8.286 Lecture 15
October 31, 2013

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE

Summary of Lecture 14: THE SPACETIME GEODESIC EQUATION

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

- Use indices μ, ν, etc., which are summed from 0 to 3, where $x^0 \equiv t$.
- Use τ to parameterize the path, where τ = proper time measured along the path.

THE SCHWARZSCHILD METRIC

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ .$$

- Describes the metric for any spherically symmetric mass distribution, for the region outside the mass distribution. M is the mass of the object, G is Newton's constant, and c is (of course) the speed of light.
- At $r = R_S$, where

$$R_S = \frac{2GM}{c^2}$$

is called the Schwarzschild horizon, the metric is singular. But the singularity is not physical, and can be removed by a different choice of coordinates. R_S is, however, a horizon: anything with $r < R_S$ can never get out.

RADIAL GEODESICS

☆ For $\mu = r$, the geodesic equation gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .$$

which simplifies to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2} .$$

It looks like Newton, but r is not really the distance from the origin, and τ is the proper time measured along the trajectory.

Solving the Radial Infall Equation

- The proper time τ needed to reach radial variable r is

$$\tau(r) = \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r}{r}} \right) + \sqrt{r(r_0 - r)} \right\} .$$

- The infalling object will be ripped apart by the singularity at $r = 0$ in a finite amount of the object's proper time.
- But from the outside, it will take an infinite amount of coordinate time t before the object reaches the horizon.