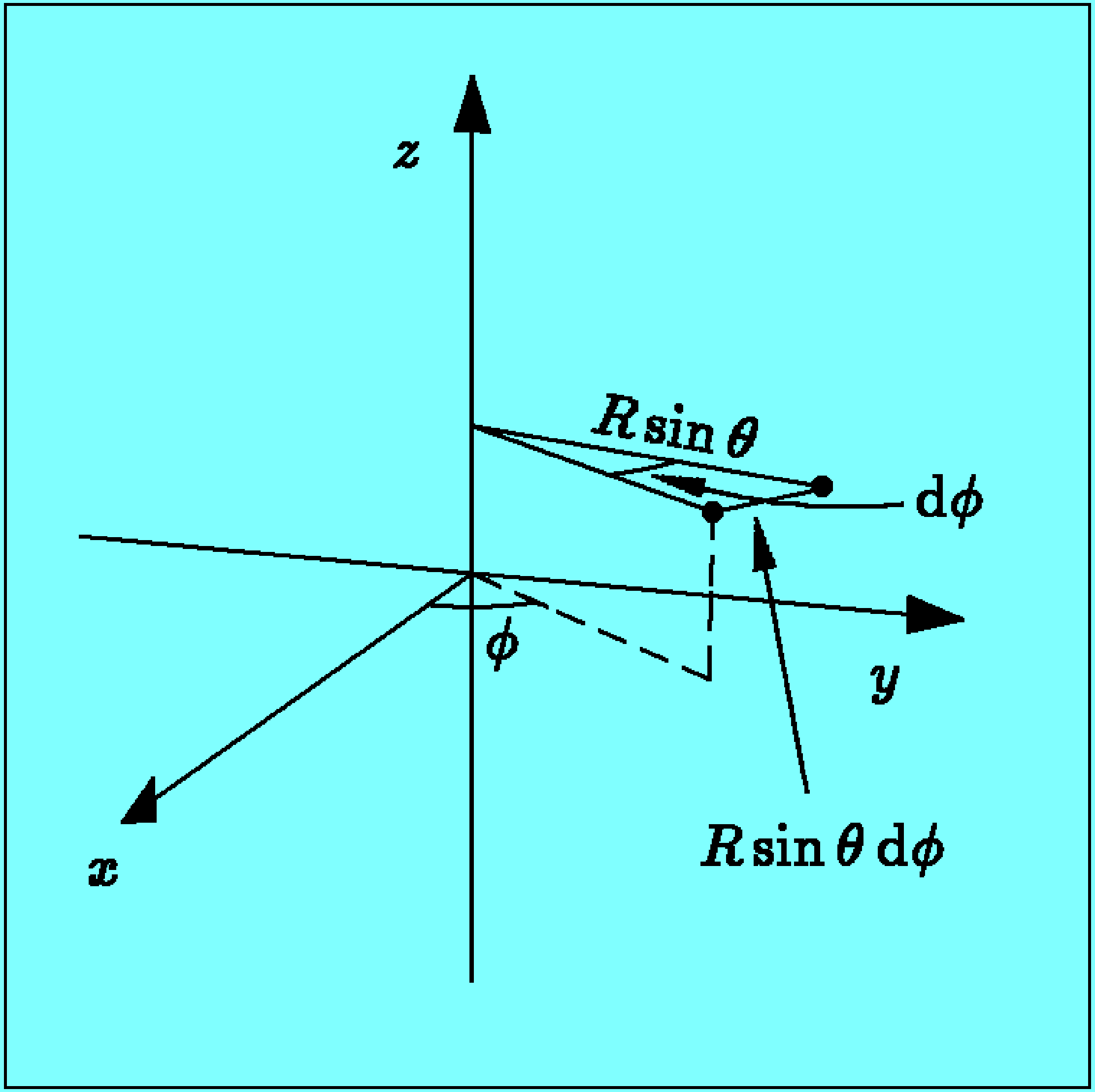
Polar Coordinates:



Varying θ:



Varying ϕ:



A Sphere in 4 Euclidean Dimensions