8.286 Lecture 14
October 29, 2013

THE GEODESIC EQUATION

Summary of Lecture 13: Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Meaning:

- If $ds^2 > 0$, it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- If $ds^2 < 0$, it is $-c^2$ times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- If $ds^2 = 0$, then the two events can be joined by a light pulse.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Application: **The Geodesic Equation.**

THE GEODESIC EQUATION

Notation: $ds^2 = g_{ij}(x^k)\,dx^i\,dx^j$.

Description of path from A to B:

$x^i(\lambda)$, where $x^i(0) = x^i_A, \quad x^i(\lambda_f) = x^i_B$.

Distance from $x^i(\lambda)$ to $x^i(\lambda + d\lambda)$:

$$ds^2 = g_{ij}(x^k(\lambda))\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\,d\lambda^2 .$$

Distance from A to B:

$$S[x^i(\lambda)] = \int_0^{\lambda_f}\sqrt{g_{ij}(x^k(\lambda))\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\,d\lambda .$$

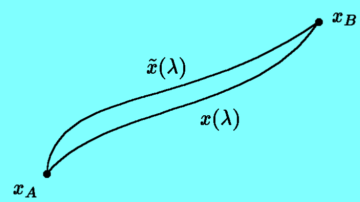
Varying the path (calculus of variations, 1696):

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

$$w^i(0) = 0 \ , \quad w^i(\lambda_f) = 0 \ ,$$

$$\left. \frac{\mathrm{d}\, S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha} \right|_{\alpha=0} = 0$$

for all $w^i(\lambda)$,



where

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\, \mathrm{d}\lambda \ , \qquad A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right) \frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ .$$

$$\left. \frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha} \right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}} \times \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij} \frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \ .$$

Integrate second term by parts!

$$\left. \frac{\mathrm{d}S}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] \right\} w^i(\lambda)\, \mathrm{d}\lambda \ .$$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \ .$$