Non-Euclidean Spaces: Spacetime Metric and Geodesic Equation

8.286 Lecture 13
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NON-EUCLIDEAN SPACES: SPACETIME METRIC AND THE GEODESIC EQUATION

Closed Universe:
\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\}, \]
where \( k > 0 \).

Open Universe: \( \text{Same thing, but with } k < 0 \! \)!

Name: Robertson-Walker metric.

Requirements: Isotropy and Homogeneity

Isotropy about the origin is obvious: \( \theta \) and \( \phi \) appear as
\[ a^2(t) r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]
exactly as on a sphere of radius \( a(t)r \).

Open Universe:
\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 + \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\}, \]
where \( \kappa = -k > 0 \).

For closed universe \( (k > 0) \), show homogeneity explicitly by showing that any point \( (r_0, \theta_0, \phi_0) \) is equivalent to the origin:
Construct a map \( (r, \theta, \phi) \rightarrow (r', \theta', \phi') \) which preserves metric, and maps \( (r_0, \theta_0, \phi_0) \) to the origin, \( (r' = 0, \theta', \phi') \).

Construct map in three steps:
\[ (r, \theta, \phi) \rightarrow (x, y, z, w) \quad \rightarrow \quad (x', y', z', w') \rightarrow (r', \theta', \phi'). \]
The same map works for \( k < 0 \), showing that any point can be mapped to the origin by a metric-preserving mapping. (We will not show it.)
We will not show it, but any 3D homogeneous isotropic space can be described by the Robertson-Walker metric, for $k$ positive, negative, or zero (flat universe).

For $k > 0$, the universe is finite. For $k \leq 0$, the universe is infinite.

The Gauss–Bolyai–Lobachevsky geometry is the 2-dimensional open universe.

In special relativity, $s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2$.

$s_{AB}^2$ is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of $s_{AB}^2$:

- If positive, it is the distance between the two events in the frame in which they are simultaneous. (Spacelike.)
- If negative, it is the time interval between the two events in the frame in which they occur at the same place. (Timelike.)
- If zero, it implies that a light pulse could travel from $A$ to $B$ (or from $B$ to $A$).