8.286 Lecture 16
November 5, 2013

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE, PART 2

Summary of Lecture 15: Relativistic Energy

$$E = mc^2$$

The conversion of 1 kg/hour would supply 1.5 times the world power production (for 2011).

A 15-gallon tank of gasoline could power the world for 2 1/2 days.

But: when U-235 undergoes fission, only about 0.09% of mass is converted to energy.

Summary of Lecture 15: Relativistic Four-Momentum

$$p^0 = \frac{E}{c} \; , \quad p^\mu = \left(\frac{E}{c} \, , \vec{p} \right) \; .$$

p^μ transforms from one inertial frame to another in the same way as $x^\mu = (ct, \vec{x})$.

$$\vec{p} = \gamma m_0 \vec{v} \; , \quad E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 \, c^2} \; ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; .$$

Lorentz-invariant square of p^μ:

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

The Hydrogen Atom:

$$m_H = m_p + m_e - \Delta E/c^2 \ ,$$

where m_H, m_p, and m_e are the masses of the hydrogen atom, the proton, and the electron, and $\Delta E = 13.6$ eV is the binding energy.

Summary of Lecture 15: The Mass of Radiation

$$\rho = u/c^2 \ ,$$

where ρ = mass density, u = energy density.

The photon has zero rest mass:

$$|\vec{p}\,|^2 - \frac{E^2}{c^2} = 0 \ , \quad \text{or} \quad E = c|\vec{p}\,| \ .$$

Summary of Lecture 15: Radiation in an Expanding Universe

Redshift:

$$n_\gamma \propto \frac{1}{a^3(t)} \, , \; E_\gamma \propto \frac{1}{a(t)} \quad \Longrightarrow \quad \rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)} \; .$$

Radiation-Dominated Era:

$$u_r = 7.01 \times 10^{-14} \text{ J/m}^3 \quad \Longrightarrow \quad \frac{\rho_{r,0}}{\rho_{m,0}} \approx 3.1 \times 10^{-4} \; .$$

$$\frac{\rho_r(t)}{\rho_m(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4} \ ,$$

$$\frac{\rho_r(t_{\rm eq})}{\rho_m(t_{\rm eq})} \equiv 1 \quad \Longrightarrow \quad \frac{a(t_0)}{a(t_{\rm eq})} = \frac{1}{3.1 \times 10^{-4}} \approx 3200 \ .$$

Assuming that $a(t) \propto t^{2/3}$ from $t_{\rm eq}$ until t_0 (crude approximation), find $t_{\rm eq} = (3.1 \times 10^{-4})^{3/2} t_0 \approx 75,000$ yr, for $t_0 = 13.8$ Gyr.

Summary of Lecture 15: Modification of the Friedmann Equations

$$\rho \propto \frac{1}{a^3} \implies \dot{\rho} = -3\frac{\dot{a}}{a}\rho\ , \qquad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot{\rho} = -4\frac{\dot{a}}{a}\rho\ .$$

$\dot{\rho}$ and pressure p:

$$dU = -p\,dV \qquad \implies \qquad \frac{d}{dt}\left(a^3\rho c^2\right) = -p\frac{d}{dt}(a^3)$$

$$\implies \qquad \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

$$\ddot a = -\frac{4\pi}{3}G\rho a \ ,$$

$$\dot\rho = -3\frac{\dot a}{a}\,\rho$$

$\left(\begin{array}{c}\text{matter-dominated}\\ \text{universe}\end{array}\right)$

Becomes inconsistent if

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ .$$

Solution:

$$\ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right) a \ .$$