8.286 Lecture 16
November 5, 2013

BLACK-BODY RADIATION
AND
THE EARLY HISTORY
OF THE UNIVERSE, PART 2
Summary of Lecture 15: Relativistic Energy

\[ E = mc^2 \]

The conversion of 1 kg/hour would supply 1.5 times the world power production (for 2011).

A 15-gallon tank of gasoline could power the world for 2 1/2 days. But: when U-235 undergoes fission, only about 0.09% of mass is converted to energy.
Summary of Lecture 15: Relativistic Four-Momentum

\[ p^0 = \frac{E}{c}, \quad p^\mu = \left( \frac{E}{c}, \vec{p} \right). \]

\( p^\mu \) transforms from one inertial frame to another in the same way as \( x^\mu = (ct, \vec{x}) \).

\[ \vec{p} = \gamma m_0 \vec{v}, \quad E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2}, \]

where

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \]
Lorentz-invariant square of $p^{\mu}$:

$$p^2 \equiv |\vec{p}|^2 - (p^0)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = - (m_0 c)^2 .$$

The Hydrogen Atom:

$$m_H = m_p + m_e - \frac{\Delta E}{c^2} ,$$

where $m_H$, $m_p$, and $m_e$ are the masses of the hydrogen atom, the proton, and the electron, and $\Delta E = 13.6$ eV is the binding energy.
Summary of Lecture 15: The Mass of Radiation

\[ \rho = \frac{u}{c^2}, \]

where \( \rho = \) mass density, \( u = \) energy density.

The photon has zero rest mass:

\[ |\vec{p}|^2 - \frac{E^2}{c^2} = 0, \quad \text{or} \quad E = c|\vec{p}|. \]
Summary of Lecture 15: Radiation in an Expanding Universe

Redshift:

\[ n_\gamma \propto \frac{1}{a^3(t)}, \quad E_\gamma \propto \frac{1}{a(t)} \quad \Rightarrow \quad \rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)}. \]

Radiation-Dominated Era:

\[ u_r = 7.01 \times 10^{-14} \text{ J/m}^3 \quad \Rightarrow \quad \frac{\rho_{r,0}}{\rho_{m,0}} \approx 3.1 \times 10^{-4}. \]
\[
\frac{\rho_r(t)}{\rho_m(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4},
\]

\[
\frac{\rho_r(t_{eq})}{\rho_m(t_{eq})} \equiv 1 \implies \frac{a(t_0)}{a(t_{eq})} = \frac{1}{3.1 \times 10^{-4}} \approx 3200.
\]

Assuming that \(a(t) \propto t^{2/3}\) from \(t_{eq}\) until \(t_0\) (crude approximation), find \(t_{eq} = (3.1 \times 10^{-4})^{3/2} t_0 \approx 75,000\) yr, for \(t_0 = 13.8\) Gyr.
Summary of Lecture 15: 
Modification of the Friedmann Equations

\[ \rho \propto \frac{1}{a^3} \implies \dot{\rho} = -3 \frac{\dot{a}}{a} \rho, \quad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot{\rho} = -4 \frac{\dot{a}}{a} \rho. \]

\[ \dot{\rho} \text{ and pressure } p: \]

\[ dU = -p \, dV \implies \frac{d}{dt} (a^3 \rho c^2) = -p \frac{d}{dt} (a^3) \]

\[ \implies \dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right). \]
Friedmann equations:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{k c^2}{a^2} \quad \text{(matter-dominated universe)}
\]

\[
\ddot{a} = -\frac{4\pi}{3} G \rho a,
\]

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} \rho
\]

Becomes inconsistent if

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right).
\]

Solution:

\[
\ddot{a} = -\frac{4\pi}{3} G \left( \rho + \frac{3p}{c^2} \right) a.
\]