8.286 Lecture 13
October 24, 2013

NON-EUCLIDEAN SPACES:
SPACETIME METRIC
AND THE
GEODESIC EQUATION
Summary of Lecture 12: 
Open Universe Metric

Closed Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\} , \]
Summary of Lecture 12: Open Universe Metric

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\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \]

where \( k > 0 \).
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Open Universe: \( \text{Same thing, but with } k < 0! \)
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where \( k > 0 \).

Open Universe: \textbf{Same thing, but with} \( k < 0 \)!

Name: \textbf{Robertson-Walker metric}.
Summary, Cont: Why is This the Open Universe Metric?

Open Universe:

\[
    ds^2 = a^2(t) \left\{ \frac{dr^2}{1 + \kappa r^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\},
\]

where \( \kappa = -k > 0 \).
Summary, Cont: Why is This the Open Universe Metric?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 + \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\}, \]

where \( \kappa = -k > 0 \).

Requirements: Isotropy and Homogeneity
Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 + \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\} , \]

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Requirements: Isotropy and Homogeneity

Isotropy about the origin is obvious: \( \theta \) and \( \phi \) appear as

\[ a^2(t) r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) , \]

exactly as on a sphere of radius \( a(t)r \).
Summary, Cont: Why is This Homogeneous?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\}, \text{ with } k < 0. \]
Summary, Cont: Why is This Homogeneous?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) \right\} \], with \( k < 0 \).

For closed universe \( (k > 0) \), show homogeneity explicitly by showing that any point \( (r_0, \theta_0, \phi_0) \) is equivalent to the origin: Construct a map \( (r, \theta, \phi) \rightarrow (r', \theta', \phi') \) which preserves metric, and maps \( (r_0, \theta_0, \phi_0) \) to the origin, \( (r' = 0, \theta', \phi') \).
Summary, Cont: Why is This Homogeneous?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\} , \quad \text{with} \quad k < 0 . \]

For closed universe \((k > 0)\), show homogeneity explicitly by showing that any point \((r_0, \theta_0, \phi_0)\) is equivalent to the origin:

Construct a map \((r, \theta, \phi) \rightarrow (r', \theta', \phi')\) which preserves metric, and maps \((r_0, \theta_0, \phi_0)\) to the origin, \((r' = 0, \theta', \phi')\).

Construct map in three steps:

\[
(r, \theta, \phi) \rightarrow (x, y, z, w) \quad \rightarrow \quad (x', y', z', w') \quad \rightarrow \quad (r', \theta', \phi') .
\]

rotation
Open Universe:

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\}, \quad \text{with } k < 0.$$ 

For closed universe ($k > 0$), show homogeneity explicitly by showing that any point $(r_0, \theta_0, \phi_0)$ is equivalent to the origin: Construct a map $(r, \theta, \phi) \rightarrow (r', \theta', \phi')$ which preserves metric, and maps $(r_0, \theta_0, \phi_0)$ to the origin, $(r' = 0, \theta', \phi')$.

Construct map in three steps:

$$(r, \theta, \phi) \rightarrow (x, y, z, w) \xrightarrow{\text{rotation}} (x', y', z', w') \rightarrow (r', \theta', \phi').$$

The same map works for $k < 0$, showing that any point can be mapped to the origin by a metric-preserving mapping.
Summary, Cont: Why is This Homogeneous?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right\} , \quad \text{with} \quad k < 0 . \]

For closed universe \((k > 0)\), show homogeneity explicitly by showing that any point \((r_0, \theta_0, \phi_0)\) is equivalent to the origin: Construct a map \((r, \theta, \phi) \rightarrow (r', \theta', \phi')\) which preserves metric, and maps \((r_0, \theta_0, \phi_0)\) to the origin, \((r' = 0, \theta', \phi')\).

Construct map in three steps:

\[(r, \theta, \phi) \rightarrow (x, y, z, w) \xrightarrow{\text{rotation}} (x', y', z', w') \rightarrow (r', \theta', \phi') .\]

The same map works for \(k < 0\), showing that any point can be mapped to the origin by a metric-preserving mapping. (We will not show it.)
We will not show it, but any 3D homogeneous isotropic space can be described by the Robertson-Walker metric, for $k$ positive, negative, or zero (flat universe).

For $k > 0$, the universe is finite. For $k \leq 0$, the universe is infinite.

The Gauss–Bolyai–Lobachevsky geometry is the 2-dimensional open universe.
In special relativity,

\[ s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2. \]

\( s_{AB}^2 \) is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of \( s_{AB}^2 \):

If positive, it is the distance between the two events in the frame in which they are simultaneous. (Spacelike.)

If negative, it is the time interval between the two events in the frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from \( A \) to \( B \) (or from \( B \) to \( A \)).