8.286 Lecture 13
October 24, 2013

NON-EUCLIDEAN SPACES: SPACETIME METRIC AND THE GEODESIC EQUATION

Summary of Lecture 12: Open Universe Metric

Closed Universe:

$$\boxed{ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \ ,}$$

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Name: **Robertson-Walker metric.**

Summary, Cont:
Why is This the Open Universe Metric?

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where $\kappa = -k > 0$.

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Requirements: Isotropy and Homogeneity

Isotropy about the origin is obvious: θ and ϕ appear as

$$a^2(t)r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \;,$$

exactly as on a sphere of radius $a(t)r$.

Summary, Cont: Why is This Homogeneous?

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For closed universe ($k > 0$), show homogeneity explicitly by showing that any point (r_0, θ_0, ϕ_0) is equivalent to the origin: Construct a map $(r, \theta, \phi) \to (r', \theta', \phi')$ which preserves metric, and maps (r_0, θ_0, ϕ_0) to the origin, $(r' = 0, \theta', \phi')$.

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Construct map in three steps:

$$(r, \theta, \phi) \to (x, y, z, w) \underset{\text{rotation}}{\longrightarrow} (x', y', z', w') \to (r', \theta', \phi') \;.$$

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The same map works for $k < 0$, showing that any point can be mapped to the origin by a metric-preserving mapping. (We will not show it.)

- We will not show it, but any 3D homogeneous isotropic space can be described by the Robertson-Walker metric, for k positive, negative, or zero (flat universe).
- For $k > 0$, the universe is finite. For $k <= 0$, the universe is infinite.
- The Gauss–Bolyai–Lobachevsky geometry is the 2-dimensional open universe.

Summary, Cont: From Space to Spacetime

In special relativity,

$$s^2_{AB} \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2\,(t_A - t_B)^2 \ .$$

s^2_{AB} is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of s^2_{AB}:

If positive, it is the distance between the two events in the frame in which they are simultaneous. (Spacelike.)

If negative, it is the time interval between the two events in the frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from A to B (or from B to A).