*8.286 Lecture 14*
*October 29, 2013*

# THE GEODESIC EQUATION

# Summary of Lecture 13: Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} .$$

Meaning:

- If $ds^2 > 0$, it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- If $ds^2 < 0$, it is $-c^2$ times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- If $ds^2 = 0$, then the two events can be joined by a light pulse.

# Adding Time to the Robertson–Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Application: **The Geodesic Equation.**

# THE GEODESIC EQUATION

Notation: $\mathrm{d}s^2 = g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j\ .$

Description of path from $A$ to $B$:

$x^i(\lambda)$, where $x^i(0) = x^i_A, \quad x^i(\lambda_f) = x^i_B\ .$

Distance from $x^i(\lambda)$ to $x^i(\lambda + \mathrm{d}\lambda)$:

$$\mathrm{d}s^2 = g_{ij}(x^k(\lambda))\,\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\,\mathrm{d}\lambda^2\ .$$

Distance from $A$ to $B$:

$$S[x^i(\lambda)] = \int_0^{\lambda_f}\sqrt{g_{ij}(x^k(\lambda))\,\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda\ .$$

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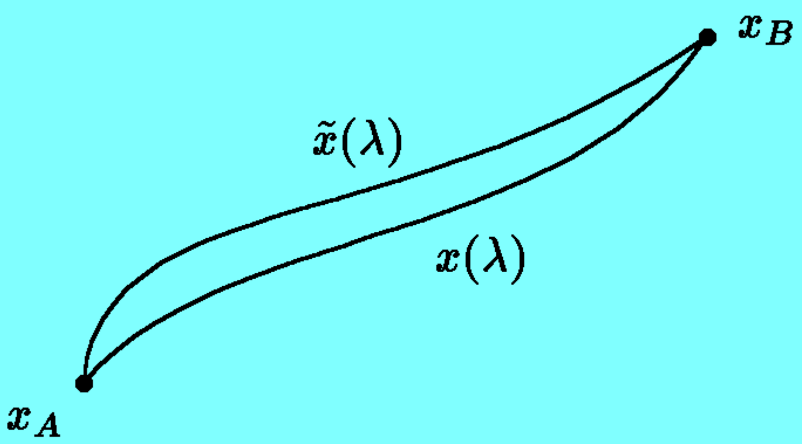
Varying the path (calculus of variations, 1696):

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

$$w^i(0) = 0 \ , \quad w^i(\lambda_f) = 0 \ ,$$

$$\left. \frac{\mathrm{d}\, S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha} \right|_{\alpha=0} = 0$$

for all $w^i(\lambda)$ ,




where

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda, \alpha)}\, \mathrm{d}\lambda \ , \qquad A(\lambda, \alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right) \frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda} \frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ .$$

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}} \times \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \; .$$

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}$$

$$\times\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

**Integrate second term by parts!**

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}$$

$$\times \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij} \frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \ .$$

**Integrate second term by parts!**

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[ \frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] \right\} w^i(\lambda)\, \mathrm{d}\lambda \ .$$

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}$$

$$\times \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij} \frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \; .$$

**Integrate second term by parts!**

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] \right\} w^i(\lambda)\, \mathrm{d}\lambda \; .$$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \; .$$