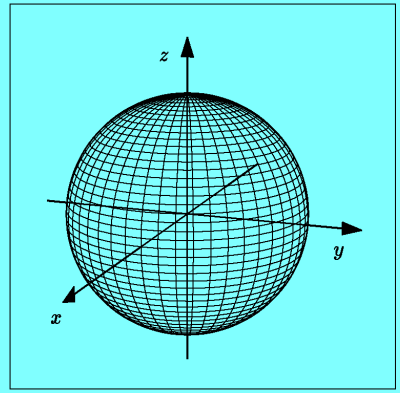
8.286 Lecture 12
October 22, 2013

NON-EUCLIDEAN SPACES: OPEN UNIVERSES AND THE SPACETIME METRIC

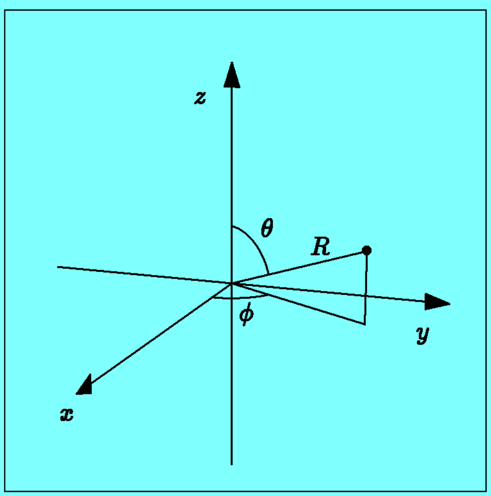
Summary of Lecture 11: Surface of a Sphere



$$x^2 + y^2 + z^2 = R^2 \ .$$

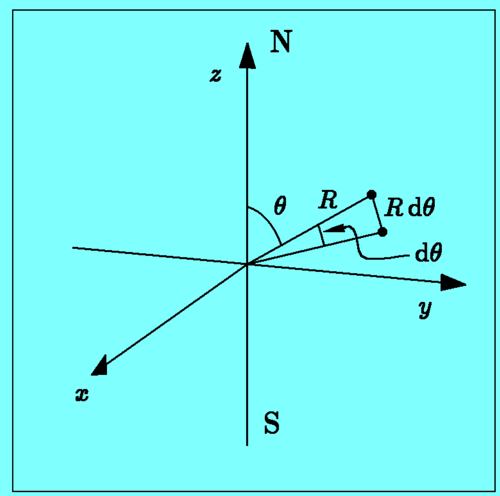
Polar Coordinates:

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta \ ,$$



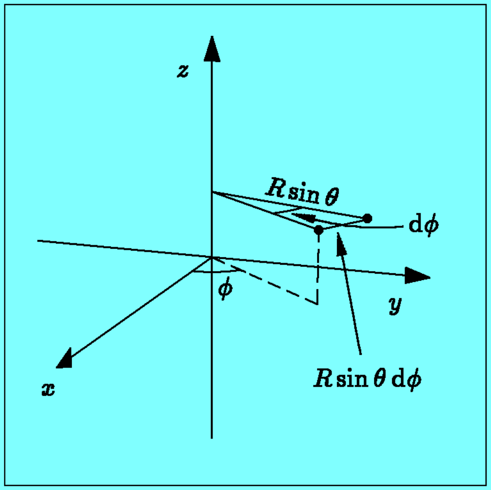
Varying θ:

$$ds = R\,d\theta$$



Varying ϕ:

$$ds = R\sin\theta\, d\phi$$



Varying θ and ϕ

Varying θ: $\quad ds = R\, d\theta$

Varying ϕ: $\quad ds = R\sin\theta\, d\phi$

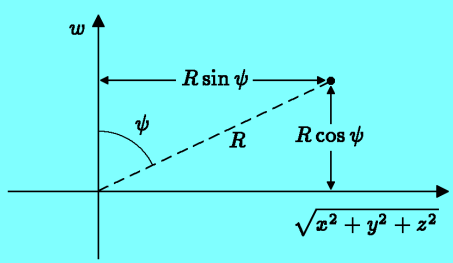
$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Review of Lecture 11: A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$x = R\sin\psi\sin\theta\cos\phi$$
$$y = R\sin\psi\sin\theta\sin\phi$$
$$z = R\sin\psi\cos\theta$$
$$w = R\cos\psi\ ,$$

$$ds = R\, d\psi$$



Metric for the Closed 3D Space

Varying ψ: $\quad ds = R\, d\psi$

Varying θ or ϕ: $\quad ds^2 = R^2\sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)$

If the variations are orthogonal to each other, then

$$ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Proof of Orthogonality of Variations

Let $d\vec{R}_\psi$ = displacement of point when ψ is changed to $\psi + d\psi$.

Let $d\vec{R}_\theta$ = displacement of point when θ is changed to $\theta + d\theta$.

- $d\vec{R}_\theta$ has no w-component $\Longrightarrow$ $d\vec{R}_\psi \cdot d\vec{R}_\theta = d\vec{R}_\psi^{(3)} \cdot d\vec{R}_\theta^{(3)}$, where (3) denotes the projection into the x-y-z subspace.
- $d\vec{R}_\psi^{(3)}$ is radial; $d\vec{R}_\theta^{(3)}$ is tangential

$$\Longrightarrow \quad d\vec{R}_\psi^{(3)} \cdot d\vec{R}_\theta^{(3)} = 0$$

Review of Lecture 11: Implications of General Relativity

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$, where R is radius of curvature.
- According to GR, matter causes space to curve.
- R cannot be arbitrary. Instead, $R^2(t) = \dfrac{a^2(t)}{k}$.
- Finally,

$$\mathrm{d}s^2 = a^2(t) \left\{ \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \right\} ,$$

where $r = \dfrac{\sin\psi}{\sqrt{k}}$. Called the Robertson-Walker metric.